POSITIVE SOLUTIONS OF A THREE POINT BOUNDARY VALUE PROBLEM

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ABSTRACT. We establish existence of positive solutions of some boundary value problems for a second order semilinear ordinary differential equation $u'' + g(t)f(u) = 0$ on $[0,1]$. The boundary conditions involve three points, $0 < \eta < 1$. The conditions on f strictly include the sub- and super-linear cases.

1. Introduction

We shall establish existence of positive solutions of a second order differential equation of the form

$$u'' + g(t)f(u) = 0 \quad (0 < t < 1) \tag{1.1}$$

with one of the following boundary conditions

$$u(0) = 0, \ \ \alpha u(\eta) = u(1), \ 0 < \eta < 1 \text{ and } \alpha\eta < 1; \qquad (BC)_1$$

$$u'(0) = 0, \ \ \alpha u(\eta) = u(1), \ 0 < \eta < 1 \text{ and } \alpha < 1; \qquad (BC)_2$$

under conditions on f which strictly include the sub- and super-linear cases.

The study of this type of boundary condition was initiated by V. Il'in and E. Moiseev [7]. Existence of solutions of more general differential equations subject to these boundary conditions has been extensively studied by Gupta and co-authors assuming sublinear growth conditions at ∞ in a number of papers, for example [4]. Existence of solutions has been discussed by Feng and Webb in the resonance cases ($\alpha\eta = 1$ for BC_1, $\alpha = 1$ for BC_2) in [2], and terms with nonlinear growth have been allowed in [3]. We refer to the cited papers for further references to the literature.

Eq. (1.1) arises from the study of positive radial solutions on an annulus of a nonlinear elliptic equation of the form

$$\triangle u + h(|x|)f(u) = 0. \tag{1.3}$$

Eq. (1.1) contains many important equations which arise from other fields, for example, the generalized Emden-Fowler equation, where $f = u^p$, $p > 0$ appears in the fields of gas dynamics, nuclear physics and chemically reacting systems, and the Thomas-Fermi equation, where $f = u^{3/2}$ and $g = t^{-1/2}$, appears in studies of atomic structures.

When g is continuous, the existence of positive solutions of Eq. (1.1) with suitable boundary conditions has been studied by Wang [10] by using theories of fixed point index, in particular norm-type cone expansion and compression theorems. The key conditions on f are either that f is superlinear, that is $\lim_{x\to 0} f(x)/x = 0$ and $\lim_{x\to\infty} f(x)/x = \infty$ or that f is sublinear, that is $\lim_{x\to 0} f(x)/x = \infty$ and $\lim_{x\to\infty} f(x)/x = 0$. However, it is known that Eq.(1.1) with $g \equiv 1$ has positive solutions for some functions which may not be superlinear. D. Guo proved such a result [5] [or [6], Example 2.3.1, p. 96)] again using norm-type cone expansion and compression theorems, when f satisfies $0 \leq$

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$\limsup_{x\to 0} f(x)/x < 8$ and $24\sqrt{3} < \limsup_{x\to 0} f(x)/x \le \infty$. By using a different nonzero fixed point theorem, these estimates were improved by Lan and Webb in [9] who obtained more general results for Eq (1.1) with one of the boundary conditions

$$u(0) = u(1) = 0, \tag{$(BC)_3$}$$

$$u(0) = u'(1) = 0, \tag{$(BC)_4$}$$

$$u'(0) = u(1) = 0. \tag{$(BC)_5$}$$

As in [9] we consider Eq. (1.1) when $g \in L^1(0,1)$ (in particular, g is allowed to have singularities), g is positive on a set of positive measure, and f satisfies either

$$0 \le \limsup_{x\to 0} f(x)/x < A \text{ and } B < \liminf_{x\to\infty} f(x)/x \le \infty$$

$$\text{or } 0 \le \limsup_{x\to\infty} f(x)/x < A \text{ and } B < \liminf_{x\to 0} f(x)/x \le \infty$$

for suitable A and B that will be explicitly calculated.

We shall prove that, under these conditions, positive solutions exist for Eq. (1.1) with $(BC)_1$ when $\alpha > 0$ and $\alpha\eta < 1$ and for Eq. (1.1) with $(BC)_2$ when $0 < \alpha < 1$. These are non-resonance cases and simple examples show that these restrictions on α are necessary.

The method is to write Eq.(1.1) + B.C. as a Hammerstein integral equation

$$u(t) = \int_0^1 k(t,s)g(s)f(u(s))\,ds \equiv Tu(t). \tag{1.5}$$

The abstract result of [9], which uses the fixed point index for compact maps and a well-known nonzero fixed point theorem, shows that T has a positive fixed point under certain assumptions on k. We verify that each of our boundary value problems give rise to a kernel (Green's function) k which satisfy the assumptions of [9].

2. Existence of positive solutions of Hammerstein integral equations

We quote some results concerning the integral equation

$$u(t) = \int_0^1 k(t,s)g(s)f(u(s))\,ds \equiv Tu(t). \tag{2.1}$$

We assume the following conditions.

(1) $k : [0,1]\times[0,1] \to [0,\infty)$ is continuous.

(2) $f : [0,\infty) \to [0,\infty)$ is continuous.

(3) $g \in L^1(0,1)$ and $g(s) \ge 0$ a.e..

Let $P = \{u \in C[0,1] : u(t) \ge 0 \text{ for } t \in [0,1]\}$. Then P is a cone in $C[0,1]$. It is well known that if g is defined on $[0,1]$ and is continuous in $[0,1]$, the map $T : P \to P$ is compact [for example, the book by M.A.Krasnosel'skii, [8]]. Lan and Webb showed this holds also in the case when g satisfies the condition (3).

Lemma 2.1. *Under the hypotheses* (1)-(3), *the map* T *defined in* (2.1) *maps* P *into* P *and is compact.*

The following well-known result (see, for example, Theorem 12.3 in Amann [1]) is also used.

Let K be a cone in a Banach space X and for $0 < \rho < r < \infty$ let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$ and $\overline{K}_{\rho,r} = \{x \in K : \rho \le \|x\| \le r\}$.

PROPOSITION 2.2. Let $T : \overline{K}_r \to K$ be a compact map. Assume that the following conditions hold.
(i) $\|Tx\| \le \|x\|$ for $x \in \partial K_r$.
(ii) There exists $e \in K$, $e \neq 0$ such that
$x \neq Tx + \lambda e$ for $x \in \partial K_\rho$ and $\lambda > 0$.
Then T has a fixed point in $\overline{K}_{\rho,r}$. [Hence not zero.]

Idea of the proof:
(i) implies index on K_r is 1,
(ii) implies index on K_ρ is 0.
The additivity property of the index then gives the index on $K_{\rho,r}$ is 1 (nonzero!) so there exists a fixed point of T in $K_{\rho,r}$.

Notation:
Let $f^\alpha = \limsup_{x\to\alpha} f(x)/x$ and $f_\alpha = \liminf_{x\to\alpha} f(x)/x$, where α denotes either 0 or ∞.

Lan-Webb make assumptions on g and on the kernel k, namely:
(G) There exist $a, b \in [0,1]$ with $a < b$ such that $\int_a^b g(s)\,ds > 0$.
(K) There exist a continuous function $\Phi : [0,1] \to \mathbb{R}^+$ and a number $\gamma \in (0,1]$ such that

$$k(t,s) \le \Phi(s) \text{ for } t,s \in [0,1] \qquad \text{and}$$

$$\gamma\Phi(s) \le k(t,s) \text{ for } t \in [a,b] \text{ and } s \in [0,1].$$

This means being able to find upper and lower bounds for $k(t,s)$ with s fixed, *of the same type*. In general we have some freedom in choosing the numbers a, b. See [9] for some optimal choices of a, b for the boundary conditions $(BC)_3$ and $(BC)_4$.

THEOREM 2.3 (Lan-Webb). *Assume that* (G), (K) *hold and define numbers* M_1, m_1 *by*

$$M_1 = \Big(\max_{0\le t\le 1} \int_0^1 k(t,s)g(s)ds\Big)^{-1}$$

$$m_1 = \Big(\min_{a\le t\le b} \int_a^b k(t,s)g(s)ds\Big)^{-1}.$$

Then Eq. (2.1) *has a solution* $u \in P$ *with* $u(t) \not\equiv 0$ *if either*

$$(h_1) \quad 0 \le f^0 < M_1 \text{ and } m_1 < f_\infty \le \infty.$$

or

$$(h_2) \quad 0 \le f^\infty < M_1 \text{ and } m_1 < f_0 \le \infty.$$

Remark 2.4. The idea of the proof is to use the cone

$$K = \{u \in P : \min\{u(t) : a \le t \le b\} \ge \gamma\|u\|\}.$$

[I believe the idea of using this type of cone is due to D.Guo.]
To apply Proposition 2.2, the function $e \equiv 1$ does.

Hypotheses (h_1) and (h_2) include and are more general than the well-known sublinear, and superlinear cases. [Linear is not allowed.] The Lan-Webb estimates for the intervals containing f^0, f_∞ etc., improved earlier ones. Norm-type compression and expansion theorem does not seem to give such a good result.

3. Positive solutions of $u'' + g(t)f(u) = 0$

We consider the boundary value problem

$$u'' + g(t)f(u) = 0, \quad \text{a.e on } [0,1], \tag{3.1}$$

with boundary conditions

$$u'(0) = 0,\ \alpha u(\eta) = u(1),\ 0 < \eta < 1,\ 0 < \alpha < 1. \tag{$(BC)_2$}$$

THEOREM 3.1. *The boundary value problem* (3.1), $(BC)_2$ *has a positive solution if* $\int_0^\eta g(s)\,ds > 0$ *and either*
(h_1) $0 \le f^0 < M_1$ *and* $m_1 < f_\infty \le \infty$, *or* (h_2) $0 \le f^\infty < M_1$ *and* $m_1 < f_0 \le \infty$, *where* $M_1 = \left(\max_{0\le t\le 1} \int_0^1 k(t,s)g(s)ds\right)^{-1}$ *and* $m_1 = \left(\min_{a\le t\le b} \int_a^b k(t,s)g(s)ds\right)^{-1}$.

To prove this we have to determine the kernel k and obtain appropriate upper and lower bounds. The solution of $u'' + y = 0$ with these BC's is (by routine integration)

$$u(t) = \frac{1}{1-\alpha}\int_0^1 (1-s)y(s)\,ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta - s)y(s)\,ds - \int_0^t (t-s)y(s)\,ds.$$

Thus the kernel is

$$k(t,s) = \frac{1}{1-\alpha}(1-s) - \begin{cases} \dfrac{\alpha}{1-\alpha}(\eta - s), & s \le \eta, \\ 0, & s > \eta, \end{cases} - \begin{cases} t-s, & s \le t, \\ 0, & s > t. \end{cases}$$

Upper bounds

Obviously $\quad k(t,s) \le \dfrac{1-s}{1-\alpha} := \Phi(s)$

Lower bounds We take $a = 0, b = \eta$. $[a = \eta, b = 1$ works too.$]$

We are looking for $\min\{k(t,s) : t \in [0,\eta]\}$ as a function of s of the same form as the upper bound.

Case 1. $s > \eta$, then $t < s$ so $k(t,s) = \dfrac{1-s}{1-\alpha}$.

Case 2. $s \le \eta$.
For $t < s$,

$$\begin{aligned} k(t,s) =& \frac{1-s}{1-\alpha} - \frac{\alpha}{1-\alpha}(\eta - s) \\ \ge& \frac{1-s}{1-\alpha} - \frac{\alpha}{1-\alpha}(1-s) = (1-\alpha)\Phi(s). \end{aligned}$$

For $t \ge s$

$$\begin{aligned} k(t,s) =& \frac{1-s}{1-\alpha} - \frac{\alpha}{1-\alpha}(\eta - s) - (t-s) \\ \ge& \frac{(1 - s - \alpha\eta + \alpha s - (1-\alpha)(1-s))}{(1-\alpha)} \\ =& \frac{\alpha(1-\eta)}{1-\alpha} \\ \ge& \alpha(1-\eta)\Phi(s). \end{aligned}$$

So we can take $\gamma = \min\{1-\alpha,\ \alpha(1-\eta)\}$.

Therefore by Theorem 2.3 we have proved that a positive solution exists.

Special case When $g(t) \equiv 1$,

$$\frac{1}{M_1} = \max_{0\le t\le 1} \int_0^1 k(t,s)\,ds = \dots \text{ so } M_1 = \frac{2(1-\alpha)}{1-\alpha\eta^2}.$$

$$\frac{1}{m_1} = \min_{0\le t\le \eta} \int_0^\eta k(t,s)\,ds = \dots \text{ hence } m_1 = \frac{1-\alpha}{\eta(1-\eta)}.$$

Remark 3.2. For the BC

$$u'(0) = 0,\ \alpha u(\eta) = u(1),\ 0 < \eta < 1. \qquad (BC)_2$$

it is necessary to have $0 < \alpha < 1$ for positive solutions to exist, as is clear from the graph of a possible solution. [Also the example $u'' + 2 = 0$ shows this.]

4. The other BC

Similar methods work for the boundary value problem

$$u'' + g(t)f(u) = 0 \quad (0 < t < 1) \qquad (4.1)$$

with boundary conditions

$$u(0) = 0,\ \alpha u(\eta) = u(1),\ 0 < \eta < 1 \text{ and } \alpha\eta < 1. \qquad (BC)_1$$

This time we take $a = \eta, b = 1$ to get appropriate lower bounds.

The simple example $u'' + 2 = 0$ shows that the hypothesis $\alpha\eta < 1$ is necessary to ensure positive solutions exist. The solution is

$$u(t) = \Big(\frac{1-\alpha\eta^2}{1-\alpha\eta}\Big)t - t^2$$

and $u(1) < 0$ if $\alpha\eta > 1$.

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