A SHARP ANGLE INEQUALITIES FOR PAIRS OF ELLIPTIC OPERATORS IN THE CASE OF DOMAIN WITH A CONICAL POINT.

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A sharp angle inequality is used in a proof of solvability of a general nonlinear elliptic problem with applying of topological methods. Our proof is founded at reducing to the inequality in the case of a domain with a smooth boundary [2].

Denotations.

G is a bounded domain with the conical point in the origin of coordinates. ∂G is a smooth boundary everywhere without of the origin of coordinates.
ρ is a $diam\, G$.
$G_k = \{x \in G : \quad \frac{\rho}{2^{k+1}} \le r \le \frac{\rho}{2^k}, \quad r = |x|\}$
Ω_k is a domain with the smooth boundary wich contain G_k. $\aleph_{2m}^{0,\lambda}(A, B, \Omega)$ is a class of linear elliptic with the constant A in Ω operators of order $2m$ with coefficients bounded in the norm $\|\cdot\|_{C^{0,\lambda}(\Omega)}$ by B. ϕ_k are functions of the unity partition: $\sum\limits_{k=0}^{\infty} \phi_k^2(x) = 1$ $\forall x \in G, \quad supp\, \phi_k \subset \Omega_k; \quad \phi_k \le \sqrt{N}, \quad x \in G_k$.

Denote

$$|D^\alpha(\phi_k r^\kappa)| \le const \cdot r^{\kappa - |\alpha|} \qquad (i_1)$$

$W_\kappa^l(G)$ is weighting space with norm

$$\|u\|_{W_\kappa^l(G)} = \left(\int\limits_G \sum_{|\alpha| \le k} r^{\kappa - 2(l - |\alpha|)} |D^\alpha u|^2 dx \right)^{\frac{1}{2}}$$

$\overset{\circ}{W}{}_\kappa^{l}(G)$ is a closure $C_0^\infty(G)$ in the norm $\|\cdot\|_{W_\kappa^l(G)}$.

Lemma 1. *For operators* $L \in \aleph_{2m}^{0,\lambda}(A, B, \Omega_k)$, $M_k \in \aleph_{2m}^{0,\lambda}(1, \tilde{C}_2, \Omega_k)$ *(where* $\tilde{C}_2$ *is some constant) and functions* $\phi_k \in C^\infty(\Omega_k)$, $u \in W_\kappa^{2m}(G) \cap \overset{\circ}{W}{}_\kappa^{m}(G)$ *following inequality is true*

$$Re \int\limits_{\Omega_k} r^\kappa \phi_k^2(x) Lu \cdot \overline{M_k u} dx \ge$$

$$\ge Re \int\limits_{\Omega_k} L(\phi_k u r^{\frac{\kappa}{2}}) \cdot \overline{M_k(\phi_k u r^{\frac{\kappa}{2}})} dx -$$

$$-C_1\|u\|_{W^{2m}_\kappa(\Omega_k)}\|u\|_{W^{2m-1}_{\kappa-2}(\Omega_k)} \tag{1}$$

Lemma 2. *Let $L\in\aleph^{0,\lambda}_{2m}(A,B,\Omega_k)$. One can designate the constants $\tilde C_1,\tilde C_2$ and operator $M_k(x,D)\in\aleph^{0,\lambda}_{2m}(1,\tilde C_2,\Omega_k)$ [2] such that $\forall u\in W^{2m}_2(\Omega_k)\cap \overset{\circ}{W}{}^{m}_2(\Omega)$ the relation*

$$Re\int\limits_\Omega Lu\cdot\overline{M_k u}dx\ \geq$$

$$\geq\ \tilde C_1\int\limits_{\Omega_k}\sum_{|\alpha|\leq 2m}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m}|D^\alpha u|^2dx-\tilde C_2\int\limits_{\Omega_k}\left(\frac{\rho}{2^k}\right)^{-4m}|u|^2dx \tag{2}$$

is valid.

Lemma 3. *If $u\in W^{2m}_\kappa(G)$ then $\phi_k r^{\frac{\kappa}{2}}u\in W^{2m}_2(G)$.*

Lemma 4. (Interpolating inequality.) *For a fixed k and for any function u from $W^l_\kappa(G)$ the inequality*

$$\|u\|^2_{W^j_{\kappa-2(l-j)}(\Omega_k)}\ \leq\ \varepsilon_1\|u\|^2_{W^l_\kappa(\Omega_k)}+C(\varepsilon_1)\|u\|^2_{W^0_{\kappa-2l}(\Omega_k)}, \tag{3}$$

where $j<l$, $\varepsilon_1>0$ is fulfiled.

Those lemmas enable prove a theorem.

Theorem 1. *Let $L\in\aleph^{0,\lambda}_{2m}(A,B,G)$. There exist such constants C_2, C_3, $\tilde C_2$ depended of known parameter A, B, λ, m, G and such operator $M=\sum\limits_{k=0}^{\infty}\phi^2_k(x)M_k(x,D)$, $\quad M_k\in\aleph^{0,\lambda}_{2m}(1,\tilde C_2,G)$ that*
$\forall u\in \overset{\circ}{W}{}^{m}_{\kappa-2m-2l}(G)\cap W^{2m}_\kappa(G)$ the inequality

$$Re\int\limits_G r^\kappa L(x,D)u\cdot\overline{M(x,D)u}dx\ \geq$$

$$\geq\ C_2\|u\|^2_{W^{2m}_\kappa(G)}-C_3\|u\|^2_{W^0_{\kappa-4m}(G)} \tag{4}$$

is valid.

The lemma 3 enables substitute (2) into (1)

$$Re\int\limits_\Omega\phi^2_k(x)r^\kappa Lu\cdot\overline{M_k u}dx\ \geq$$

$$\geq\ C_4\int\limits_{\Omega_k}\sum_{|\alpha|\leq 2m}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m}|D^\alpha(\phi_k u r^{\frac{\kappa}{2}}|^2dx-$$

$$-C_5\int\limits_{\Omega_k}\left(\frac{\rho}{2^k}\right)|\phi_k r^{\frac{\kappa}{2}}u|^2dx-$$

$$-C_3\|u\|_{W^{2m-1}_{\kappa-2}(\Omega_k)}\|u\|_{W^{2m}_{\kappa}(\Omega_k}. \tag{5}$$

Consider first addend of the right member of (5).

$$\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m}|D^{\alpha}(\phi_k r^{\frac{\kappa}{2}})|^2dx \;=$$

$$=\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m}|\phi_k r^{\frac{\kappa}{2}}D^{\alpha}u+$$

$$+\sum_{\beta<\alpha;\,\beta+\gamma=\alpha}c_{\beta\gamma}D^{\beta}u\,D^{\gamma}(\phi_k r^{\frac{\kappa}{2}})|^2dx\;\ge$$

$$\ge\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m}|\phi_k r^{\frac{\kappa}{2}}D^{\alpha}u|^2-$$

$$-4\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m}|\phi_k r^{\frac{\kappa}{2}}D^{\alpha}u|*$$

$$*|\sum_{\beta<\alpha;\,\beta+\gamma=\alpha}c_{\beta\gamma}D^{\beta}u\,D^{\gamma}(\phi_k r^{\frac{\kappa}{2}})|dx\;\ge$$

$$\ge\;N\cdot\int\limits_{G_k}\sum_{|\alpha|\le 2m}r^{\kappa-2(2m-|\alpha|)}|D^{\alpha}u|^2dx-$$

$$-\varepsilon_2\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m}r^{\kappa-2(2m-|\alpha|)}|D^{\alpha}u|^2dx-$$

$$-C(\varepsilon_2)\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m}r^{2|\alpha|-4m}\sum_{\beta<\alpha;\,\beta+\gamma=\alpha}r^{\kappa-2\gamma}|D^{\beta}u|^2dx\;\ge$$

$$\ge\;N\cdot\|u\|^2_{W^{2m}_{\kappa}(G_k)}-$$

$$-\varepsilon_2\cdot\|u\|^2_{W^{2m}_{\kappa}(\Omega_k)}-$$

$$-C_6\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m}r^{2|\alpha|-4m}\sum_{\beta<\alpha;\,\beta+\gamma=\alpha}r^{\kappa-2\alpha+2\beta+2\alpha-4m}|D^{\beta}u|^2dx\;\ge$$

$$\ge\;N\cdot\|u\|^2_{W^{2m}_{\kappa}(G_k)}\;-\;\varepsilon_2\cdot\|u\|^2_{W^{2m}_{\kappa}(\Omega_k)}\;-\;C_7\|u|^2_{W^{2m-1}_{\kappa-2}(\Omega_k)}\;\ge$$

$$\ge\;N\cdot\|u\|^2_{W^{2m}_{\kappa}(G_k)}\;-\;\varepsilon_3\cdot\|u\|^2_{W^{2m}_{\kappa}(\Omega_k)}\;-\;C_8\|u|^2_{W^{0}_{\kappa-4m}(\Omega_k)}\,.$$

Here Caushy inequality and the interpolating inequality were used. Due to this inequalities we can estimate third addend of the right member of (5) by means of

$$-\varepsilon_4\|u\|^2_{W^{2m}_{\kappa}(\Omega_k)}\;-\;C_9\|u\|^2_{W^{0}_{\kappa-4m}(\Omega_k)}.$$

Now we can write down (5) as

$$Re \int\limits_{\Omega_k} \phi_k^2(x) r^\kappa Lu \cdot \overline{M_k u} dx \ \geq$$

$$\geq \ C_{10} \|u\|^2_{W^{2m}_\kappa (G_k)} \ - \ \varepsilon_5 \|u\|^2_{W^{2m}_\kappa (\Omega_k)} \ - \ C_{11} \|u\|^2_{W^0_{\kappa-2m}(\Omega_k)}$$

To the left it is possible the substitution Ω_k at G. Sum with respect to k and substitute Ω_k at $G_{k-1} \cap G_k \cap G_{k+1}$ to the right. Thereby we are getting (4).

We introduce $\overline{\aleph}^{l,\lambda}_{2m}(A, B, G)$ as a class of a linear elliptic operators with the constant of ellipticity A and with coefficients having the derivatives of the order $d\,(d \leq l)$ bounded by $B \cdot r^{-d}$. Thereto the derivatives of the order l are Holder continuous with the constant λ and bounded by $B \cdot r^{-l-\lambda}$. Let

$$[u, v]_{W^l_\kappa(G)} \ = \int\limits_G \sum_{|\alpha|,|\beta| \leq l} c_{\alpha,\beta}(x) r^{\kappa - 2l + |\alpha| + |\beta|} D^\alpha u \overline{D^\beta v} dx, \qquad (6)$$

where $c_{\alpha\beta}(x)$ are real infinity differentiable functions of following construction

$$c_{\alpha\beta}(x) \ = \sum_{k=0}^{\infty} \tilde{c}_{\alpha\beta}\left(x \cdot \frac{2^k}{\rho}\right) \cdot r_k^{-(\kappa - 2l + |\alpha| + |\beta|)} \ = \sum_{k=0}^{\infty} c^k_{\alpha\beta}(x),$$

where $r_k \ = \ \frac{r}{\rho/2^k}$. Here $\tilde{c}_{\alpha\beta}$ are real infinity differentiable functions satisfying to the conditions

$$C_{12} \|\tilde{u}\|^2_{W^l_2(\Omega')} \ \leq \int\limits_{\Omega'} \sum_{|\alpha|,|\beta| \leq l} \tilde{c}_{\alpha\beta}(x') D^\alpha \tilde{u} \overline{D^\beta \tilde{u}} dx' \ \leq$$

$$\leq \ C_{13} \|\tilde{u}\|^2_{W^l_2(\Omega')},$$

where $\Omega' \ = \ \frac{2^k}{\rho}\Omega_k$; $\tilde{c}_{\alpha\beta}$ have finite support and $\forall \Omega" \subset R^n \setminus \Omega'$

$$\int\limits_{\Omega"} \sum_{|\alpha|,|\beta| \leq l} \tilde{c}_{\alpha\beta}(x') D^\alpha \tilde{u} \overline{D^\beta \tilde{u}} dx' \ \geq \ 0.$$

It can shou that $c_{\alpha\beta}$ satisfy to the conditions

$$C_{14} \|u\|^2_{W^l_\kappa(G)} \ \leq \ [u, u]_{W^l_\kappa(G)} \ \leq \ C_{15} \|u\|_{W^l_\kappa(G)}.$$

Lemma 5.

For operators $L \in \overline{\aleph}^{l,\lambda}_{2m}(A, B, \Omega_k)$, $M_k \in \overline{\aleph}^{l,\lambda}_{2m}(1, \tilde{C}_2, \Omega_k)$ *and functions* $\phi_k \in C^\infty(\Omega_k)$, $u \in W^{2m+l}_\kappa(\Omega_k) \cap \overset{\circ}{W}{}^m_{\kappa - 2m - 2l}(\Omega_k)$ *the relation*

$$Re[Lu, \phi_k^2 M_k u]_{W^l_\kappa(\Omega_k)} \ \geq\geq \ Re \int\limits_{\Omega_k} \sum_{|\alpha|,|\beta| \leq l} c_{\alpha\beta}(x) r^{|\alpha| + |\beta| - 2l} D^\alpha L(x, D)(\phi_k r^{\frac{\kappa}{2}} u) *$$

$$*\overline{D^\beta M_k(x,D)(\phi_k r^{\frac{\kappa}{2}} u)}dx \; - \; C_{16}\|u\|_{W_\kappa^{l+2m}(\Omega_k)}\|u\|_{W_{\kappa-2}^{l+2m-1}(\Omega_k)} \qquad (7)$$

is valid.

Lemma 6. *Let $L \in \overline{\aleph}_{2m}^{l,\lambda}(A,B,\Omega_k)$. There exist such constants $C_1', C_2' > 0$ and operator $M_k(x,D) \in \overline{\aleph}_{2m}^{l,\lambda}(1, C_2', \Omega_k)$ that*
$\forall u \in W_2^{2m+l}(\Omega_k) \cap \overset{\circ}{W}{}_2^m(\Omega_k)$ following inequality is fulfiled.

$$Re\int\limits_{\Omega_k} \sum_{|\alpha|,|\beta|\le l} c_{\alpha\beta}^k(x)\left(\frac{\rho}{2^k}\right)^{|\alpha|+|\beta|-2l} D^\alpha L u \overline{D^\beta M_k u} dx \ge$$

$$\ge C_1'\int\limits_{\Omega_k} \sum_{|\alpha|\le 2m+l}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m-2l} |D^\alpha u|^2 dx -$$

$$-C_2'\int\limits_{\Omega_k}\left(\frac{\rho}{2^k}\right)^{-4m-2l}|u|^2 dx \qquad (8)$$

Theorem 2. *Let $L \in \overline{\aleph}_{2m}^{l,\lambda}(A,B,G)$. There exist such real infinity differentiable functions $c_{\alpha\beta}$ $(|\alpha|, |\beta| \le l)$ satisfying to the condition*

$$C_{14}\|u\|^2_{W_\kappa^l(G)} \le [u,u]_{W_\kappa^l(G)} \le C_{15}\|u\|^2_{W_\kappa^l(G)},$$

the positive constants K_1, K_2, C_2' depending of known parameter only and such operator $M = \sum_{k=0}^\infty \phi_k^2(x) M_k(x,D)$, $M_k(x,D) \in \overline{\aleph}_{2m}^{l,\lambda}(1, C_2', G)$ that for any function $u \in W_\kappa^{2m+l}(G) \cap \overset{\circ}{W}{}^m_{\kappa-2m-2l}(G)$ the inequality

$$Re[Lu, Mu]_{W_\kappa^l(G)} \ge K_1\|u\|^2_{W_\kappa^{2m+l}(G)} - K_2\|u\|^2_{W^0_{\kappa-4m-2l}(G)} \qquad (9)$$

is valid.

On the lemma 3 $\phi_k r^{\frac{\kappa}{2}} u \in W_\kappa^{2m+l}(G)$. Therefore

$$Re[Lu, \phi_k^2 M_k u]_{W_\kappa^l(G)} \ge$$

$$\ge C_{17}\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m+l}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m-2l}|D^\alpha(\phi_k r^{\frac{\kappa}{2}} u)|^2 dx -$$

$$-C_{18}\int\limits_{\Omega_k}\left(\frac{\rho}{2^k}\right)^{-4m-2l}|\phi_k r^{\frac{\kappa}{2}} u|^2 dx -$$

$$-C_{19}\|u\|_{W_\kappa^{2m+l}(\Omega_k)}\|u\|_{W_{\kappa-2}^{2m+l-1}(\Omega_k)} \qquad (10)$$

Consider the first addend of the right member of (10). We shall be use interpolating inequality and inequality $|a+b| \ge |a|^2 - 4|a|\cdot|b|$.

$$\int\limits_{\Omega_k}\sum_{|\alpha|\le 2m+l}\left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m-2l}|D^\alpha(\phi_k r^{\frac{\kappa}{2}} u)|^2 dx =$$

$$= \int\limits_{\Omega_k} \sum_{|\alpha|\le 2m+l} \left(\frac{\rho}{2^k}\right)^{2|\alpha|-4m-2l} |\phi_k r^{\frac{\kappa}{2}} D^\alpha u -$$

$$- \sum_{\gamma<\alpha;\,\gamma+\delta=\alpha} c_{\gamma\delta} D^\delta(\phi_k r^{\frac{\kappa}{2}}) D^\gamma u|^2 dx \ \ge$$

$$\ge\ N \cdot C_{20} \int\limits_{G_k} \sum_{|\alpha|\le 2m+l} r^{\kappa-2(2m+l-|\alpha|)} |D^\alpha u|^2 dx -$$

$$- C_{21} \int\limits_{\Omega_k} \sum_{|\alpha|\le 2m+l} r^{2|\alpha|-4m-2l} \sum_{\gamma<\alpha;\,\gamma+\delta=\alpha} \left\{ r^{\frac{\kappa}{2}-|\alpha|+|\gamma|} |D^\gamma u| \right\} *$$

$$* \left\{ r^{\frac{\kappa}{2}} |D^\alpha u| \right\} dx \ \ge$$

$$\ge\ N \cdot C_{20} \|u\|^2_{W^{2m+l}_\kappa(G_k)} -$$

$$- \varepsilon_6 \|u\|^2_{W^{2m+l}_\kappa(\Omega_k)} - C_{22} \|u\|^2_{W^{2m+l-1}_{\kappa-2}(\Omega_k)} \ \ge\ N \cdot C_{20} \|u\|^2_{W^{2m+l}_\kappa(G_k)} -$$

$$- \varepsilon_7 \|u\|^2_{W^{2m+l}_\kappa(\Omega_k)} -$$

$$- C_{23} \|u\|^2_{W^0_{\kappa-4m-2l}(\Omega_k)} \tag{11}$$

We apply Caushy inequality and inequality (3) to third addend of the right member of (10). Thereby we get

$$Re[Lu, \phi_k^2 M_k u]_{W^l_\kappa(\Omega_k)} \ \ge$$

$$\ge\ N \cdot C_{20} \|u\|^2_{W^{2m+l}_\kappa(G_k)} - \varepsilon_8 \|u\|^2_{W^{2m+l}_\kappa(\Omega_k)} - C_{24} \|u\|^2_{W^0_{\kappa-4m-2l}(\Omega_k)} \tag{12}$$

from (10). In the left member we substitute Ω_k at G and in the right member substitute Ω_k at $G_{k-1} \cup G_k \cup G_{k+1}$. Summing with respect to k gives (9).

References

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