AN ELLIPTIC PROBLEM WITH A LAYER

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1. Introduction.

We are interested in the following problem: let Ω be an open set in R^N we denote by Σ the set $\Omega \cap \{x_N = 0\}$ and we assume $\Sigma \neq \emptyset$. Moreover we denote $x = (x', x_N) = (x_1, x_2, ..., x_{N-1}, x_N)$. We consider the problem

$$\int DuDv\ m(dx) + \int D'uD'v\ \sigma(dx) = 0 \tag{1.1}$$

$$u \in H^1_{loc}(\Omega)\ \ \textit{with trace in}\ H^1_{loc}(\Sigma)$$

$$\forall v \in H^1_{loc}(\Omega)\ \ \textit{with trace in}\ H^1_{loc}(\Sigma),\ \ \textit{with supp}(v) \subseteq \Omega$$

where m denotes the Lebesgue measure on R^N and σ denotes the Lebesgue measure on R^{N-1}; moreover we denote by D the gradient in R^N and by D' the tangential gradient on $R^{N-1}_{x'}$. If u verify (1.1) we say that u is a *solution* of (1.1). If we replace in (1.1) the equality by the inequality $\leq$ ($\geq$) and we consider only positive test functions v we say that u is a *subsolution* (*supersolution*) of (1.1) in Ω.

The aim of this paper is to study the local regularity for a solution of (1.1).

If we consider a ball that does not intersect Σ the problem of the regularity of u reduce to the problem of the regularity of an harmonic function; then in particular Harnack inequality for nonegative u and Hölder continuity for u hold. Problems arise in the case of sets having a non empty intersection with Σ (due to the different rescaling by the usual dilation of the two terms in (1.1)).

We also observe that the bilinear form in (1.1) defines a strongly local regular Dirichlet form on $L^2(\Omega, m + \delta_\Sigma)$,[3], but the measure $m + \delta_\Sigma$ does not verify a doubling property then the regularity theory in [1][2] does not apply.

To study the local regularity of u in $B(x_0, r)$ with, $x_0 \in R^{N-1}_{x'}$ we modify the definition of a ball defining $B(x_0, r) = \{x : |x' - (x_0)'|^4 + |x_N - (x_0)_N|^2 < r^4\}$ and we write $S(x_0, r) = B(x_0, r) \cap \{x_N = 0\}$. We define a cut-off function between $B(x_0, tr)$ and $B(x_0, sr)$, $s, t \in [\frac{1}{2}, 1)$ $s < t$, as $\eta(x) = \phi(d(x - x_0))$ where $d(x) = (|x'|^4 + x_N^2)^{\frac{1}{4}}$, $\phi(\rho) = 1$ for $\rho \leq sr$, $\phi(\rho) = 0$ for $\rho \geq tr$, $0 \leq \phi \leq 1$ and $|\phi'| \leq \frac{C}{(t-s)r}$. Then

$$|D'\eta| \leq \frac{C}{(t-s)r}\ \ \textit{on}\ S(x_0, r),\ \ |D\eta| \leq \frac{C}{(t-s)r^2}\ \ \textit{on}\ B(x_0, r).$$

With such a modification we obtain:

Theorem 1.1. *Let u be a nonegative solution of (1.1) in $B(x_0, 4r)$, $x_0 \in R_{x'}^{N-1}$; then*

$$sup_{B(x_0,r)} u \le C \ inf_{B(x_0,r)} u$$

where C is a constant depending only on N.

Theorem 1.1. *Let u be a solution of (1.1) in Ω; then u is locally Hölder continuous in Ω.*

In Section 2 we prove suitable Poincaré and Sobolev type inequalities, that play a fundamental role in the proof, by a Moser type iteration method, of local L^∞ estimates for solutions (or subsolutions) of (1.1), given in section 3. In section 4 we prove the results in Theorems 1.1 and 1.2. The result in Theorem 1.2 is an easy consequence of the one in Theorem 1.1; the proof of the result in Theorem 1.1 uses an iteration method introduced by Moser,[6], that allow use to consider estimates only on concentric balls $B(x_0, r)$; this last opportunity is usefull due to the different forms of the balls in the case $x_0 \in \Sigma$ or $x_0 \notin \Sigma$.

2. Poincaré and Sobolev type inequalities.

It is well known that a fundamental role in the local regularity theory of harmonic functions relative to an uniformly elliptic operator is played by the usual Poncaré and Sobolev inequalities.

The goal of this section is to prove suitably adapted Poincaré and Sobolev inequalities relative to the problem in consideration.

Proposition 2.1. *Let u be a function in $H^1(B(x_0, r))$, $x_0 \in R_{x'}^{N-1}$, with a trace in $H^1(S(x_0, r))$ and u_r be the average of u on $S(x_0, r)$ relative to the measure σ; then*

$$\fint_{B(x_0,r)} |u - u_r|^2 \ m(dx) + \fint_{S(x_0,r)} |u - u_r|^2 \ \sigma(dx) \le$$

$$\le C[r^4 \fint_{B(x_0,r)} |D_N u|^2 \ m(dx) + r^2 \fint_{S(x_0,r)} |D' u|^2 \ \sigma(dx)]$$

where $\fint_{B(x_0,r)} m(dx)$ ($\fint_{S(x_0,r)} \sigma(dx)$) denotes the average on the set $B(x_0, r)$ ($S(x_0, r)$) relative to the measure m (σ).

The result in Proposition 2.1 isa consequence of the following Sobolev type inequality:

Proposition 2.2. *Let u be a function in $H^1(B(x_0, r))$, $x_0 \in R_{x'}^{N-1}$, with a trace in $H^1(S(x_0, r))$.*
(a) Let $N > 3$; there exists $q > 2$ such that

$$[\fint_{B(x_0,r)} |u - u_r|^q \ m(dx) + \fint_{S(x_0,r)} |u - u_r|^q \ \sigma(dx)]^{\frac{2}{q}} \le$$

$$\le C[r^4 \fint_{B(x_0,r)} |D_N u|^2 \ m(dx) + r^2 \fint_{S(x_0,r)} |D' u|^2 \ \sigma(dx)]$$

(b) Let $N = 3$; for every $q > 2$ we have

$$[\fint_{B(x_0,r)} |u - u_r|^q \ m(dx) + \fint_{S(x_0,r)} |u - u_r|^q \ \sigma(dx)]^{\frac{2}{q}} \leq$$

$$\leq C[r^4 \fint_{B(x_0,r)} |D_N u|^2 \ m(dx) + r^2 \fint_{S(x_0,r)} |D'u|^2 \ \sigma(dx)]$$

(c) Let $N = 2$; then

$$osc_{B(x_0,r)} u \leq$$

$$\leq C[r^4 \fint_{B(x_0,r)} |D_N u|^2 \ m(dx) + r^2 \fint_{S(x_0,r)} |D'u|^2 \ \sigma(dx)]^{\frac{1}{2}}$$

Proof. We prove the result for the case (a); the proof in the cases (b) and (c) is analogous.

It is enough to prove the result in the case $r = 1$ and we write $B(x_0, r) = B$, $S(x_0, r) = S$.

Let $s = \frac{2N-2}{N-3}$, we have

$$\int_S |u - u_1|^s \ \sigma(dx)]^{\frac{2}{s}} \leq$$

$$\leq C \int_S |D'u|^2 \ \sigma(dx),$$

where we denote by C possibly different constants depending only on N.

Let $q = \frac{2+s}{2} = \frac{2N-4}{N-3}$, by easy computations we obtain

$$sup_{((x_0)_N - 1, (x_0)_N + 1)} \int_{B \cap \{x_N = t\}} |u - u_1|^q \ \sigma(dx) \leq$$

$$\leq C[(\int_B |D_N u|^2 \ m(dx))^{\frac{q}{2}} + (\int_S |u - u_1|^s \ \sigma(dx))^{\frac{q}{s}}] \leq$$

$$\leq C[(\int_B |D_N u|^2 \ m(dx))^{\frac{q}{2}} + (\int_S |D'u|^2 \ \sigma(dx))^{\frac{q}{2}}]$$

and the result follows.

3. The local L^∞ estimate for subsolutions.

We prove at first an L^∞ estimate for nonegative subsolution of (1.1) and finally we prove the general L^∞ estimate for nonegative solutions of (1.1)

Proposition 3.1. *Let u be a function in $H^1(B(x_0.r))$, $x_0 \in R^{N-1}_{x'}$, with a trace in $H^1(S(x_0,r))$. Assume that u is a nonnegative subsolution in a neighbourhood of $B(x_0,r)$; then there exists constants d and C such that for $\alpha \in [\frac{1}{2},1)$ and $p \geq 2$ we have*

$$(sup_{B(x_0,\alpha r)}u)^p \leq \frac{C}{(1-\alpha)^d}[\fint_{B(x_0,r)} u^p m(dx) + \fint_{S(x_0,r)} u^p \sigma(dx)]^{\frac{1}{p}}$$

Proof. Let $\beta \geq 1$ and $0 < M < +\infty$; we define

$$H_M(t) = t^\beta \quad for \;\; t \in [0,M]$$

$$H_M(t) = M^\beta + \beta M^{\beta-1}(t-M) \quad for \;\; t > M$$

The function $H_M(t)$ is Lipschitz-continuous for every fixed M.

We assume that u is Lipschitz continuous (if it is not the case we use an approximation of u in $H^1(B(x_0,r))$ and in $H^1(S(x_0,r))$ by a sequence $\{u_k\}$ of nonegative Lipschitz-continuous functions).

For a fixed M we define

$$\phi(x) = \eta(x)^2 \int_0^{u(x)} H_M'(t)^2 dt$$

where η is a Lipschitz continuous function with support in $B(x_0,r)$ to be choosen. We observe that ϕ are nonegative Lipschitz continuous functions defined in $B(x_0,r)$ and

$$D_i\phi = \eta^2 H_M'(u)^2 D_i u + 2\eta D_i\eta \int_0^{u(x)} H_M'(t)^2 dt \tag{3.1}$$

for $i = 1,2,...,N$. Since u is a subsolution we have

$$\int Du\ \eta^2 H_M'(u)^2 Du\ m(dx) + \tag{3.2}$$

$$+\int Du\ 2\eta D\eta (\int_0^{u_k} H_M'(t)^2 dt)\ m(dx) + \int Du\ \eta^2 H_M'(u)^2 Du\ \sigma(dx) +$$

$$+\int Du\ 2\eta D\eta (\int_0^{u} H_M'(t)^2 dt)\ \sigma(dx) \leq 0.$$

We observe that

$$|D_i u\ 2\eta D_i\eta (\int_0^u H_M'(t)^2 dt)| \leq$$

$$\leq \frac{1}{2} D_i u\ \eta^2 H_M'(u)^2 D_i u +$$

$$+2|D_i\eta|^2 (\frac{1}{H_M'(u)} \int_0^u H_M'(t)^2 dt)^2 \leq$$

$$\leq \frac{1}{2} D_i u\ \eta^2 H'_M(u)^2 D_i u + +2|D_i \eta|^2 (u H'_M(u))^2$$

From (3.2) it follows

$$\frac{1}{2}\int |D(H_M(u))|^2 \eta^2\ m(dx) + \tag{3.3}$$

$$+\frac{1}{2}\int |D(H_M(u))|^2 \eta^2\ \sigma(dx) \leq$$

$$\leq 2\int |D\eta|^2 (u H'_M(u))^2\ m(dx) + 2\int |D\eta|^2 (u H'_M(u))^2\ \sigma(dx).$$

We choose now η as the cut-off function between the $B(x_0, sr)$ and $B(x_0, tr)$. From (3.3) we obtain

$$\int_{B(x_0,sr)} |D(H_M(u))|^2\ m(dx) + \tag{3.4}$$

$$+\int_{S(x_0,sr)} |D(H_M(u))|^2\ \sigma(dx) \leq$$

$$\leq \frac{c_2}{(t-s)^2 r^4} \int_{B(x_0,tr)} (u H'_M(u))^2\ m(dx) +$$

$$+\frac{c_2}{(t-s)^2 r^2} \int_{S(x_0,tr)} (u H'_M(u))^2\ \sigma(dx).$$

Using the Sobolev inequality we obtain

$$[\ \fint_{B(x_0,sr)} |H_M(u) - (H_M(u))_{sr}|^q\ m(dx) +$$

$$+\ \fint_{S(x_0,sr)} |H_M(u) - (H_M(u))_{sr}|^q\ \sigma(dx)]^{\frac{1}{q}} \leq$$

$$c_3 \frac{s}{t-s} [\ \fint_{B(x_0,tr)} (u H'_M(u))^2\ m(dx) +\ \fint_{S(x_0,tr)} (u H'_M(u))^2\ \sigma(dx)]^{\frac{1}{2}}.$$

We use the inequality $H(t) \leq t H'(t)$ and we obtain

$$[\ \fint_{B(x_0,sr)} |H_M(u)|^q\ m(dx) +\ \fint_{S(x_0,sr)} |H_M(u)|^q\ \sigma(dx)]^{\frac{1}{q}} \leq$$

$$c_5 (\frac{s}{t-s} + 1) [\ \fint_{B(x_0,tr)} (u H'_M(u))^2\ m(dx) +$$

$$+\ \fint_{S(x_0,tr)} (u H'_M(u))^2\ \sigma(dx)]^{\frac{1}{2}}.$$

We observe that $(\frac{s}{t-s}+1) \leq 2\frac{s}{t-s}$. We take into account the definition of H_M and we let $M \to +\infty$; then

$$[\fint_{B(x_0,sr)} u^{\beta q}\ m(dx) + \fint_{S(x_0,sr)} u^{\beta q}\ \sigma(dx)]^{\frac{1}{q}} \leq$$

$$c_6\beta\frac{s}{t-s}[\fint_{B(x_0,tr)} u^{2\beta}\ m(dx) + \fint_{S(x_0,tr)} u^{2\beta}\ \sigma(dx)]^{\frac{1}{2}}.$$

We write $2\beta = \nu$, $q = 2\tau$ $(\tau > 1)$ and we obtain

$$[\fint_{B(x_0,sr)} u^{\tau\nu}\ m(dx) + \fint_{S(x_0,sr)} u^{\tau\nu}\ \sigma(dx)]^{\frac{1}{\tau\nu}} \leq \tag{3.5}$$

$$(c_6\nu\frac{s}{t-s})^{\frac{2}{\nu}}[\fint_{B(x_0,tr)} u^{\nu}\ m(dx) + \fint_{S(x_0,tr)} u^{\nu}\ \sigma(dx)]^{\frac{1}{\nu}}.$$

From (3.5) an iteration method of Moser's type (see for example [3]) give the result.

Proposition 3.2. *Let u be a local nonegative solution of our problem in $B(x_0, 2r)$, $x_0 \in R^{N-1}_{x'}$. Then there exists constants d, $\tau > 1$ and C such that for $\alpha \in [\frac{1}{2}, 1)$ and every real p we have*

$$(sup_{B(x_0,\alpha r)}u)^p \leq$$

$$\leq \frac{C}{(1-\alpha)^d}(1+|p|)^{\frac{2\tau}{\tau-1}}[\fint_{B(x_0,r)} u^p m(dx) + \fint_{S(x_0,r)} u^p \sigma(dx)]^{\frac{1}{p}}$$

Proof. It is enough to prove the result in the case $-\infty < p < 2$ and $u \geq \epsilon > 0$.

By Proposition 3.1 u is bounded in $B(x_0, r)$; we define $\phi = \eta^2 u^\beta$,with $\beta \leq 1$ and we can prove that ϕ is in $H^1(B(x_0, r))$ and its trace is in $H^1(S(x_0, r))$.

We recall that

$$D_i\phi = \eta^2\beta u^{\beta-1}D_i u + 2\eta D_i\eta\ u^\beta$$

$$D_i(u^{\frac{\beta+1}{2}}) = \frac{\beta+1}{2}u^{\frac{\beta-1}{2}}D_i u.$$

Then for $\beta \neq -1$ we obtain

$$|\frac{\beta}{\beta+1}|^2\int_{B(x_0,r)} |D(u^{\frac{\beta+1}{2}})|^2\eta^2\ m(dx)+ \tag{3.6}$$

$$+|\frac{\beta}{\beta+1}|\int_{S(x_0,r)} |D'(u^{\frac{\beta+1}{2}})|^2\eta^2\ \sigma(dx)) \leq$$

$$\leq \int_{B(x_0,r)} |D(u^{\frac{\beta+1}{2}})D\eta|u^{\frac{\beta+1}{2}}\eta\ m(dx)+$$

$$+\int_{S(x_0,r)} |D'(u^{\frac{\beta+1}{2}})D\eta|u^{\frac{\beta+1}{2}}\eta\ \sigma(dx).$$

From (3.6) we easily obtain for $\beta \neq 0, -1$

$$\int_{B(x_0,r)} |D(u^{\frac{\beta+1}{2}})|^2 \eta^2 \ m(dx) + \int_{S(x_0,r)} |D'(u^{\frac{\beta+1}{2}})|^2 \eta^2 \ \sigma(dx) \leq$$

$$\leq (\frac{\beta+1}{\beta})^2 | \int_{B(x_0,r)} |D\eta|^2 u^{\beta+1} \ m(dx) +$$

$$+ (\frac{\beta+1}{\beta})^2 \int_{S(x_0,r)} |D'\eta|^2 u^{\beta+1} \ \sigma(dx).$$

Then, taking again η as the cut-off function between $B(x_0, sr)$ and $B(x_0, tr)$, $\frac{1}{2} \leq s < t < 1$, we have

$$\int_{B(x_0,sr)} |D(u^{\frac{\beta+1}{2}})|^2 \ m(dx) + \int_{S(x_0,sr)} |D'(u^{\frac{\beta+1}{2}})|^2 \ \sigma(dx) \leq$$

$$\leq (\frac{\beta+1}{\beta})^2 \frac{1}{(t-s)^2 r^4} \int_{B(x_0,tr)} u^{\beta+1} \ m(dx) +$$

$$+ (\frac{\beta+1}{\beta})^2 \frac{1}{(t-s)^2 r^2} \int_{S(x_0,tr)} u^{\beta+1} \ \sigma(dx).$$

We use now the Sobolev inequality in Proposition 2.2; then by the same methods as in Proposition 3.1 we have

$$[⨍_{B(x_0,sr)} u^{\frac{\beta+1}{2}q} \ m(dx) + ⨍_{S(x_0,sr)} u^{\frac{\beta+1}{2}q} \ \sigma(dx)]^{\frac{1}{q}} \leq \tag{3.7}$$

$$\leq c | \frac{\beta+1}{\beta} | (\frac{s}{t-s} + 1) [(⨍_{B(x_0,tr)} u^{\beta+1} \ m(dx) +$$

$$+ ⨍_{S(x_0,tr)} u^{\beta+1} \ \sigma(dx)]^{\frac{1}{2}}.$$

Setting $\beta + 1 = \nu$ and $q = 2\tau$ we have for any $-\infty < \nu \leq 2$, $\nu \neq 0, -1$

$$[⨍_{B(x_0,sr)} u^{\tau\nu} \ m(dx) + ⨍_{S(x_0,sr)} u^{\tau\nu} \ \sigma(dx)]^{\frac{1}{\tau|\nu|}} \leq \tag{3.8}$$

$$\leq c^{\frac{2}{|\nu|}} (| \frac{\nu}{\nu-1} | \frac{s}{t-s} + 1)^{\frac{2}{|\nu|}} [(⨍_{B(x_0,tr)} u^{\nu} \ m(dx) +$$

$$+ ⨍_{S(x_0,tr)} u^{\nu} \ \sigma(dx)]^{\frac{1}{|\nu|}}.$$

From (3.8) the results follows by a Moser's type iteration argument (see for example [3]).

Proposition 3.3. *Let the assumptions of Proposition 3.2 hold and assume that* $u \geq \epsilon > 0$. *For* $\alpha \in [\frac{1}{2}, 1)$ *define* k *by*

$$logk = \fint_{S(x_0,\alpha r)} logu\ \sigma(dx),$$

$x_0 \in \Sigma$; *then for* $\lambda > 0$ *we have*

$$m\{x \in B(x_0, \alpha r); \quad |log(\frac{u(x)}{k})| > \lambda\} \geq \frac{C}{1-\alpha} m(B(x_0, \alpha r))$$

$$\sigma\{x \in S(x_0, \alpha r); \quad |log(\frac{u(x)}{k})| > \lambda\} \geq \frac{C}{1-\alpha} \sigma(S(x_0, \alpha r))$$

where C *is a constant that does not gepend on* ϵ.

Proof. By easy computations we obtain

$$\int |D(logu)|^2 \eta^2\ m(dx) + \int |D'(logu)|^2 \eta^2\ \sigma(dx) \leq$$

$$\leq 4 \int |D\eta|^2\ m(dx) + \int |D'\eta|^2\ \sigma(dx) \leq$$

$$\leq \frac{c}{(1-\alpha)^2} (\frac{m(B(x_0, r))}{r^4} + \frac{\sigma(S(x_0, r))}{r^2}).$$

where η is the cut-off function between $B(x_0, \alpha r)$ and $B(x_0, r)$. By Proposition 2.1 we obtain

$$\fint_{B(x_0,\alpha r)} |logu - logk|^2\ m(dx) \leq \frac{c}{(1-\alpha)^2} \frac{m(B(x_0, r))}{r^4} \tag{3.9}$$

$$\fint_{S(x_0,\alpha r)} |logu - logk|^2\ \sigma(dx) \leq \frac{c}{(1-\alpha)^2} \frac{\sigma(S(x_0, r))}{r^2}). \tag{3.10}$$

From (3.10) and (3.11) the result easily follows.

4. Proof of Theorems 1.1 and 1.2.

We are now in position to prove Theorem 1.1.

Lemma 4.1. *Let* m, μ, C, $\theta \in [\frac{1}{2}, 1)$ *be positive constants and let* $w > 0$ *be a function in* $H^1(B(x_0, r))$, $x_0 \in R^{N-1}_{x'}$, *such that*

$$sup_{B(x_0,sr)} w^p \leq \tag{4.1}$$

$$\frac{C}{(t-s)^d} \frac{1}{m(B(x_0, r))} \int_{B(x_0,tr)} w^p\ m(dx) +$$

$$+\frac{C}{(t-s)^d}\frac{1}{\sigma(S(x_0,r))}\int_{S(x_0,tr)} w^p\ \sigma(dx)$$

for all $\frac{1}{2}\leq\theta\leq s<t\leq 1,\quad 0<p<\mu^{-1}$. *Moreover, let*

$$m(x\in B(x_0,r);\ logw\geq\lambda)\leq\frac{C\mu}{\lambda}m(B(x_0,r)) \tag{4.2}$$

$$\sigma(x\in S(x_0,r);\ logw\geq\lambda)\leq\frac{C\mu}{\lambda}\sigma(B(x_0,r)) \tag{4.3}$$

for all $\lambda>0$. *Then there exists a constant* $\gamma=\gamma(\theta,d,C)$ *such that*

$$sup_{B(x_0,\theta r)}u\leq\gamma^{\mu}$$

Proof. We assume, without loss of generality $r=1$. Replacing w by w^{μ} and λ by $\lambda\mu$ we reduce us to the case $\mu=1$.

Define

$$\phi(s)=sup_{B(x_0,s)}logw,\quad\theta\leq s<1;$$

we observe that $\phi(s)$ is a nondecreasing function.

We now prove that the following inequality holds:

$$\phi(s)\leq\frac{3}{4}\phi(t)+\frac{\gamma_1}{(t-s)^{2d}} \tag{4.4}$$

where $\theta\leq s<t\leq 1$ and γ_1 is a constant depending on θ, d, C.

We decompose $B(x_0,t)$ and $S(x_0,t)$ into the sets where $logw>\frac{1}{2}\phi(t)$ and where $logw\leq\frac{1}{2}\phi(t)$; then taking into account (4.2) and (4.3) we obtain

$$\int_{B(x_0,t)} w^p\ m(dx)\leq(e^{p\phi(t)}\frac{2C}{\phi(t)}+e^{p\frac{\phi(t)}{2}})m(B(x_0,1))$$

$$\int_{S(x_0,t)} w^p\ \sigma(dx)\leq(e^{p\phi(t)}\frac{2C}{\phi(t)}+e^{p\frac{\phi(t)}{2}})\sigma(S(x_0,1)).$$

Summing up the two inequalities we have

$$\frac{1}{m(B(x_0,1))}\int_{B(x_0,t)} u^p\ m(dx)+\frac{1}{\sigma(S(x_0,1))}\int_{S(x_0,t)} u^p\ \sigma(dx)\leq$$

$$\leq 2(e^{p\phi(t)}\frac{2C}{\phi(t)}+e^{p\frac{\phi(t)}{2}}).$$

We choose now p such that the two terms in the right-hand side are equal:

$$p=\frac{2}{\phi(t)}log(\frac{\phi(t)}{2C})$$

provided the term in the right-hand side is less than $\mu^{-1} = 1$; this last inequality requires

$$\phi(t) > c_1 \tag{4.5}$$

where c_1 depends only on C.

In that case we have

$$\frac{1}{m(B(x_0,1))}\int_{B(x_0,t)} u^p\ m(dx) + \frac{1}{\sigma(S(x_0,1))}\int_{S(x_0,t)} u^p\ \sigma(dx) \leq$$
$$\leq 4e^{p\frac{\phi(t)}{2}},$$

hence by (4.1) we obtain

$$\phi(s) \leq \frac{1}{p} log(\frac{4C}{(t-s)^d} e^{p\frac{\phi(t)}{2}}) =$$
$$= \frac{1}{p} log(\frac{4C}{(t-s)^d}) + \frac{1}{2}\phi(t)$$

Then, taking into account the fixed value of p, the above inequality becomes

$$\phi(s) \leq \frac{1}{2}\phi(t)\{\frac{log(\frac{4C}{(t-s)^d})}{log(\frac{\phi(t)}{2C})}) + 1\}.$$

If

$$\phi(t) \geq \frac{32C^3}{(t-s)^{2d}} \tag{4.6}$$

we obtain

$$\phi(s) \leq \frac{3}{4}\phi(t)$$

then (4.4) holds.

If (4.5) or (4.6) does not hold; then

$$\phi(s) \leq \phi(t) \leq \frac{\gamma_1}{(t-s)^{2d}},$$

$\theta \leq s < t \leq 1$, where γ_1 is a constant depending on $C,\ c_1,\ d,\ \theta$; so (4.4) holds again.

We have so proved the inequality (4.4); the result now follows by iteration as in Lemma 3 in [6].

We are now in position to prove the result of Theorem 1.1. We assume, without loss of generality, $u \geq \epsilon > 0$. We use the result in Proposition 4.1 for $\frac{u}{k}$ and $\frac{k}{u}$; Where $logk = \fint_{S(x_0,r)} logu\ \sigma(dx)$. The assumptions of proposition 4.1 hold in $B(x_0, 2r)$ by Proposition 3.2 and 3.3; then

$$sup_{B(x_0,r)}\frac{u}{k} \leq C, \quad sup_{B(x_0,r)}\frac{k}{u} \leq C$$

Then the result follows.

The result in Theorem 1.2 can be proved from Theorem 1.1 by standard methods (see [5]).

References

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