An extension of Gronwall's inequality

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1 Introduction

The Gronwall inequality is a well-known tool in the study of differential equations, Volterra integral equations, and evolution equations [2]. It is often used to establish *a priori* bounds which are used in proving global existence, uniqueness, and stability results. There are a number of versions of the result, a well-known version was first proved by Bellman according to the book [3]. One classical version reads as follows.

Theorem 1.1. *Suppose that* $\phi(t) \le c_0(t) + \int_0^t c_1(s)\phi(s)\,ds$ *for a.e.* $t \in [0,T]$*, where* c_0 *is non-negative and non-decreasing,* $c_1 \in L^1_+$*, and* $\phi \in L^\infty_+$*. [Subscript + denotes functions that are* ≥ 0 *a.e.] Then*

$$\phi(t) \le c_0(t) \exp\left(\int_0^t c_1(s)\,ds\right) \ \ a.e. \ . \tag{1.1}$$

Another somewhat different inequality is the following, apparently first given by Liang Ou-Iang [5], and also given in [2],(both authors consider continuous functions).

Theorem 1.2. *Suppose that* $\phi^2(t) \le c_0^2 + 2\int_0^t h(s)\phi(s)\,ds$ *for a.e.* $t \in [0,T]$*, where* c_0 *is a constant,* $h \in L^1_+$*, and* $\phi \in L^\infty_+$*. Then, for a.e.* $t \in [0,T]$*,*

$$\phi(t) \le c_0 + \int_0^t h(s)\,ds.$$

If $h \in L^2$, by using the inequality $h\phi \le h^2/2 + \phi^2/2$ this case could be reduced to the classical case but would give a different type of conclusion.

We have discovered a 'nonlinear' version of Gronwall's inequality that contains both of these results and many other inequalities of Gronwall type given in the literature, see for example [3].

To describe the type of results we prove consider the following simple case. Suppose that a non-negative L^∞ function ϕ satisfies a.e. the inequality

$$\phi^2(t) \le c_0^2 + \int_0^t 2c_1(s)\phi(s) + 2c_2(s)\phi^2(s)\,ds,$$

where c_i are non-negative L^1 functions. Then we prove that

$$\phi(t) \le \left[c_0 + \textstyle\int_0^t c_1(s)\,ds\right] \exp\left(\textstyle\int_0^t c_2\right) \text{ a.e..}$$

This result contains both of the results just cited and also a special case of a result of Dafermos [1], see below. We are interested in more general inequalities and in obtaining *explicit* L^∞ bounds for ϕ which depend only on c_0 and integrals of the given functions c_i. Also we consider the case when the inequalities hold a.e. since this is a commonly occurring situation.

2 Extended Gronwall inequality

We give a general result which contains Theorems 1.1 and 1.2 and many other results as well as giving new ones. To make the formula a little easier we insert some constants.

Theorem 2.1. *Let $k \geq 1$ be an integer. Suppose that*

$$\phi^k(t) \leq c_0^k(t) + \int_0^t kc_1(s)\phi(s) + kc_2(s)\phi^2(s) + \cdots + kc_k(s)\phi^k(s)\,ds, \text{ for a.e. } t \in [0,T], \tag{2.1}$$

where $c_0 \geq 0$ is non-decreasing, $\phi \in L^\infty_+$, and $c_i \in L^1_+$ for $i \geq 1$. Then,

$$\phi(t) \leq W_k(c_0, c_1, \ldots, c_{k-1}) \exp\left(\int_0^t c_k\right) \ a.e. \tag{2.2}$$

where the function W_k is defined recursively by

$$\begin{aligned} W_1(c_0) =& c_0 \\ W_{k+1}(c_0, c_1, \ldots, c_k) =& W_k\Big(\big[c_0^k + k\textstyle\int_0^t c_1\big]^{1/k}, c_2, \ldots, c_k\Big), \quad k \geq 1. \end{aligned}$$

We use in the proof the classical Gronwall inequality quoted above. For completeness, and because we use similar arguments without further explanation, we give a proof here.

Proof of Theorem 1.1. Fix τ in $[0,T]$ and let $w(t) := c_0(\tau) + \int_0^t c_1(s)\phi(s)\,ds$. Then w is absolutely continuous (AC), $\phi(t) \leq w(t)$ for a.e. $t \in [0,\tau]$ and

$$w'(t) = c_1(t)\phi(t) \leq c_1(t)w(t) \text{ for a.e. } t \in [0,\tau].$$

Write $C_1(t) := \int_0^t c_1(s)\,ds$. The function $w\exp(-C_1)$ is AC because C_1 is AC, exp is Lipschitz on bounded intervals, so the composition $\exp(-C_1)$ is AC, and a product of AC functions is also AC. Therefore, for a.e. $t \in [0,\tau]$,

$$[w(t)\exp(-C_1(t))]' \leq 0, \tag{2.3}$$

and hence, for all $t \in [0,\tau]$,

$$w(t)\exp(-C_1(t)) \leq w(0).$$

Since $w(0) = c_0(\tau)$ this gives $w(t) \leq c_0(\tau)\exp(C_1(t))$ for all $t \in [0,\tau]$. In particular, $w(\tau) \leq c_0(\tau)\exp(C_1(\tau))$ and, as τ is arbitrary, (1.1) holds. $\square$

Proof of Theorem 2.1. The proof is by induction. As before we replace $c_0(t)$ by $c_0(\tau)$ and reduce the proof to the case where c_0 is a constant which we suppose to be positive (otherwise add $\varepsilon > 0$). The case $k = 1$ is the classical result proved above. Suppose the inequality holds for an integer k and consider the case for $k+1$. Thus suppose that for a.e. $t \in [0, T]$,

$$\phi^{k+1}(t) \leq c_0^{k+1} + \int_0^t (k+1)c_1(s)\phi(s) + (k+1)c_2(s)\phi^2(s) + \cdots + (k+1)c_{k+1}(s)\phi^{k+1}(s)\, ds.$$

Write, for $t \in [0, T]$,

$$u^{k+1}(t) := c_0^{k+1} + \int_0^t (k+1)c_1(s)\phi(s) + (k+1)c_2(s)\phi^2(s) + \cdots + (k+1)c_{k+1}(s)\phi^{k+1}(s)\, ds.$$

Since u^{k+1} is AC and $u \geq c_0 > 0$, it follows that u is AC. Therefore u' exists a.e. and for a.e. $t \in [0, T]$,

$$\begin{aligned} u^k u' &= c_1\phi + c_2\phi^2 + \cdots + c_{k+1}\phi^{k+1} \\ &\leq c_1 u + c_2 u^2 + \cdots + c_{k+1}u^{k+1}. \end{aligned}$$

Hence

$$u^{k-1}u' \leq c_1 + c_2 u + \cdots + c_{k+1}u^k, \text{ a.e. } .$$

So we have for all $t \in [0, T]$,

$$u^k(t) - u^k(0) \leq \int_0^t kc_1 + kc_2 u + \cdots + kc_{k+1}u^k,$$

that is,

$$u^k(t) \leq \left[c_0^k + \int_0^t kc_1\right] + \int_0^t kc_2 u + \cdots + kc_{k+1}u^k.$$

This shows, by the result for the case k, that for a.e. $t \in [0, T]$,

$$u(t) \leq W_k([c_0^k + \textstyle\int_0^t kc_1]^{1/k}, c_2, \ldots, c_k) \exp\left(\textstyle\int_0^t c_{k+1}\right).$$

As u is continuous this holds for all $t \in [0, T]$ and therefore holds almost everywhere for ϕ. This completes the inductive step. $\square$

The special case $k = 2$ includes the classical case and the result quoted in Theorem 1.2 since we may take $c_2 = 0$. It includes a special case of a result of Dafermos [1] who shows that if

$$\phi^2(t) \leq M^2\phi^2(0) + \int_0^t 2Ng(s)\phi(s) + 2\gamma\phi^2(s)\, ds$$

where N, M, γ are non-negative constants and $\phi \in L_+^\infty$, $g \in L_+^1$ then

$$\phi(t) \leq \big(M\phi(0) + N\int_0^t g(s)\, ds\big) \exp(\gamma t).$$

Pachpatte [4] has an extension of this special case of Dafermos's result for continuous functions when the integral is replaced by a suitable multiple integral, which has applications to certain higher order ODE's. In Dafermos, [1], an extra term is included with γ replaced by $\gamma + \beta t$ in the first inequality. We shall use the same idea to give another result below which will extend Dafermos's result and allow some t dependence on the functions c_i. We first consider some special cases of the inequality proved in Theorem 2.1.

Example 2.2. If $u \le a_0 + \int_0^t a_1(s) u^{m/k}(s)\, ds$ where $m < k$ are positive integers, and $a_1 \in L^1_+,\ u \in L^\infty_+$, then

$$u \le \big[a_0^{1-m/k} + \int_0^t (1-m/k) a_1(s)\, ds\big]^{\frac{1}{1-m/k}}.$$

As usual we may suppose that $a_0 > 0$. Write $v = u^{1/k}$ so that the inequality reads

$$v^k \le a_0 + \int_0^t a_1(s) v^m(s)\, ds.$$

This is now a special case of Theorem 2.1 so we can write

$$v \le W_k(a_0^{1/k}, 0, \dots, a_1/k, 0, \dots, 0),$$

where a_1/k occurs in the $(m+1)$-st place. Thus,

$$\begin{aligned} v \le& W_{k-m+1}(a_0^{1/k}, a_1/k, 0, \dots, 0) \\ =& W_{k-m}\big(\big[a_0^{\frac{k-m}{k}} + \int_0^t (k-m) a_1(s)/k\, ds\big]^{\frac{1}{k-m}}, 0, \dots, 0\big). \end{aligned}$$

Since $W_r(c_0, 0, \dots, 0) = c_0$ for every r, this establishes the result.

By an approximation argument this result shows the following: if

$$u(t) \le a_0 + \int_0^t a_1(s) u^\alpha(s)\, ds, \text{ for a.e. } t \in [0, T],$$

where $0 < \alpha < 1$, then

$$u(t) \le \big[a_0^{1-\alpha} + \int_0^t (1-\alpha) a_1(s)\, ds\big]^{\frac{1}{1-\alpha}}, \text{ a.e..}$$

This last inequality is well-known and is usually shown via a comparison theorem: If h is continuous and $u(t) \le u_0 + \int_0^t h(s, u(s))\, ds$ then $u(t) \le v(t)$ where v is the solution of the differential equation $v' = h(t, v), v(0) = u_0$; ["the faster runner goes further"].

Example 2.3. Given the inequality

$$u \le a_0 + \int_0^t a_1 u^{m/k} + a_2 u^{j/k}$$

we can immediately write down an inequality for u, namely

$$u^{1/k} \le W_k(a_0^{1/k}, 0, \dots, 0, a_1/k, 0, \dots, 0, a_2/k, 0, \dots, 0),$$

where a_1/k (respectively a_2/k) occurs in the $(m+1)$ (resp. $(j+1)$) place. Hence, as above, we obtain

$$u^{1/k} \le W_{k-m-(j-m)}\Big(\big\{\big[a_0^{\frac{k-m}{k}} + \int_0^t (k-m)a_1(s)/k\,ds\big]^{\frac{k-j}{k-m}} + \tfrac{k-j}{k}\int_0^t a_2(s)\,ds\big\}, 0, \dots, 0\Big),$$

that is,

$$u \le \big\{\big[a_0^{1-\frac{m}{k}} + \int_0^t (1-m/k)a_1(s)\,ds\big]^{\frac{1-j/k}{1-m/k}} + (1-j/k)\int_0^t a_2(s)\,ds\big\}^{1/(1-j/k)}.$$

By an approximation argument this gives the following result. The inequality

$$u \le a_0 + \int_0^t a_1 u^\alpha + a_2 u^\beta,$$

where $0 < \alpha < \beta < 1$, implies

$$u \le \big\{\big[a_0^{1-\alpha} + \int_0^t (1-\alpha)a_1(s)\,ds\big]^{\frac{1-\beta}{1-\alpha}} + (1-\beta)\int_0^t a_2(s)\,ds\big\}^{1/(1-\beta)}.$$

It is not clear whether the comparison result applies to this situation, because possibly noncontinuous functions are involved. Also one encounters the problem of solving the differential equation $v' = a_1 v^\alpha + a_2 v^\beta$ where a_i need not be constants.

For example, in the simple case when we have

$$u(t) \le a_0 + \int_0^t a_1(s)u^{1/3} + a_2(s)u^{2/3}\,ds$$

we obtain the estimate $u \le w := W_3(a_0^{1/3}, a_1/3, a_2/3)^3$ which can be computed directly to be

$$w(t) = \Big(\big[a_0^{2/3} + 2A_1(t)/3\big]^{1/2} + A_2(t)/3\Big)^3,$$

where $A_i(t) = \int_0^t a_i(s)\,ds$. To apply the comparison result we have to take $a_0 > 0$ or else find a maximal (or minimal) solution because of lack of uniqueness. A small calculation shows that $w' \ge a_1 w^{1/3} + a_2 w^{2/3}$ so that $w(t) \ge v(t)$ where v is the solution of $v' = a_1 v^{1/3} + a_2 v^{2/3}$, $v(0) = a_0$. In other words the estimate we obtain is not as good as could (in theory) be obtained from the comparison result. However, even with a_1 and a_2 constant, the solution v in this case is found only in implicit form, so the estimate obtained from the comparison result is of less practical use than our explicit estimate.

Example 2.4. The inequality

$$u(t) \leq a_0 + \int_0^t a_1(s)u^{m/k}(s) + a_2(s)u(s)\,ds$$

has been studied by Perov when m/k is replaced by $\alpha, 0 \leq \alpha < 1$. [In fact he has a result also for $\alpha > 1$.] We can immediately write down an inequality for u, namely

$$u^{1/k} \leq W_k(a_0^{1/k}, 0, \ldots, 0, a_1/k, 0, \ldots, 0)\exp\bigl(\int_0^t a_2(s)/k\,ds\bigr).$$

where a_1/k is in the $(m+1)$-place. As in example 2.2 this gives

$$u^{1/k} \leq \bigl[a_0^{1-m/k} + \int_0^t (1-m/k)a_1(s)\,ds\bigr]^{\frac{1}{k-m}} \exp\bigl(\int_0^t a_2(s)/k\,ds\bigr).$$

Hence we obtain

$$u \leq \bigl[a_0^{1-m/k} + \int_0^t (1-m/k)a_1(s)\,ds\bigr]^{\frac{1}{1-m/k}} \exp\bigl(\int_0^t a_2(s)\,ds\bigr).$$

By a simple approximation argument we can then obtain: the inequality

$$u(t) \leq a_0 + \int_0^t a_1(s)u^{\alpha}(s) + a_2(s)u(s)\,ds$$

implies that

$$u \leq \bigl[a_0^{1-\alpha} + \int_0^t (1-\alpha)a_1(s)\,ds\bigr]^{\frac{1}{1-\alpha}} \exp\bigl(\int_0^t a_2(s)\,ds\bigr).$$

This is a minor change from Perov's result.

Remark 2.5. We could, of course, write down other consequences of the theorem but such inequalities await application. Another 'nonlinear' Gronwall inequality is that attributed to Bihari but apparently proved earlier by Lasalle [3].

If $u(t) \leq a + b\int_0^t k(s)g(u(s))\,ds$ where g is nondecreasing, then

$$u(t) \leq G^{-1}[G(a) + b\int_0^t k(s)\,ds]$$

where $G(s) = \int_0^s \frac{d\xi}{g(\xi)}$. This can, in theory, be applied to some of the situations we include, but the problems of calculating G and G^{-1} do not appeal in general. Also such inequalities do not fit our aim of giving explicit bounds in terms of the given data.

We now give a further extension which allows for noninteger powers.

Theorem 2.6. *Let $m \geq 1$ be an integer and $p \geq m$ be a real number. Suppose that*

$$\phi^p(t) \leq c_0^p(t) + \int_0^t pc_1(s)\phi^{p-m}(s) + pc_2(s)\phi^{p-m+1}(s) + \cdots + pc_{m+1}(s)\phi^p(s)\,ds, \quad (2.4)$$

for a.e. $t \in [0, T]$, where $c_0 \geq 0$ is non-decreasing, $\phi \in L_+^\infty$, and $c_i \in L_+^1$, $i \geq 1$. Then,

$$\phi(t) \leq W_{m+1}(c_0, c_1, c_2, \ldots, c_m)\exp\big(\int_0^t c_{m+1}\big), \ \ a.e. \ . \quad (2.5)$$

Proof. As usual we may and do suppose that c_0 is a positive constant and we omit 'almost everywhere' in the following. Write

$$u^p(t) = c_0^p + \int_0^t pc_1(s)\phi^{p-m}(s) + pc_2(s)\phi^{p-m+1}(s) + \cdots + pc_{m+1}(s)\phi^p(s)\,ds.$$

Then we obtain

$$u^{p-1}u' \leq c_1 u^{p-m} + \cdots + c_{m+1}u^p$$

so that

$$u^{m-1}u' \leq c_1 + c_2 u + \cdots + c_{m+1}u^m.$$

Hence $u^m \leq c_0^m + \int_0^t mc_1 + \int_0^t mc_2 u + \cdots + mc_{m+1}u^m$, which gives

$$u(t) \leq W_m([c_0^m + \textstyle\int_0^t mc_1]^{1/m}, c_2, \ldots, c_m)\exp\big(\textstyle\int_0^t c_{m+1}\big),$$

and yields the conclusion. □

Remark 2.7. When $p - m = j/k$ it is possible to reduce the inequality in the hypothesis of Theorem 2.6 to a special case of the one in Theorem 2.1. For example, given the inequality

$$u^{5/2}(t) \leq c_0^{5/2} + \int_0^t \tfrac{5}{2}c_1(s)u^{1/2}(s) + \tfrac{5}{2}c_2(s)u^{3/2}(s) + \tfrac{5}{2}c_3(s)u^{5/2}(s)\,ds$$

we can say at once from Theorem 2.6 that

$$u \leq W_3(c_0, c_1, c_2)\exp\big(\int_0^t c_3(s)\,ds\big).$$

However writing $v = u^{1/2}$ the given inequality may be written

$$v^5(t) \leq (c_0^{1/2})^5 + \int_0^t 5(c_1/2)v + 5(c_2/2)v^3 + 5(c_3/2)v^5.$$

Theorem 2.1 then gives the conclusion

$$v \leq W_5(c_0^{1/2}, c_1/2, 0, c_2/2, 0)\exp\big(\int_0^t c_3(s)/2\,ds\big).$$

A small calculation checks that

$$W_5(c_0^{1/2}, c_1/2, 0, c_2/2, 0) = W_3(c_0, c_1, c_2)^{1/2}$$

so the apparent two conclusions are but one. However, using Theorem 2.6 is simpler here.

3 A further extension

We now give a result of the same type where the functions c_i are allowed to depend on t. The extra idea used is that employed by Dafermos [1] in proving the following:

$$\phi^2(t) \leq M^2\phi^2(0) + \int_0^t 2Ng(s)\phi(s) + (2\gamma + 4\beta t)\phi^2(s)\,ds$$

where N, M, γ are non-negative constants and $\phi \in L^\infty_+,\ g \in L^1_+$ then

$$\phi(t) \leq \big(M\phi(0) + N\int_0^t g(s)\,ds\big)\exp(\alpha t + \beta t^2),$$

where $\alpha = \gamma + \beta/\gamma$.

We will prove the following result.

Theorem 3.1. *Let $k \geq 1$ be an integer. Suppose that for $i = 1, 2, \ldots, k$, and for $s, t \in [0, T]$, we have $0 \leq g_i(s, t) \leq h(s)$ where h is an L^1 function, and that $\partial g_i/\partial t$ exists and satisfies $0 \leq \partial g_i(s, t)/\partial t \leq A g_i(s, t)$ for some non-negative constant A. Suppose that*

$$\phi^k(t) \leq c_0^k(t) + k\int_0^t g_1(s,t)\phi(s) + \cdots + g_k(s,t)\phi^k(s)\,ds, \text{ for a.e. } t \in [0,T], \tag{3.1}$$

where $c_0 \geq 0$ is non-decreasing, $\phi \in L^\infty_+$. Then,

$$\phi(t) \leq W_k(c_0(t), g_1(t,t), \ldots, g_{k-1}(t,t))\exp\left(\int_0^t [g_k(s,s) + A/k]\,ds\right). \tag{3.2}$$

Proof. Let

$$u^k(t) = c_0^k + \int_0^t kg_1(s,t)\phi(s) + kg_2(s,t)\phi^2(s) + \cdots + kg_k(s,t)\phi^k(s)\,ds,$$

where we suppose that c_0 is a positive constant. The hypotheses made allow us to differentiate under the integral sign to obtain

$$\begin{aligned}
u^{k-1}u' &= \sum_{i=1}^k g_i(t,t)\phi^i(t) + \int_0^t \sum_{i=1}^k \frac{\partial g_i(s,t)}{\partial t}\phi^i(s)\,ds \\
&\leq \sum_{i=1}^k g_i(t,t)u^i(t) + \int_0^t \sum_{i=1}^k A g_i(s,t)\phi^i(s)\,ds \\
&\leq \sum_{i=1}^{k-1} g_i(t,t)u^i(t) + [A/k + g_k(t,t)]u^k(t).
\end{aligned}$$

This yields the inequality

$$u^k(t) \leq u^k(0) + \int_0^t \sum_{i=1}^{k-1} kg_i(s,s)u^i(s) + [A + kg_k(s,s)]u^k(s)\,ds.$$

Theorem 2.1 now applies to give the conclusion

$$\phi(t) \le u(t) \le W_k(c_0, g_1(t,t), \ldots, g_{k-1}(t,t)) \exp\Big(\int_0^t [A/k + g_k(s,s)]\, ds\Big).$$

$\square$

This includes Dafermos's result for we may take $g_1(s,t) = Ng(s)$ and $g_2(s,t) = \gamma + 2\beta t$. Then $\partial g_1/\partial t = 0$ and $\partial g_2(s,t)/\partial t = 2\beta \le A(\gamma + 2\beta t)$ for $A = 2\beta/\gamma$.

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