ON THE BEHAVIOR OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS IN THE NEIGHBORHOOD OF CONICAL BOUNDARY POINTS

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1. Introduction: Preliminaries.

Let $G \subset \mathbb{R}^n$ be a bounded domain with bounary ∂G, which is smooth for $x \in \partial G \backslash \{0\}$. For $x \in \mathbb{R}^n$, we denote the spherical coordinates by $(r, \omega) = (r, \omega_1, \dots, \omega_{n-1})$ with $r = |x|$, $\omega \in S^{n-1}$. It is assumed that G coincides in the neighborhood of 0 with a cone with opening $\Omega \subset S^{n-1}$. More precisely, using the notation

$$G_a^b = G \cap \{(r, \omega) : a < r < b,\ \omega \in \Omega\} \subset \mathbb{R}^n \quad \text{for } a \geq 0,\ b > 0,$$

we assume that G_0^d is identical to the corresponding cone for a $d > 0$. We denote also $\Gamma_a^b = \partial G \cap \{(r, \omega) : a < r < b,\ \omega \in \partial\Omega\} \subset \mathbb{R}^n \quad \text{for } a \geq 0,\ b > 0,$ — lateral surface of G_a^b; $\Omega_\rho = G \cap \{|x| = \rho\}$; $G_\varepsilon = G \setminus G_0^\varepsilon$, $\Gamma_\varepsilon = \partial G \setminus \Gamma_0^\varepsilon$, $\forall \varepsilon > 0$.

We consider the Dirichlet problem (DP):

$$\begin{cases} \mathfrak{L}_m u := -\text{div}\ (|\nabla u|^{m-2} \nabla u) = -a_0(x) u |u|^{q-1} + f(x) & \text{in } G, \\ u(x) = 0 & \text{on } \partial G \setminus \{0\}, \end{cases}$$

where $1 < m < \infty$, $q > 0$ and $a_0(x) \geq 0$, $f(x)$ are measurable functions in G.

Let $L^p(G)$ and $W^{k,p}(G)$, $p \geq 1$, be the usual Lebesgue and Sobolev spaces. $W_0^{1,p}(G)$ denotes the space of functions in $W^{1,p}(G)$ that vanish on ∂G in the sense of traces. We define the wight space $V_{p,\alpha}^k(G)$ as the space of functions with finite norm

$$\|u\|_{V_{p,\alpha}^k(G)} = \left(\int\limits_G \sum_{|\beta| \leq k} r^{p(|\beta| - k + \frac{\alpha}{2})} |D^\beta u|^p dx \right)^{\frac{1}{p}}, \ p \geq 1,$$

where $k \geq 0$ is integer.

Definition. A function $u(x)$ is called a **generalized** solution of (DP), if $u \in W^{1,m}(G_\varepsilon) \cap L^{q+1}(G_\varepsilon)$ $\forall \varepsilon > 0$ satisfies

$$\int\limits_G \{|\nabla u|^{m-2} u_{x_i} \eta_{x_i} + a_0(x) u |u|^{q-1} \eta - f(x) \eta\}\, dx = 0 \tag{II}$$

for any $\eta \in W^{1,m}(G) \cap L^{q+1}(G)$ having a compact in G support, and $u(x) = 0 \quad x \in \Gamma_\varepsilon \quad \forall \varepsilon > 0$ in the sense of traces.

*The paper was written while the first author was visiting the University of Würzburg. Financial support of the DAAD is gratefully acknowledged.

A function $u(x)$ is called a **weak** solution of (DP), if $u \in W_0^{1,m}(G) \cap L^{q+1}(G)$ and satisfies (II) for all $\eta \in W_0^{1,m}(G) \cap L^{q+1}(G)$.

The properties of generalized solutions of (DP) in the neighborhood of isolated singularities are studied by many authors (see e.g. [8] and the literature cited there). The behavior of solutions near conical boundary points are treated only in special cases: in [7], [4] for $a_0(x) \equiv 0$, in [1] for bounded solutions, in [5], [2] for $m = 2$. The purpose of the present article is to extend the results of [5], [2] to the more general quasilinear case $m \neq 2$.

The first eigenvalue problem which characterizes the singular behavior of solutions of (DP) can be derived by inserting in $\mathfrak{L}_m v = 0$ functions of the form $v = r^\lambda \phi(\omega)$ which leads to

$$\mathfrak{D}(\lambda, \phi) = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega, \tag{EVP1}$$

where $\mathfrak{D}(\lambda, \phi) = -\mathrm{div}_\omega\{(\lambda^2\phi^2 + |\nabla_\omega \phi|^2)^{\frac{m-2}{2}} \nabla_\omega \phi\} -$

$$-\lambda\{\lambda(m-1) + n - m\}(\lambda^2\phi^2 + |\nabla_\omega \phi|^2)^{\frac{m-2}{2}} \phi.$$

For this eigenvalue problem, there exists a solution (λ_0, ϕ) such that $\lambda_0 > \max\left\{0, \frac{m-n}{m-1}\right\}$, $\phi > 0$ in Ω, $\phi^2 + |\nabla_\omega \phi|^2 > 0$ in $\overline{\Omega}$, (see [7], [4]); $\lambda_0 = \lambda_0(\Omega)$ is the smallest positive eigenvalue of (EVP1).

For technical reasons, we introduce a further eigenvalue problem,

$$-\mathrm{div}_\omega(|\nabla_\omega \psi|^{m-2} \nabla_\omega \psi) = \mu |\psi|^{m-2} \psi \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega. \tag{EVP2}$$

From the direct method of the calculus of variations we know that there exists a solution (μ, ψ) of (EVP2) with $\mu > 0$ and $\psi > 0$ in Ω. From the corresponding variational principle we derive the so–called *Wirtinger's inequality*

$$\int_\Omega |\psi|^m \, d\omega \le \frac{1}{\mu(m)} \int_\Omega |\nabla_\omega \psi|^m \, d\omega \quad \forall \psi \in W_0^{1,m}(\Omega)$$

with a sharp constant $\frac{1}{\mu(m)}$. We set $\mu_0 = \mu(n)$; $\mu_0 = \mu_0(\Omega)$ is the smallest positive eigenvalue of (EVP2) under $m = n$.

2. Integral estimates of solutions.

The aim of this section is to present integral estimates for the solutions of (DP). Moreover, the weak comparison principle is not used in the proof, so that it may also apply to the case of elliptic systems.

THEOREM 2.1. *Let $a_0(x) \in L_{\frac{m}{m-1-q}}(G)$, if $0 < q < m-1$ and $0 < a_0 \le a_0(x) \le a_1$ $(a_0, a_1 - \text{const.})$, if $q \ge m-1$. Let $f(x) \in V^0_{\frac{m}{m-1},2}(G)$. Then the weak solution of the problem (DP) $u(x) \in V^1_{m,0}(G)$ and it holds the inequality*

$$\int_G \left(|\nabla u|^m + r^{-m}|u|^m + a_0(x)|u|^{1+q}\right) dx \le c(n, G) \int_G |rf|^{\frac{m}{m-1}} dx.$$

Proof. The assertion arises from (II) with $\eta(x) = u(x)\Theta\left(\frac{|x|}{\varepsilon}\right)$ $\forall \varepsilon > 0$, where $\Theta(t)$ be nonnegative infinitely differentiable function and such that $\Theta(t) = 0$ for $t < 1$, $\Theta(t) = 1$ for $t > 2$.

COROLLARY 2.2. *Let $m > n$. Under suppositions of Theorem 2.1 a weak solution $u(x)$ of (DP) is the Hölder-continuous in $\overline{G}$.*

THEOREM 2.3. *Let $m = n$ and let the following condition be satisfied:* $\int\limits_{G_0^\rho} |rf|^{n/(n-1)}\,dx \le c\rho^{\varkappa}$. *Let* $\chi_0 = \frac{2\sqrt{\mu_0}}{(1+\mu_0)^{(n-2)/n}}$. *Then for any weak solution of (DP) is satisfied the bound* $\int\limits_{G_0^\rho} |\nabla u|^n\,dx \le$

$$\le c(n,\mu_0,\Omega)\begin{cases} (\rho/d)^{\chi_0}, & \text{if } \chi_0 < \varkappa,\\ (\rho/d)^{\chi_0}\ln^{\frac{n}{n-1}}(d/\rho), & \text{if } \chi_0 = \varkappa, \quad \rho\in(0,d),\\ (\rho/d)^{\varkappa}, & \text{if } \chi_0 > \varkappa. \end{cases}$$

Remark. It is well known that if $m = n = 2$, then $\mu_0 = \lambda_0^2 = \frac{\pi^2}{\omega_0^2}$, where ω_0 is the quantity of the angle with the vertex 0. In this case the assertion of Theorem was proved in [2] (see Theorem 2.2 in it).

The proof of the theorem will be carried out due to following lemma.

LEMMA 2.4. *Let $1 < m \le n$. For any function $u \in W_0^{1,m}(G)$ we have*

$$\int\limits_\Omega \left\{\rho u u_r + \tfrac{n-m}{2}u^2\right\}|\nabla u|^{m-2}\,d\omega \le \frac{\rho^2}{\chi}\int\limits_\Omega |\nabla u|^m\,d\omega,$$

where $\chi = \frac{m-n+\sqrt{4\mu_0+(n-m)^2}}{(1+\mu_0)^{(m-2)/m}}$.

Proof. Lemma is proved by anology with inequality (42) [3] and with the help of the Young inequality.

Remark. For $m = n = 2$ the constant χ is sharp.

Proof of the theorem 2.3. Let $V(\rho) = \int\limits_{G_0^\rho} |\nabla u|^n dx$. ¿From (DP) and Lemma 2.4 it follows that $V(\rho)$ satisfies the differential inequality $V(\rho) \le \frac{\rho}{\chi_0}V'(\rho) + c\rho^{\varkappa\frac{n-1}{n}}V^{\frac{1}{n}}(\rho)$. In view of Theorem 2.1 as an initial condition for the differential inequality we can use $V_0 = V(d) \le \int\limits_G |\nabla u|^n\,dx \le c\int\limits_G |rf|^{n/(n-1)}\,dx$. On putting $W(\rho) = V^{\frac{n-1}{n}}(\rho)$, we obtain differential inequality for $W(\rho)$:

$$\begin{cases} W(\rho) \le \frac{n}{n-1}\frac{\rho}{\chi_0}W'(\rho) + c\rho^{\varkappa\frac{n-1}{n}}, & 0<\rho<d\\ W(d) = V_0^{\frac{n-1}{n}} \end{cases}$$

Solwing the Cauchy problem for the corresponding equation, we get

$$W^*(\rho) = \left(\frac{\rho}{d}\right)^{\chi_0\frac{n-1}{n}}\Bigg(V_0^{\frac{n}{n-1}} + $$
$$+\,\varkappa\chi_0\begin{cases} \frac{n-1}{n}\ln\frac{d}{\rho}, & \text{if } \chi_0 = \varkappa,\\ \frac{d^{\frac{n-1}{n}(\varkappa-\chi_0)} - \rho^{\frac{n-1}{n}(\varkappa-\chi_0)}}{\varkappa-\chi_0}, & \text{if } \chi_0 \ne \varkappa \end{cases}\Bigg).$$

It is well known that the solution of differential inequality can be estimated by the solution $W^*(\rho)$ of the corresponding equation: $W(\rho) \le W^*(\rho)$ and hence we obtain finally the required estimate. The theorem 2.3 is proved.

LEMMA 2.5. *Let* $q > m - 1$, $0 < a_0 \le a_0(x) \le a_1$ (a_0, a_1 - const). *Let* $|f(x)| \le k_1|x|^{\beta}$, $x \in G_0^d$, *where* $\beta > -1$ *if* $m > n$, $\beta > -m$ *if* $m \le n$. *Then for any generalized solution* $u(x)$ *of (DP) are hold the inequalities*

$$||u||_{p;G^{\rho}_{\rho/2}} \le c(a_0, m, n, p, q, k_1)\rho^{\frac{n}{p} - \frac{m}{q-m+1}} \quad \forall p > m,$$

$$\int\limits_{G^{\rho}_{\rho/2}} \left(|\nabla u|^m + |u|^{1+q}\right) dx \le c(a_0, m, n, q, k_1)\rho^{n - \frac{(1+q)m}{1+q-m}}.$$

Proof. Desired inequalities are obtain from (II) with $\eta(x) =$
$= |u|^t \mathrm{sgn} u \zeta^s(|x|)$ under suitably chosen numbers $t \ge 1$, $s > 0$ and the cut-off function $\zeta(r)$.

COROLLARY 2.6. *Let* $q > \frac{mn}{n-m} - 1$, $1 < m < n$ *and the hypothesis of the Lemma 2.5 about the functions* $a_0(x)$, $f(x)$ *are hold. Then for any generalized solution* $u(x)$ *of (DP) the inequality*

$$\int\limits_{G_0^{\rho}} \left(|\nabla u|^m + r^{-m}|u|^m + |u|^{1+q}\right) dx \le c(a_0, n, m, q, k_1, d),$$

$\forall \rho \in (0, d)$ *is valid. Moreover, if* $\beta > -\frac{n}{s}$ *with some* $s > \frac{n}{m}$ *and* $\lambda_0 < \frac{\beta+m}{m-1}$, *then* $u(x)$ *is the Hölder-continuous in* $\overline{G}$.

Proof. The properties of $u(x)$ are proved in Theorem 7.1 of chapt. IV and Theorem 2.2 of chapt. IX [6] in virtue Lemma 2.5.

3. A solvability property of the operator $\mathfrak{D}$ from (EVP1).

In order to construct a barrier function which can be used in the weak comparison principle, a solvability property of the operator $\mathfrak{D}$ associated to the eigenvalue problem (EVP1) is proved.

THEOREM 3.1. *For* $0 \le \lambda < \lambda_0$ *there exists a solution* ϕ *of the problem*

$$\mathfrak{D}(\lambda, \phi) = 1 \quad in\ \Omega, \qquad \phi = 0 \quad on\ \partial\Omega, \tag{3.1}$$

with $\phi > 0$ *in* Ω.

This theorem will be proved in a sequence of lemmas. In the proofs of these lemmas we frequently use the fact that every solution (λ, ϕ) of (3.1) corresponds to a solution of $\mathfrak{L}_m(r^{\lambda}\phi) = r^{(\lambda-1)(m-1)-1}$ in G_0^d, which, by local regularity of the Pseudo–Laplace equation, implies that $\phi \in C^{\beta}(\overline{\Omega}) \cap W_0^{1+\varepsilon,m}(\Omega)$ for $\beta, \varepsilon > 0$.

LEMMA 3.2. *The problem (3.1) is solvable for all* $0 \le \lambda < \lambda_0$.

Proof. We prove that Fredholm's alternative holds for (3.1) in the sense that, if (3.1) is not solvable, then λ is an eigenvalue of D. For this purpose, we choose a sufficiently large $\alpha \in \mathbb{R}$ such that the problem $D(\lambda, \phi) + \alpha|\phi|^{m-2}\phi = g$ in Ω, $\phi = 0$ on $\partial\Omega$ is uniquely solvable for all $g \in H^{-1,m'}(\Omega)$, $\frac{1}{m} + \frac{1}{m'} = 1$ and denote the solution operator by $\phi = \Phi g$. By the regularity of D, $\Phi\colon C^{\beta}(\overline{\Omega}) \to C^{\beta}(\overline{\Omega})$ is a compact operator for a $\beta > 0$. Moreover, Φ is homogeneous of degree $\frac{1}{m-1}$. The problem $D(\lambda, \phi) = f$ in Ω, $\phi = 0$ on $\partial\Omega$ is then equivalent to

$$\phi - \alpha F\phi = \Phi f, \tag{3.2}$$

where $F\phi = \Phi(|\phi|^{m-2}\phi)$ is compact and homogeneous of degree 1. The operator $Id - \alpha F$ is studied on the unit ball $B_1 = \{\phi \in C^{\beta}(\overline{\Omega}) : ||\phi||_{C^{\beta}} \le 1\}$. If $0 \notin (Id - \alpha F)(\partial B_1)$ then

K. Borsuk's theorem states that (3.2) is solvable for sufficiently small f. Since (3.2) is equivalent to $D(\lambda, \phi) = f$ and $D(\lambda, \cdot)$ is homogeneous of degree $m-1$ we can solve $D(\lambda, \phi) = f$ for all f.

LEMMA 3.3. *Let (λ, ϕ) be a solution of (3.1). Then $\phi(\omega) \neq 0$ for all $\omega \in \Omega$.*

Proof. Let $K = \{(r, \omega) : \ 1 < r < 2, \omega \in \Omega\}$. If (λ, ϕ) is a solution of (3.1) then $v = r^{\lambda}\phi(\omega)$ solves $L_m v = r^{(\lambda-1)(m-1)-1}$ in K, $v = 0$ on $(1, 2) \times \partial\Omega$, $v = c_r\phi$ for $r = 1, 2$. Assume that $\phi(\omega_0) = 0$ for $\omega_0 \in \Omega$. We apply the weak comparison principle on the domain K using the function v. It follows that every solution of $L_m u = f$ in K, $u = v$ on ∂K with $f \in C_0^\infty(K)$ satisfies $u(r, \omega_0) \leq 0$ which is a contradiction.

LEMMA 3.4. *For sufficiently small $\lambda \geq 0$ the solution of (3.1) is unique and satisfies $\phi > 0$ in Ω.*

Proof. The operator $\mathfrak{D}(0, \cdot)$ is strictly monotone on $W_0^{1,m}(\Omega)$. Hence, problem (3.1) is uniquely solvable and the comparison principle implies $\phi > 0$ in Ω. Since $\mathfrak{D}(\lambda, \cdot)$ is continuous in λ the conclusion also holds for sufficiently small $\lambda \geq 0$.

LEMMA 3.5. *There exists a constant $c = c(\lambda_1)$ such that $||\phi||_{1,m} \leq c$ for all solutions (λ, ϕ) of (3.1) satisfying $0 \leq \lambda \leq \lambda_1 < \lambda_0$.*

Proof. Assuming the converse we obtain a sequence (λ_i, ϕ_i) solving (3.1) with $\lambda_i \to \lambda$, $||\phi_i||_{1,m} \to \infty$. For the normalized functions $\tilde{\phi}_i = \frac{\phi_i}{||\phi_i||_{1,m}}$ we obtain that $D(\lambda_i, \tilde{\phi}_i) \to 0$ in $W^{-1,m'}(\Omega)$ and, by regularity, $||\tilde{\phi}_i||_{1+\epsilon,m} \leq c$. Hence, we can extract a subsequence $\{\tilde{\phi}_{i_k}\}$ such that $\tilde{\phi}_{i_k} \to \phi$ in $W_0^{1,m}(\Omega)$ and $D(\lambda, \phi) = 0$ with $||\phi||_{1,m} = 1$. This contradicts the fact that there is no eigenvalue of D in the interval $[0, \lambda_1]$.

Proof of Theorem 3.1. Lemma 3.5 implies a kind of continuity of solutions (λ, ϕ) in the following sense. If $\lambda_i \to \lambda$ with $0 \leq \lambda_i, \lambda < \lambda_0$, then there exists a subsequence $\{\phi_{i_k}\}$ such that $\phi_{i_k} \to \phi$ in $C^\alpha(\overline{\Omega})$, where (λ, ϕ) is a solution of (3.1). Hence, by Lemmas 3.3 and 3.4 there exists a solution (λ, ϕ) with $\phi > 0$ in Ω for all $0 \leq \lambda < \lambda_0$.

4. Estimates of solutions for singular f.

In [7], [4] it is proved that the weak solution $u \in W_0^{1,m}(G)$ of (DP) can be bounded by $|u(x)| \leq cr^{\lambda_0}$, if $a_0(x) \equiv 0$ and the condition $|f(x)| \leq cr^\beta$, $\beta > (\lambda_0 - 1)(m-1) - 1$ is satisfied. The proof of this is based on the weak comparison principle for (DP). Here we shall obtain the estimates of modulus of the solutions of (DP). Let $d > 0$ be a small fixed number. We also suppose that $|f(x)| \leq k_1|x|^\beta$, $\beta > -\frac{n}{s}$ with some $s > \frac{n}{m}$.

Observe that a function $v = r^\alpha\phi(\omega)$ is a weak solution $v \in W_0^{1,m}$, if $\phi(\omega)$ is sufficiently smooth and $\alpha > \frac{m-n}{m}$. Since $\mathfrak{L}_m v \sim r^{\alpha(m-1)-m}$ and the right-hand side of (DP) $-a_0(x)v|v|^{q-1} + f(x) \sim r^{\alpha q} + r^\beta$, hence we obtain that $r^{\alpha(m-1)-m} \sim r^{\alpha q} + r^\beta$. This arguments suggest the following theorems to us.

THEOREM 4.1. *Let $1 < m \leq n$, $q > 0$ be given. Let $0 \leq a_0(x) \leq a_1 =$ const. Let $u(x)$ be a weak solution of (DP). Then the following assertions are hold:*
1) if $\lambda_0 < \frac{\beta+m}{m-1}$, then $|u(x)| \leq c_0|x|^{\lambda_0}$, $x \in G_0^d$;
2) if $\lambda_0 > \frac{\beta+m}{m-1}$, then $|u(x)| \leq c_0|x|^{\frac{\beta+m}{m-1}}$, $x \in G_0^d$;

Proof. The operator $\mathfrak{L}_{m,q}u := \mathfrak{L}_m u + a_0(x)u|u|^{q-1}$ satisfies the weak comparison principle which states that $\mathfrak{L}_{m,q}u \leq \mathfrak{L}_{m,q}v$ in G_0^d, $u \leq v$ on $\partial G_0^d \Rightarrow u \leq v$ in $\overline{G_0^d}$. Then the first assertion follows from the Theorem 2 [4]. To proof the second assertion we choose the barrier function $v(x) = A|x|^{\lambda}\phi(\omega)$ where $A > 0$, $\lambda = \frac{\beta+m}{m-1} < \lambda_0$ and (λ, ϕ) is the solution of (3.1).

THEOREM 4.2. *Let $1 < m < n$, $q > m-1$ be given. Let $0 < a_0 \le a_0(x) \le a_1$, ($a_0$, a_1* — const). *Let $u(x)$ be any generalized solution of (DP). If $\lambda_0 < \frac{\beta+m}{m-1}$, $q > \frac{mn}{n-m} - 1$, then $|u(x)| \le c_0|x|^{\lambda_0}$, $x \in G_0^d$.*

Proof. Firstly, from the Lemma 2.5 with $p \to \infty$ we get the bound $|u(x)| \le c|x|^{\frac{m}{m-1-q}}$. Then we construct the barrier function and use the method by contradiction (just similarly as in [5]) in view of the Corollary 2.6.

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