ISOPERIMETRIC INEQUALITIES AND REGULARITY AT SHRINKING POINTS FOR PARABOLIC PROBLEMS

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1. Introduction. In this paper we consider a noncylindrical in time domain Q_T which is given by

$$Q_T = \{(x,t), t \in (t_0, T), x \in \Omega(t) \subset \mathbb{R}^N\}. \tag{1.1}$$

We assume that for any τ, $t_0 < \tau < T$, the domain Q_τ is homeomorphic to a cylinder $\Omega \times (t_0, \tau)$ where Ω has the classical regularity in parabolic problems. We also assume that Q_T "shrinks" at P, that is

$$\overline{Q}_T \cap \{t = T\} = P, \tag{1.2}$$

where P is the point $(x = 0, t = T)$; P is called the "shrinking point" of Q_T.

Our concern is the regularity at P of solutions of various parabolic problems in Q_T. Among all papers devoted to the question of regularity of boundary points for parabolic problems, let us quote e.g. [7], [10] to [17], [22], [23]. For example, let us consider the heat equation with diffusion coefficient K (positive constant)

$$u_t - K\,\Delta u = 0 \quad \text{in} \quad Q_T, \tag{1.3}$$

coupled with a boundary condition on the lateral boundary Σ_T and an initial condition on $\Omega(t_0)$. The point P is said "regular" for the heat equation (1.3) if for all continuous boundary and initial conditions, the solution u of (1.3) is continuous at P. Otherwise P is said "irregular".

In [12], L.C. Evans and R.F. Gariepy proved that the regularity of P depends on the local form of Q_T near P. Their criterium for regularity is based on the properties of the fundamental solution of the heat equation, through the concept of thermal capacity. Similar results are obtained for linear equations in divergence form with C^1-Dini continuous coefficients in [13]. A sufficient condition for regularity for nonlinear uniformly parabolic equations in divergence form can be found in [14].

Let us consider again the heat equation (1.3). In radial symmetry, that is if for any t, $t_0 < t < T$, $\Omega(t)$ is a ball of center 0 and radius $R(t)$, if the boundary condition only depends on t and if the initial condition only depends on $r = \|x\|$, I. Petrovsky [22] studied the regularity of P for different classes of functions $R(t)$. More precisely, let $g(t) = \log|\log(T-t)|$. It is proved in [22] that if

$$(T-t)R^{-2}(t)g(t) \to +\infty \text{ (resp. 0)},$$

then P is regular (resp. irregular) and this is true independently of the diffusion coefficient K. However there are domains Q_T, such that the regularity condition depends on K. For example if

$$R^2(t) = a(T-t)g(t)$$

with a positive constant, then P is regular if and only if $a \geq 4K$ (see [22]).

In the papers [20], [21] the above result (for the simple heat equation in radial symmetry) is used to get criteria of continuity or discontinuity at P for bounded solutions $(C \leq u(x,t) \leq S)$ of more general parabolic problems, such as

$$\begin{cases} u_t - \operatorname{div} a(x,t,u,\nabla u) + Bu = f \ \text{ in } \ Q_T, \\ u = C \ \text{ on } \ \Sigma_T, \\ u|_{t=t_0} = u_0 \ \text{ in } \ \Omega(t_0), \end{cases} \tag{1.4}$$

(B, C constant), in general (i.e. non radial) shrinking domains Q_T as defined in the beginning. It turns out that the regularity of solutions of such problems (1.4) depends (in general) on the boundary conditions and also on the whole domain Q_T (not only on its local form near P).

The comparison of (1.4) with a heat equation in radial symmetry is obtained by means of symmetrization. This allows to compare for each t, $\|u(t) - C\|$ to $\|v(t) - C\|$ where v solves a heat equation. One gets a sufficient continuity (at P) condition of u, for (1.4) in general domains Q_T and a sufficient discontinuity condition for a particular form of (1.4), namely $u_t - A(t)\Delta\varphi(u) + Bu = f$ in Q_T for radial domains Q_T. These results are presented in Section 2.

In Section 3 the continuity result is generalized under convenient assumptions to the problem

$$\begin{cases} u_t - \operatorname{div} \, a(x,t,u,\nabla u) + \\ \qquad + b(x,t,u,\nabla u)u + d(x,t,u,\nabla u) = h(x,t) \ \text{ in } \ Q_T, \\ u = C \ \text{ on } \ \Sigma_T, \\ u|_{t=t_0} = u_0 \ \text{ in } \ \Omega(t_0). \end{cases} \tag{1.5}$$

Let us finally remark that, since for the heat equation the continuity (discontinuity) at P strongly depends on the diffusion coefficient , then for a general nonlinear problem the continuity strongly depends on the upper and lower bounds of the solution itself. Moreover for given initial and boundary data, the continuity shall depend, in general, on the whole domain and not only on its local form near P. This will be illustrated in Section 4 for a generalized nondegenerate porous media equation (see [21]).

Let us also stress that the above results can be applied to the homogeneous Dirichlet problem for some classes of degenerate parabolic equations, such as the generalized porous media equation. Roughly speaking the results on continuity for the heat equation which do not depend on the diffusion coefficient extend to the degenerate equation (see Sect.5 and [21]).

2. Basic notations and statement of the comparison results. We begin by recalling here the definitions and main properties of rearrangements which will be used later on to construct the "symmetrized" problem.

For Ω (Lebesgue) measurable subset of $\mathbb{R}^N$ and $k : \Omega \to \mathbb{R}$ measurable, let us denote $|\Omega|$ the measure of $|\Omega|$ and $k^* : [0, |\Omega|] \to \overline{\mathbb{R}}$ the monotone nonincreasing rearrangement of k:

$$\begin{aligned} k^*(0) &= \text{ess sup } k(\leq +\infty), \\ k^*(|\Omega|) &= \text{ess inf } k(\geq -\infty), \\ k^*(s) &= \text{Inf}\{\theta \in \mathbb{R}, |k > \theta| \leq s\} \ , \ s \in (0, |\Omega|) \end{aligned} \tag{2.1}$$

(here $|k > \theta|$ is the measure of $\{x \in \Omega, k(x) > \theta\}$). We denote by $\widetilde{\Omega}$ the ball of $\mathbb{R}^N$, centered at the origin, having same measure as Ω :

$$\widetilde{\Omega} = \left\{ x \in \mathbb{R}^N, \alpha_N \|x\|^N < |\Omega| \right\}, \tag{2.2}$$

where α_N is the measure of the unit ball in $\mathbb{R}^N$, and we define $\widetilde{k} : \overline{\widetilde{\Omega}} \to \overline{\mathbb{R}}$ by

$$\widetilde{k}(x) = k^* \left(\alpha_N \|x\|^N \right). \tag{2.3}$$

For the usual properties of rearrangements, the reader is refered to (e.g.) [18]. In particular it is classical that a function $k : \Omega \to \mathbb{R}$ and its rearrangements k^* and $\widetilde{k}$ are equimeasurable:

$$\begin{aligned} \forall \theta \in \mathbb{R}, \quad |k > \theta| &= |k^* > \theta| = \left|\widetilde{k} > \theta\right|, \\ |k = \theta| &= |k^* = \theta| = \left|\widetilde{k} = \theta\right| \cdots \end{aligned}$$

which implies that for every Borel function F which is bounded above or below

$$\int_\Omega F(k) dx = \int_0^{|\Omega|} F(k^*) \, ds = \int_{\widetilde{\Omega}} F\left(\widetilde{k}\right) dx. \tag{2.4}$$

We also recall the Hardy–Littewood inequality

$$\int_\Omega k \, \ell \, dx \leq \int_0^{|\Omega|} k^* \, \ell^* \, ds = \int_{\widetilde{\Omega}} \widetilde{k} \, \widetilde{\ell} \, ds \tag{2.5}$$

(for k in $L^p(\Omega)$, ℓ in $L^{p'}(\Omega)$, $1 \leq p \leq \infty$, $(1/p) + (1/p') = 1$) and the Polya Czegö inequality

$$\int_{\widetilde{\Omega}} \left\|\nabla \widetilde{k}\right\|^2 dx = \int_0^{|\Omega|} \left(N \, \alpha_N^{\frac{1}{N}} s^{1-\frac{1}{N}} \frac{du^*}{ds} \right)^2 ds \leq \int_\Omega \|\nabla k\|^2 \, dx \tag{2.6}$$

for k in $H_0^1(\Omega)$, $k \geq 0$.

Now let us state precisely the Dirichlet problem we consider in the following, together with the required assumptions :

$$\begin{cases} \mathcal{L}u = u_t - \operatorname{div} a\left(x,t,u,\nabla\varphi(u)\right) + Bu = f \\ \qquad\qquad \text{in } Q_\tau = \{(x,t), t_0 < t < \tau, x \in \Omega(t)\}, \\ u = C \text{ on } \Sigma_\tau = \{(x,t), t_0 < t < \tau, x \in \partial\Omega(t)\}, \\ u|_{t=t_0} = u_0(x) \text{ in } \Omega(t_0). \end{cases} \tag{2.7}$$

Here B and C are constants, ∇ (respectively div) denotes the gradient (respectively divergence) with respect to x and $Q_\tau, a, \varphi, f, u_0$ satisfy (throughout the paper) the assumptions

(H.1) (i) the domain Q_τ can be transformed by a inversible C^2 mapping into a cylindrical domain $\Omega\times(t_0,\tau)$, Ω having the usual regularity required in the parabolic problems ;

(ii) $a : Q_\tau \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ (defined a.e. in Q_τ and everywhere in $\mathbb{R}\times\mathbb{R}^N$) satisfies the uniform parabolicity condition

$$\underline{A}\,\|\xi\|^2 \le \alpha(t,\eta)\,\|\xi\|^2 \le a(x,t,\eta,\xi)\cdot\xi \le \beta(\eta)\,\|\xi\|^2 \tag{2.8}$$

($\underline{A}$ positive constant, β bounded on bounded sets) a.e. in Q_τ and for every $\eta\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$;

(iii) $\varphi : \mathbb{R}\to\mathbb{R}$ is a strictly increasing C^1 function such that $\varphi' > 0$;

(iv) $f \in L^1(Q_\tau)$, $u_0 \in L^\infty(\Omega(t_0))$.

In this Section, we are concerned with sub and supersolutions of problem (2.7), which we define below.

DEFINITION 1. A function $u : Q_\tau \to \mathbb{R}$ is called a *regular subsolution* (respectively *supersolution*) of (2.7) if

a) $u \in L^\infty(Q_\tau), \nabla\varphi(u) \in L^2(Q_\tau)^N$ (i.e. $\nabla u \in L^2(Q_\tau)^N$), u_t and $\operatorname{div} a\left(\cdot,u,\nabla\varphi(u)\right) \in L^1(Q_\tau)$,

b)

$$\begin{cases} \mathcal{L}u \le f \text{ in } Q_\tau, \\ u \le C \text{ on } \Sigma_\tau, \\ u|_{t=t_0} \le u_0 \text{ in } \Omega(t_0) \end{cases}$$

(respectively if all the above inequalities are reversed). Obviously a regular solution of (2.7) is both a regular sub and supersolution.

Now let us define the "symmetrized problem"

$$\begin{cases} \widetilde{\mathcal{L}}v = v_t - A(t)\Delta\varphi(v) + Bv = \widetilde{f} \\ \qquad\qquad \text{in } \widetilde{Q}_\tau = \left\{(x,t), t_0 < t < \tau,\ x \in \widetilde{\Omega}(t)\right\}, \\ v = C \text{ on } \widetilde{\Sigma}_\tau = \left\{(x,t), t_0 < t < \tau,\ x \in \partial\widetilde{\Omega}(t)\right\}, \\ v|_{t=t_0} = \widetilde{u_0} \text{ in } \widetilde{\Omega}(t_0), \end{cases} \tag{$\widetilde{2.7}$}$$

where $A(t)$ will be precised hereafter, $\widetilde{u_0}$, $\widetilde{\Omega}(t)$ are defined from u_0, $\Omega(t)$ according to the previously given definitions (2.1) to (2.3) and where $\widetilde{f}$ is the rearrangement of f with respect to x : for any fixed t, $\widetilde{f}(t) = \widetilde{f(t)}$. Since $\left(\widetilde{\mathcal{Q}}_\tau, \widetilde{\mathcal{L}}\right)$ are special forms of $(\mathcal{Q}_\tau, \mathcal{L})$, the regular sub and supersolution of $\widetilde{(2.7)}$ are well defined according to Definition 1.

For k radial, defined in $\Omega = \widetilde{\Omega}$, let us denote

$$\overline{k}(s) = k(x) \text{ for } s = \alpha_N \|x\|^N. \tag{2.9}$$

For $k(t)$ radial defined in $\widetilde{\Omega}(t)$, we write

$$\overline{k}(s,t) = \overline{k(t)}(s).$$

Finally we define the "admissible radial functions"

DEFINITION 2. A radial function k defined in $\widetilde{\mathcal{Q}}_\tau$ is said admissible if for any fixed t, it is differentiable with respect to the radius at the origin or if, $\overline{k}$ being defined as above, the distributional derivative of $\varphi(\overline{k})$ is a function defined almost everywhere and

$$\underline{\lim}_{s\to 0} s^{2-\frac{2}{N}} \frac{\partial}{\partial s}\varphi(\overline{k}) \leq 0.$$

We remark that the last inequality holds true if (e.g.) there exists a small neighborhood $N(t)$ of 0 such that

- either $k(t)$ is nonincreasing with respect to the radius in $N(t)$,
- or $\nabla\ \varphi \circ k(t) \in L^\infty(N(t))$ – since $\underline{\lim}\, s^{2-\frac{2}{N}} \frac{\partial}{\partial s}\varphi(\overline{k}) = \underline{\lim} N^{-1} \alpha_N^{-\frac{1}{N}} s^{1-\frac{1}{N}} \frac{\partial}{\partial r}\varphi(\overline{k})$ $= 0$–.

Remark 1. In the above definition k is supposed to be radially symmetric but non necessarily monotone. However if k is monotone nonincreasing with respect to the radius, then $\overline{k}$ coincides a.e. with k^* and k is admissible.

At last we can state the comparison result obtained in [20]

THEOREM 1. *Assume u is a regular subsolution of (2.7) in the sense of Definition 1, with $u \geq C$ in $\mathcal{Q}_\tau$. Let S be such that $u \leq S$ in $\mathcal{Q}_\tau$ and consider $\widetilde{(2.7)}$ with $A(t) = \operatorname{Inf}\{\alpha(\eta,t), \eta \in [C,S]\}$ (see (2.8)). Assume v is a supersolution of $\widetilde{(2.7)}$ which is regular, radial and admissible in the sense of Definitions 1 and 2. Then one has a.e. $t \in (t_0,\tau)$, $\forall p \in [1,\infty]$, $\|u(t) - C\|_{L^p(\Omega(t))} \leq \|v(t) - C\|_{L^p(\widetilde{\Omega}(t))}$ $(< \infty)$.*

Remark 2. In the examples, one has often a priori estimates for the solutions, so that Theorem 1 can apply also for solutions provided an upper bound is known.

We can derive lower estimates for $\|u(t) - C\|$, with u supersolution of (2.7), under the following reinforced assumption

(H.2) • (H.1) holds together with

- $\Omega(t) = \widetilde{\Omega}(t)$, $\forall t \in (t_0, \tau)$,

- $a(x,t,\eta,\xi) \equiv A(t)\xi$ for a given function $A \geq \underline{A}$ (so that $\mathcal{L}u = \widetilde{\mathcal{L}}u$),
- $f \in C^0\left(\overline{Q}_\tau\right)$, $\left(u_0 \in C^0\left(\overline{\Omega(t_0)}\right)\right)$, f and u_0 radially symmetric, monotone nonincreasing with respect to the radius and such that $u_0 \geq C$ on $\partial\Omega(t_0)$, $f \geq BC$ in Q_τ.

Then we can state the following :

THEOREM 2. *Assume that* (2.7) *has the particular form given by* (H.2) *and that* u *is a radial monotone nonincreasing along radii regular supersolution of* (2.7) (*see Definition* 1). *Assume* $u \leq S$, *define*

$$M' = \operatorname{Max}\{\varphi'(\eta), C \leq \eta \leq S\}$$

and let w *be the solution of the following heat equation*

$$\begin{cases} \mathcal{H}w = w_t - A(t)M'\Delta w + Bw = f = \widetilde{f} \text{ in } Q_\tau = \widetilde{Q}_\tau, \\ w = C \text{ on } \Sigma_\tau = \widetilde{\Sigma}_\tau, \\ w|_{t=t_0} = u_0 = \widetilde{u}_0 \text{ in } \Omega(t_0) = \widetilde{\Omega}(t_0). \end{cases} \tag{2.7'}$$

Then a.e. $t \in (t_0,\tau), \forall p \in [1,\infty], \|w(t) - C\|_{L^p(\Omega(t))} \leq \|u(t) - C\|_{L^p(\Omega(t))}$ $(< +\infty)$.

Let us remark that u is bounded below in Q_τ by C by assumption (it is nonincreasing along radii) and that the assumption (H.2) on f, u_0 garantee that also w is radial, monotone nonincreasing along radii and $w \geq C$ in Q_τ.

3. Application to the continuity of (sub-)solutions at a shrinking point (general result). Let $T > t_0$. In what follows we assume that for any τ, $t_0 < \tau < T$, the domain Q_τ used in (2.7) satisfies (H.1.i) and we consider

$$Q_T = \bigcup_{t_0<\tau<T} Q_\tau , \ \Sigma_T = \bigcup_{t_0<\tau<T} \Sigma_T. \tag{3.1}$$

We assume that Q_T "*shrinks*" at $P = (0,T)$, that is

$$\overline{Q}_T \cap \{t = T\} = P = (0,T) \tag{3.2}$$

and we consider problem (2.7) in Q_T instead of Q_τ, that is

$$\begin{cases} \mathcal{L}u = u_t - \operatorname{div} a(x,t,u,\nabla\varphi(u)) + Bu = f \text{ in } Q_T, \\ u = C \text{ on } \Sigma_T, \\ u|_{t=t_0} = u_0 \text{ in } \Omega(t_0), \end{cases} \tag{3.3}$$

where φ, u_0 are as in (H.1) and where now $f \in L^1(Q_T)$ and $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies

$$\underline{A}\|\xi\|^2 \leq \alpha(t,\eta)\|\xi\|^2 \leq a(x,t,\eta,\xi)\cdot\xi \leq \beta(\eta)\|\xi\|^2 \tag{3.4}$$

a.e. $x \in Q_T$, for every $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

For a function $u : Q_T \to \mathbb{R}$, the respective notions of "admissibility", "regularity", being "sub (or super)-solution for (3.3)" mean that, for any τ, $t_0 < \tau < T$, $u|_{Q_\tau}$ is respectively admissible, regular, sub (or super)-solution for (2.7) in the sense given in Section 2.

Moreover we will say that a function u, which is regular in Q_T in the above sense and satisfies $u|_{\Sigma_T} = C$, is *continuous* at the shrinking point P if

$$\operatorname*{ess\,sup}_{T-2h<t<T-h} ||u(t) - C||_{L^\infty(\Omega(t))} \to 0 \tag{3.5}$$

when h goes to zero.

Clearly Theorem 1 has applications to the continuity of (sub-)solutions of (3.3). Let us define

$$\begin{cases} \widetilde{\mathcal{L}}v = v_t - A(t)\Delta\varphi(v) + Bv = \widetilde{f} \ \text{ in } \ \widetilde{Q}_T, \\ v = C \ \text{ on } \ \widetilde{\Sigma}_T, \\ v|_{t=t_0} = \widetilde{u}_0 \ \text{ in } \ \widetilde{\Omega}(t_0). \end{cases} \tag{$\widetilde{3.3}$}$$

From Theorem 1 applied in Q_{T-h}, we get

COROLLARY 1. *Assume u is a regular subsolution of (3.3) with $C \le u \le S$ in Q_T. Let $A(t) = \operatorname{Inf}\{\alpha(t,\eta), \eta \in [C,S]\}$. As soon as ($\widetilde{3.3}$) has a regular, radial and admissible supersolution which is continuous at P (in the sense of (3.5)), then also u is continuous at P.*

From Theorem 2, one gets also a discontinuity result. Briefly speaking, in radial symmetry and with $a(x,t,\eta,\xi) = A(t)\xi$, discontinuity for w implies discontinuity for u.

The same method can be applied to slightly different equations such as (1.5) in Q_T, where Q_T is as before, shrinking at P, a, u_0 are as before and

$$\begin{aligned} |b(x,t,\eta,\xi)| &\le B(\eta), \\ |d(x,t,\eta,\xi)| &\le D(\eta), \\ |h(x,t)| &\le H, \end{aligned} \tag{H.1$'$}$$

for a.e. $(x,t) \in Q_T$, every $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, with B, D bounded on bounded sets, H positive constant.

Defining regular subsolutions of (1.5) as above,we get

COROLLARY 2. *Assume (H.1$'$), assume u is a regular subsolution of (1.5) such that $C \le u \le S$ in Q_T and let the function A be given by $A(t) = \operatorname{Inf}\{\alpha(t,\eta), \eta \in [C,S]\}$. Then u is continuous at P as soon as P is regular for the heat equation*

$$w_t - A(t)\Delta w = 0 \ \text{ in } \ \widetilde{Q}_T. \tag{3.6}$$

Let us remark that considering $-u$ instead of u, the above corollary holds also when u is a solution of (1.5) such that $u \le C$ in Q_T.

Moreover if the operator $\mathcal{M}$ possesses a comparison principle, one can give continuity results of the type above also for the solutions corresponding to general continuous boundary data ($u = \psi$ on Σ_T), simply constructing sub- and super-solutions which satisfy Corollary 2.

4. The nondegenerate porous media equation. As an application of the previous results we consider the nondegenerate porous media equation in Q_T with $\psi \in C^2(\mathbb{R})$, $\psi'(\eta) > 0$ for $\eta \in \mathbb{R}$:

$$\begin{cases} u_t - \Delta\psi(u) = 0 \text{ in } Q_T, \\ u = C \text{ on } \Sigma_T, \\ u|_{t=t_0} = u_0 \text{ in } \Omega(t_0) \end{cases} \tag{4.1}$$

To focus the attention on the shrinking point P, we assume u to be a classical solution of (4.1). For example we assume

$$u_0 \in C^1\left(\overline{\Omega(t_0)}\right),\ u_0|_{\partial\Omega(t_0)} = C,\ C \le u_0 \le S. \tag{H.3}$$

Then (4.1) has a unique solution u and for any τ, $t_0 < \tau < T$, $u|_{Q_\tau}$ is a classical solution which is regular in the sense of Definition 1 . Moreover $C \le u \le S$ in Q_T.

Let us define

$$\begin{aligned} K_1 = K_1(C,S) = \min(\psi'(\eta), C \le \eta \le S) > 0, \\ K_2 = K_2(C,S) = \max(\psi'(\eta), C \le \eta \le S) \ge K_1. \end{aligned} \tag{4.2}$$

Then Theorem 1 applied to the present case gives us the following result:

THEOREM 3. *Assume* (H.3). *Let* v *be the solution of*

$$\begin{cases} v_t - K_1\Delta v = 0 \text{ in } \widetilde{Q}_T, \\ v = C \text{ on } \widetilde{\Sigma}_T, \\ v|_{t=t_0} = \widetilde{u}_0 \text{ in } \widetilde{\Omega}(t_0), \end{cases}$$

with K_1 *as in* (4.2), $\widetilde{\Omega}(t_0)$ *defined from* $\Omega(t_0)$ *as above and* $\widetilde{Q}_T$, $\widetilde{\Sigma}_T$ *defined by symmetrization in an obvious way. If* v *is continuous at* P, *then so is the solution* u *of* (4.1).

Since $\widetilde{Q}_T$ is radially symmetric, we can use the results of Section 2 and obtain a sufficient condition for the continuity of u at P, depending only on the radius $R(t)$ of $\widetilde{\Omega}(t)$, obviously defined as $|\Omega(t)| = \alpha_N R(t)^N$. This gives

COROLLARY 3. *Assume* (H.3). *Let* $g(t) = \log|\log(T-t)|$.

(i) *If, with* K_1 *defined in* (4.2),

$$R^2(t) \le 4K_1(T-t)g(t),\ \forall t, t_0 < t < T, \tag{4.3}$$

then the solution u *of* (4.1) *is continuous at* P.

(ii) *This is the case (for any* C *and* S *) if*

$$\lim_{t \nearrow T} \frac{(T-t)g(t)}{R^2(t)} = +\infty. \tag{4.4}$$

In the radially symmetric case, discontinuity results can be obtained by applying Theorem 2 . We assume that

$$\Omega(t) = \widetilde{\Omega}(t), \forall t, t_0 < t < T \tag{H.4}$$

and we consider that

$$u_0 = \widetilde{u}_0 \text{ is as in (H.3)}. \tag{H.5}$$

Then the solution u of (4.1) is also radially symmetric, monotone nonincreasing with respect to the radius $(u = \widetilde{u})$ and we can apply Theorem 2 to u, getting the following

THEOREM 4. *Assume* (H.4), (H.5). *Let* w *be the solution of*

$$\begin{cases} w_t - K_2 \Delta w = 0 \text{ in } Q_T = \widetilde{Q}_T, \\ w = C \text{ on } \Sigma_T = \widetilde{\Sigma}_T, \\ w|_{t=t_0} = u_0 = \widetilde{u}_0 \text{ in } \Omega(t_0) = \widetilde{\Omega}(t_0), \end{cases}$$

with K_2 *as in* (4.2). *If* w *is discontinuous at* P, *then so is the solution* u *of* (4.1).

Again by means of the results of Section 2 we have

COROLLARY 4. *Assume* (H.4). *Let* $C < S$ *be given constants and let* $K_2 = K_2(C, S)$ *be given by* (4.2).
(i) *If, for some* $l > 1$,

$$R^2(t) \geq 4\lambda K_2 (T - t) g(t), \forall t, t_0 < t < T, \tag{4.5}$$

then there exists u_0 *satisfying* (H.5) *such that the corresponding solution* u *of* (4.1) *is discontinuous at* P.
(ii) *If*

$$\lim_{t \nearrow T} \frac{(T-t)g(t)}{R^2(t)} = 0, \tag{4.6}$$

then, for any C *and* S, *there exists* $t_0 < T$ *(close enough to* T *) and* u_0 *satisfying* (H.5) *such that the corresponding solution of* (4.1) *is discontinuous at* P.

Hence for domains satisfying (4.4) or (4.6) we have the same result for (4.1) as for the heat equation (for any diffusion coefficient).

On the contrary let us stress here the fact that there are classes of functions R for which the continuity (or discontinuity) of u depends on C and S, that is of the bounds of u. While for linear problems one can speak of regularity of a boundary point for an equation (independently of the boundary and initial data), for nonlinear equations one has to speak of regularity for a given boundary value problem. Moreover one can see that, also when the boundary value C is fixed, the continuity of u at P can strongly depend on the upper bound of u_0. For example, let us consider a domain Q_T for which

$$R^2(t) = 4(1+a)(T-t)g(t), \qquad t < T, \tag{4.7}$$

where a is a given positive constant.

To fix the ideas consider problem (4.1) with $C = 0$, $R(t)$ given by (4.7) and assume that ψ satisfies the following condition

$$\begin{gathered} \psi'(0) > 1 + a \text{ and there exists a positive number } m \\ \text{such that } \psi'(m) = 1,\ \psi'(\eta) \geq 1 \text{ for } 0 \leq \eta \leq m,\ \psi'(\eta) \leq 1 \text{ for } \eta \geq m. \end{gathered} \tag{4.8}$$

Remark that the boundary value $C = 0$ is such that $\psi'(0) > 1 + a$, so that the point P would be regular for the heat equation with diffusion coefficient $\psi'(0)$. For (4.1), it turns out that if the initial datum is small enough, then u is continuous at P, but without the above restriction, one can construct explicit discontinuous subsolutions and prove by the way the existence of discontinuity solutions. One can give also examples of different domains which coincide around P for which one has either continuity or discontinuity, with same boundary and initial data. Precisely we have (see [21]) :

PROPOSITION 1. *Assume* (4.7), (4.8). *Let* $0 < \delta < m$ *be the smallest root of* $\psi'(\delta) = 1 + a$.
(i) *If* $S \leq \delta$, *then every solution of* (4.1), (H.3) *with* $C = 0$ *is continuous at* P.
(ii) *There exist* t_0 *and* u_0 *satisfying* (H.5) *such that the corresponding solution of* (4.1), (H.4) *with* $C = 0$ *is discontinuous at* P.
(iii) *For* u_0 *as in point* (ii), *one can have either continuity or discontinuity at* P *for the solution of* (4.1) *with* $C = 0$ *and initial datum* u_0 *for different radial domains which coincide in a neighborhood of* P.

Let us remark that the value $\psi'(m) = 1$ is immaterial : it can be changed provided $R(t)$ changes accordingly, substituting $\psi'(m) + a$ to $1 + a$. More important is that the above proposition (see (iii)) implies that in general the regularity of a shrinking point P is no longer a local property as it is for linear equations.

5. Two classes of degenerate parabolic equations. In the same context of section 4 , let us consider now the nonnegative bounded solutions of the following two classes of equations

$$u_t - \Delta\psi(u) = 0 \quad \text{in} \quad Q_T, \tag{5.1}$$

where $\psi \in C^1[0,\infty) \cap C^2(0,\infty), \psi'(s) > 0$ for $s > 0$, $\psi'(0) = 0$ (a model is obtained with $\psi(s) = s^m$, $m > 1$) and

$$v_t - v\Delta v + \gamma |\nabla v|^2 = 0 \quad \text{in} \quad Q_T, \tag{5.2}$$

where γ is a given constant.

Both equations (5.1) and (5.2) are of degenerate parabolic type since they lose their parabolicity at points where u or v vanishes. Therefore one has to consider weak solutions - see [1] for a survey on equation (5.1) and [4] to [8], for equation (5.2). A particular weak solution can be constructed as a limit of a monotone nonincreasing sequence of positive classical approximating solutions and to this (maximal) solution we will refer hereafter.

We consider here the case of homogeneous boundary data, with $0 \leq u_0(x) \leq S$ or $0 \leq v_0(x) \leq S$ (and hence $0 \leq u(x,t) \leq S$ or $0 \leq v(x,t) \leq S$). Let us stress the fact that in general the results obtained for this problem do not apply to general continuous boundary data as can be seen with an explicit counterexample (see [21]).

Applying the results of the previous section to the classical approximations and then passing to the limit, we get results of the following style:

THEOREM 5. *Let* $g(t) = \log|\log(T-t)|$. *If* $R(t)$ *is such that*

$$\lim_{t \nearrow T} \frac{(T-t)g(t)}{R^2(t)} = +\infty,$$

then any weak solution of (5.1), (5.2) *is continuous at* P.

Let ut remark that this both explains and generalizes the results of [7]. In symmetrical domains, we can also give discontinuity results for equation (5.1) using Theorem 2 :

THEOREM 6. *Let* $g(t) = \log|\log(T-t)|$, $K_2 = K_2(S) = \max\{\psi'(\eta), 0 \le \eta \le S\}$. *If*

$$R^2(t) \ge K(T-t)g(t)$$

for $K > 4K_2$, *then there exists* $u_0 = \tilde{u}_0$, $0 \le u_0 \le S$ *such that the viscosity solution of equations* (5.1), (5.2) *in* $Q_T = \tilde{Q}_T$ *is discontiuous at* P.

Let us remark that in nonsymmetrical domains it is not possible to give general discontinuity conditions not depending of the particular form of $\Omega(t)$. In fact equations (5.1), (5.2) have finite speed of propagation, so there can be subsets of Q_T where $u \equiv 0$ depending of the form of $\Omega(t)$ (and independently in general of its measure).

Remark 3. Let us remark that the continuity result of this Section applies also to the porous media equation with source or absorption as long as we deal with nonnegative bounded solutions.

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