ATTRACTORS OF REACTION-DIFFUSION EQUATIONS WITH NONMONOTONE NONLINEARITY

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ABSTRACT. In this paper we study the existence of global compact attractors for nonlinear parabolic equations of reaction-diffusion type. The studied equations are generated by a difference of subdifferential maps and are not assumed to have a unique solution for each initial state. Applications are given to inclusions modelling combustion in porous media and processes of transmission of electrical impulses in nerve axons.

1. Introduction. The main purpose of this paper is to study the existence of a compact global attractor for the next equation

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) + f_1(u) - f_2(u) \ni h, \text{ in } \Omega \times (0,\infty), \\ u\mid_{\partial\Omega} = 0, \\ u\mid_{t=0} = u_0, \end{cases}$$

where $p \geq 2$, $h \in L_2(\Omega)$, $f_i : \mathbb{R} \to 2^{\mathbb{R}}$, $i = 1, 2$, are maximal monotone maps and $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary $\partial\Omega$. The case $f_2(u) = \omega u$, $\omega \geq 0$, is well known. For such function the theory of maximal monotone operators provides a theorem of existence and uniqueness of solutions. The existence of a global compact attractor under some dissipative conditions is obtained then applying the theory of attractors for semigroups. For other functions f_2 the uniqueness is not guarantied. So, the theory of global attractors for semigroups of operators cannot be applied in this case. Therefore, we apply the theory of global attractors for multivalued semiflows developed in [3]-[4] and prove the existence of a global compact attractor under certain conditions.

We shall recall briefly the main definitions from the theory of multivalued dynamical systems. Let X be a complete metric space with the metric denoted by ρ and 2^X ($P(X)$, $\mathcal{B}(X)$) be the set of all (nonempty, nonempty bounded) subsets of X. For $x \in X$, $A, B \subset X$ we set $d(x,B) = \inf_{y\in B}\{\rho(x,y)\}$, $d(A,B) = \sup_{x\in A}\{d(x,B)\}$.

DEFINITION 1. The m-map $G : \mathbb{R}^+ \times X \to P(X)$ is called a multivalued semiflow if the next conditions are satisfied:

1. $G(0,\cdot) = I$ is the identity map;
2. $G(t_1 + t_2, x) \subset G(t_1, G(t_2, x))$, $\forall t_1, t_2 \in \mathbb{R}^+$, $\forall x \in X$,

where $G(t,B) = \underset{x\in B}{\cup} G(t,x)$, $B \subset X$.

DEFINITION 2. It is said that the set $\Re \subset X$ is a global attractor of the m-semiflow G if

1. $d(G(t,B),\Re) \to 0$ as $t \to +\infty$, $\forall B \in \mathcal{B}(X)$;
2. G is negatively semi-invariant, i.e., $\Re \subset G(t,\Re)$, $\forall t \geq 0$.

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In applications it is desirable for the global attractor to be compact and invariant (i.e. $\Re = G(t, \Re)$, $\forall t \geq 0$).

This paper is organized as follows. In Section 2 we obtain some abstract results concerning the existence of a compact global attractor for differential inclusions generated by a difference of subdifferential maps. In Section 3 we apply this abstract result to parabolic equations of reaction-diffusion type.

2. Abstract setting. Let H be a real Hilbert space supplied with the norm $|\cdot|$ and the scalar product $(\cdot,\cdot)$. Consider the differential inclusion

$$\begin{cases} \frac{du(t)}{dt} + \partial\psi^1(u(t)) - \partial\psi^2(u(t)) \ni h, \ t \geq 0 \\ u(0) = u_0, \end{cases} \tag{1}$$

where $h \in H$ and $\partial\psi^i$, $i = 1, 2$, are the subdifferentials of the proper, convex, lower semicontinuous functions $\psi^i : H \to]-\infty, +\infty]$.

DEFINITION 3. The function $u(\cdot) \in C([0,T], H)$ is called a strong solution of (1) if:

1. $u(0) = u_0$;
2. $u(\cdot)$ is absolutely continuous on $(0,T)$;
3. There exist functions $g^i(t)$, $g^i(t) \in \partial\psi^i(u(t))$, a.e. on $(0,T)$, such that

$$\frac{du(t)}{dt} + g^1(t) - g^2(t) = h, \text{ a.e. } t \in (0,T). \tag{2}$$

Let us consider the next conditions:

(A1) $\forall L < +\infty$ the level set

$$H_L = \{u \in H : \psi^1(u) + |u| \leq L\}$$

is compact.

(A2) $D(\psi^1) \subset D(\partial\psi^2)$, $0 \in D(\psi^1)$, and there exist $C \geq 0, 0 \leq k < 1, \frac{1}{2} < \gamma \leq 1$, such that $\forall u \in D(\psi^1)$

$$[\partial\psi^2(u), u] \leq k\psi^1(u) + C(|u|^2 + 1), \tag{3}$$

$$|\partial\psi^2(u)|^+ \leq M(|u|)\left(1 + |\psi^1(u)|^{1-\gamma}\right), \tag{4}$$

where $[\partial\psi^2(u), u] = \sup\limits_{v\in\partial\psi^2(u)} (v, u)$, $|\partial\psi^2(u)|^+ = \sup\limits_{v\in\partial\psi^2(u)} |v|$ and M is an increasing function.

THEOREM 1. [5, Theorem 5.3] *Let* A1–A2 *hold. Then for any* $u_0 \in \overline{D(\psi^1)}$ *and* $T > 0$ *there exists at least one strong solution* $u(\cdot)$ *of (1) satisfying for* $i = 1, 2$,

$$\sqrt{t}\,\frac{du}{dt} \in L_2(0,T;H), \tag{5}$$

$$\psi^i(u(t)) \in L_1(0,T), \tag{6}$$

$$t\psi^i(u(t)) \in L_\infty(0,T), \tag{7}$$

$$\psi^i(u(t)) \text{ are absolutely continuous on } (0,T], \tag{8}$$

$$g^1(t) \in L_2(\delta,T;H), g^2(t) \in L_\infty(\delta,T;H) \cap L_1(0,T;H), \forall\delta > 0. \tag{9}$$

LEMMA 1. *Under the conditions of Theorem 1,* $g^2(t) \in L_2(0,T;H)$.

Remark. In [5] it was assumed also that $\psi^i \geq 0$. However, this condition is not necessary, because without loss of generality we can assume that

$$\min\left\{\psi^i(u) : u \in H\right\} = \psi^i\left(x_0^i\right) = 0.$$

We note that for each $u_0 \in \overline{D(\psi^1)}$ there exists a solution $u(\cdot)$, but it can be non-unique. Hence, it is not possible to define a semigroup of operators.

Let $\mathcal{D}(u_0, T)$ be the set of all strong solutions of (1) such that $u(0) = u_0$ and $g^2(t) \in L_2(0, T; H)$. Denote

$$\mathcal{D}(u_0) = \cup_{T>0}\mathcal{D}(u_0, T).$$

Theorem 1 and Lemma 1 allow us to define a multivalued semiflow $G : \mathbb{R}^+ \times \overline{D(\psi^1)} \to P\left(\overline{D(\psi^1)}\right)$ in the following way:

$$G(t, u_0) = \{u(t) : u(\cdot) \in \mathcal{D}(u_0)\},\ t \geq 0, u_0 \in \overline{D(\psi^1)}.$$

LEMMA 2. *Let* A1–A2 *hold. For any* $t_i \geq 0$, $i = 1, 2$, $G(t_2 + t_1, u_0) = G(t_2, G(t_1, u_0))$.

THEOREM 2. *Let* A1–A2 *hold. Suppose that there exist* $\delta > 0$, $M > 0$ *such that* $\forall u \in D(\partial\psi^1)$, $|u| > M$, $\forall y_1 \in \partial\psi^1(u)$, $\forall y_2 \in \partial\psi^2(u)$,

$$(y_1 - y_2 - h, u) \geq \delta.$$

Then G has the global invariant compact attractor $\Re$. It is the minimal closed set attracting all bounded sets and maximal among all bounded negatively semi-invariant subsets of G.

3. Applications. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Consider the next differential inclusion

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) + f_1(u) - f_2(u) \ni h, \text{ in } \Omega \times (0, \infty), \\ u\mid_{\partial\Omega} = 0, \\ u\mid_{t=0} = u_0, \end{cases} \tag{10}$$

where $p \geq 2$, $h \in L_2(\Omega)$ and $f_i : \mathbb{R} \to 2^{\mathbb{R}}$, $i = 1, 2$, are maximal monotone maps with $D(f_i) = \mathbb{R}$.

Let us define the operator $A : D(A) \to L_2(\Omega)$,

$$A(u) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right),$$

with $D(A) = \left\{u \in W_0^{1,p}(\Omega) : A(u) \in L_2(\Omega)\right\}$ and the operators $B_i : D(B_i) \to L_2(\Omega)$, $i = 1, 2$,

$$B_1(u) = \{y \in L_2(\Omega) : \ y(x) \in A(u) + f_1(u(x)), \text{ a.e. in } \Omega\},$$

$$B_2(u) = \{y \in L_2(\Omega) : \ y(x) \in f_2(u(x)), \text{ a.e. in } \Omega\}.$$

The theory of subdifferential maps in Hilbert spaces (see [1]) provides the existence of proper, convex, lower semicontinuous functions ψ^i such that $B_i = \partial\psi^i, i = 1, 2$. Moreover, $\overline{D(\psi^i)} = L_2(\Omega)$, $i = 1, 2$. Therefore, Equation (10) is a particular case of (1).

Let us consider the following conditions:

(F1) Let $p > 2$ and $\exists \alpha > 0$, $K_1, K_2 \geq 0$ such that

$$|f_2(s)|^+ \leq K_1 + K_2 |s|^{1+\alpha}, \tag{11}$$

where $|f_2(s)|^+ = \sup_{y \in f_2(s)} |y|$ and

$$\alpha < \frac{p-2}{2}, \text{ if } n \leq p,$$

$$\alpha < \frac{p-2}{2}, 2(1+\alpha) \leq \frac{np}{n-p}, \text{ if } n > p.$$

(F2) Let $p = 2$ and $\exists K_1, K_2, M \geq 0, \varepsilon > 0$, such that

$$|f_2(s)|^+ \leq K_1 + K_2 |s|, \tag{12}$$

$$(y_1 - y_2) s \geq (-\lambda_1 + \varepsilon) s^2 - M, \forall y_i \in f_i(s), i = 1, 2, \tag{13}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

THEOREM 3. *Let either* F1 *or* F2 *be satisfied. Then* G *has the global invariant attractor* $\Re$, *which is compact in* $L_2(\Omega)$ *and bounded in* $W_0^{1,p}(\Omega)$. *It is the minimal closed set attracting all bounded sets. It is maximal among all bounded negatively semi-invariant sets.*

Inclusion (1) contains as particular cases boundary problems of physical interest. Let us consider some of them.

EXAMPLE 1

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) - |u|^{\alpha} u = h, \text{ in } \Omega \times (0, \infty), \\ u\,|_{\partial\Omega} = 0, \\ u\,|_{t=0} = u_0, \end{cases}$$

where $p > 2$ and α is taken from $F1$. In this case $f_1 \equiv 0, f_2(s) = |s|^{\alpha} s$.

EXAMPLE 2

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - f(u) \in \lambda H(u-1), \text{ in } (0, \pi) \times (0, \infty), \\ u(0) = u(\pi) = 0, \\ u\,|_{t=0} = u_0, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and non-decreasing, $\lambda > 0$ and

$$H(z) = \begin{cases} 0, & \text{if } z < 0, \\ [0,1], & \text{if } z = 0, \\ 1, & \text{if } z > 0. \end{cases}$$

Suppose also that there exist $K_1 \geq 0, 0 \leq K_2 < 1$ such that

$$|f(s)| \leq K_1 + K_2 |s|.$$

In this case $f_1 \equiv 0$, $f_2(s) = f(s) + \lambda H(s-1)$. This inclusion is used for modelling processes of combustion in porous media (see [2]).

EXAMPLE 3

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u \in H(u-a), \text{ in } (0, \pi) \times (0, \infty), \\ u(0) = u(\pi) = 0, \\ u\,|_{t=0} = u_0, \end{cases}$$

where $a \in \left(0, \frac{1}{2}\right)$. In this case $f_1(s) = s$, $f_2(s) = H(s-a)$.

EXAMPLE 4

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u(u-1)(u-a) \in H(u-a), \text{ in } (0,\pi) \times (0,\infty), \\ u(0) = u(\pi) = 0, \\ u\mid_{t=0} = u_0, \end{cases}$$

where again $a \in \left(0, \frac{1}{2}\right)$. Let $f(s) = s(s-1)(s-a)$. In this case $f_1(s) = f(s) + \frac{1}{3}(a^2 - a + 1)s$, $f_2(s) = \frac{1}{3}(a^2 - a + 1)s + H(s-a)$.

Inclusions (3)–(4) are used as models of conduction of electrical impulses in nerve axons (see [6]-[7]).

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