NONEXISTENCE OF GLOBAL NONNEGATIVE SOLUTIONS TO QUASILINEAR PARABOLIC INEQUALITIES

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1. Introduction

In this note we review some recent results (obtained jointly with S. I. Pohozaev) concerning blow-up of nonnegative solutions to parabolic inequalities of the following type:

$$\rho(x,t)\partial_t(u^k) \geq \sum_{i,j=1}^{n} \partial_{x_i}\left[a_{ij}(x,t,u)\, f(|\nabla u|)\,\partial_{x_j} u\right] + c(x,t,u)\, u^q \tag{1.1}$$

in $\mathbb{R}^n \times (0,\infty)$; here $k > 0$, $q > 1$ and ρ, a_{ij}, f, c are given functions (ρ, f, c positive, $a_{ij} = a_{ji}$; precise assumptions are made in the following). We refer the reader to [5] for the proofs.

In particular, the present results apply to solutions of the Cauchy problem for parabolic equations, allowing us to investigate *critical exponents* for blow-up (e.g., see [2], [6]). A similar approach has been used to prove nonexistence theorems of Liouville type for elliptic inequalities (see [3]). The underlying ideas of the method suggest a general approach to nonexistence problems, which leads to the concept of *nonlinear capacity* (see [4]).

2. Mathematical framework

Let S_T denote the strip $\mathbb{R}^n \times (0,T]$ ($T \in (0,\infty]$); set $S \equiv S_\infty$. The following assumptions will be used:

(*a*) $\rho \in C(S_T)$, $a_{ij} \in C(S_T \times [0,\infty))$, $c \in C(S_T \times [0,\infty))$, $f \in C([0,\infty))$;

(*b*) $\rho > 0$, $c > 0$, $\rho(x,\cdot)$ nondecreasing for any $x \in \mathbb{R}^n$;

(*c*) there exist $A_0 = A_0(x,t,u)$, $A_1 = A_1(x,t,u) \in C(S_T \times [0,\infty))$ such that $0 \leq A_0 \leq A_1$ and there holds:

$$A_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\,\xi_i\,\xi_j\,, \quad \Big|\sum_{i,j=1}^{n} a_{ij}\,\xi_i\,\eta_j\Big| \leq A_1\,|\xi|\,|\eta|$$

for any $\xi, \eta \in \mathbb{R}^n$;

(*d*) $f \geq 0$ in $[0,\infty)$ and for any $t \geq 0$ there holds either

$$(i) \quad 0 \leq f(t) \leq c_0\,, \qquad \text{or} \qquad (ii) \quad c_1\, t^{p-2} \leq f(t) \leq c_2\, t^{p-2}$$

($c_0 > 0$; $0 < c_1 \leq c_2$, $p > 2$) .

The above assumptions will be collectively referred to as *Assumption* (H).

Concerning solutions to inequality (1.1) we make the following definitions.

DEFINITION 2.1. By a *strong solution* to inequality (1.1) in S_T we mean any nonnegative function $u \in C(S_T)$ such that (its distributional derivatives of the first order in time and up to the second order in the space variables are defined almost everywhere in S_T and) inequality (1.1) is satisfied almost everywhere in S_T.

DEFINITION 2.2. Let $\alpha \in (-k, 0)$. By a solution of class P_α to (1.1) in S_T we mean any nonnegative function $u \in C(S_T)$ such that for any test function $\psi \geq 0$ with support in $\bar{S}_T$ there holds:
(i)

$$\int\int_{supp\,u} A_1 f(|\nabla u|)\,|\nabla u|\,u^\alpha\,|\nabla\psi| < \infty\,; \tag{2.1}$$

(ii)

$$|\alpha| \int\int_{supp\,u} \Big[\sum_{i,j=1}^{n} a_{ij}\partial_{x_i}u\,\partial_{x_j}u\Big] f(|\nabla u|)\,u^{\alpha-1}\psi + \int\int_{supp\,u} c\,u^{q+\alpha}\psi \leq$$

$$\leq \int\int_{supp\,u} \Big[\sum_{i,j=1}^{n} a_{ij}\partial_{x_i}u\,\partial_{x_j}\psi\Big] f(|\nabla u|)\,u^\alpha - \frac{k}{k+\alpha}\int\int_{supp\,u} \rho\,u^{k+\alpha}\partial_t\psi\,. \tag{2.2}$$

A solution of class P_α to (1.1) is said to be global, if it is such a solution in S_T for any $T > 0$.

It is easy to prove that, due to Assumption (H), to condition (2.1) and to the assumption $\alpha > -k$, every integral in inequality (2.2) is finite. Hence Definition 2.2 is well posed.

Concerning the relationship between the above definitions, the following result is easily proved.

PROPOSITION 2.3. *Let u be a strong solution to inequality (1.1) in S_T, such that the pointwise limit $u(\cdot,0) := \lim_{t\to 0} u(\cdot,t)$ is defined and continuous in $I\!R^n$; let condition (2.1) be satisfied. Then u is a solution of class P_α $(\alpha \in (-k,0))$.*

3. Results

Let us introduce the following quantities:

$$D = D(x,t,u) := \Big(\frac{A_1^p}{A_0^{p-1}c^{\frac{\mu-1}{\mu}}}\Big)^\mu\,, \qquad E = E(x,t,u) := \Big(\frac{\rho}{c^{\frac{\nu-1}{\nu}}}\Big)^\nu\,, \tag{3.1}$$

where

$$\mu := \frac{q+\alpha}{q-p+1}\,, \qquad \nu := \frac{q+\alpha}{q-k} \qquad (\alpha < 0)\,; \tag{3.2}$$

here $p = 2$ if condition (i), respectively $p > 2$ if condition (ii) of Assumption (H)-(d) holds.

Our main nonexistence result can be stated as follows.

THEOREM 3.1. *Let $k > 0$, $p \geq 2$, $q > \max\{p-1, k\}$ and Assumption (H) be satisfied. Assume that for some $\alpha \in (-\min\{p-1,k\}, 0)$ there exists $\lambda > 0$ such that:*

$$R^{n+\frac{2}{\lambda}-p\mu} \int\int_{\{1\leq\xi_\lambda\leq 2\}} \Big[\sup_{u\geq 0} D(R\xi,\, R^{\frac{2}{\lambda}}\tau, u)\Big]\,d\xi\,d\tau \longrightarrow 0 \quad as\ R\to\infty\,, \tag{3.3}$$

$$R^{n+\frac{2}{\lambda}-\frac{2\nu}{\lambda}} \int\int_{\{1\leq\xi_\lambda\leq 2\}} \Big[\sup_{u\geq 0} E(R\xi,\, R^{\frac{2}{\lambda}}\tau, u)\Big]\,d\xi\,d\tau \longrightarrow 0 \quad as\ R\to\infty\,, \tag{3.4}$$

where

$$\xi_\lambda := |\xi|^{2\eta} + |\tau|^{\lambda\eta}, \quad \eta := \max\{\frac{1}{\lambda}, 1\} \qquad (\xi \in I\!R^n, \tau > 0; \lambda > 0)\,. \tag{3.5}$$

Then the only global solution of class P_α to inequality (1.1) is trivial.

In the proof of Theorem 3.1 the following lemma plays a major role. We refer the reader to [5] for its lengthy but elementary proof (the main tool being Hölder inequality).

Lemma 3.2. *Let $k > 0$, $p \geq 2$, $q > \max\{p-1, k\}$ and Assumption (H) be satisfied. Let u be a solution of class P_α to (1.1) in S_T ($\alpha \in (-\min\{p-1,k\}, 0)$). Then there exist $k_1 > 0$, $k_2 > 0$ (depending on k, p, q, α, f) such that*

$$\frac{|\alpha|}{p} \int\int_{\operatorname{supp} u} A_0 \,|\nabla u|^p\, u^{\alpha-1}\psi + \frac{1}{\mu\nu}\int\int_{\operatorname{supp} u} c\, u^{q+\alpha}\psi \leq$$

$$\leq k_1 \int\int_{\operatorname{supp} u} D\, \frac{|\nabla\psi|^{p\mu}}{\psi^{p\mu-1}} + k_2 \int\int_{\operatorname{supp} u} E\, \frac{|\partial_t\psi|^{\nu}}{\psi^{\nu-1}} \tag{3.6}$$

for any test function $\psi \geq 0$ with support in $\bar{S}_T$.

The main idea of the proof of Theorem 3.1 can now be described as follows: first we prove suitable a priori estimates for solutions of class P_α to (1.1) (see inequality (3.6)); then we combine the above estimates with a scaling argument to complete the proof.

Let us mention some applications of the above result.

Theorem 3.3. *Let the following condition:*

$$1 < q < 1 + \frac{2}{n} \tag{3.7}$$

be satisfied. Then there exists $\tilde{\alpha} \in (-1, 0)$ such that for any $\alpha \in (\tilde{\alpha}, 0)$ the only global solution of class P_α to the inequality:

$$\partial_t u \geq \sum_{i=1}^{n} \partial_{x_i}\left[\frac{1}{(1+|\nabla u|^2)^\theta}\,\partial_{x_i} u\right] + u^q \tag{3.8}$$

in S ($\theta \geq 0$) is trivial.

It is worth observing that condition (3.7) coincides with the well-known Fujita condition for the semilinear Cauchy parabolic problem:

$$\begin{cases} \partial_t u = \Delta u + u^q & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = u_0 \quad \text{in } \mathbb{R}^n \times \{0\}\,, \end{cases} \tag{3.9}$$

where u_0 is nonnegative, continuous and bounded in $\mathbb{R}^n$ (see [1]).

Theorem 3.4. *Let $m \geq 1$ and there hold:*

$$m < q < m + \frac{2}{n}\,. \tag{3.10}$$

Then there exists $\tilde{\alpha} \in (-\frac{1}{m}, 0)$ such that for any $\alpha \in (\tilde{\alpha}, 0)$ the only global solution of class P_α to the inequality:

$$\partial_t u \geq \Delta(u^m) + u^q \tag{3.11}$$

in S is trivial.

Theorem 3.5. *Let $p \geq 2$ and there hold:*

$$p - 1 < q < p - 1 + \frac{p}{n}\,. \tag{3.12}$$

Then there exists $\bar{\alpha} \in (-1, 0)$ such that for any $\alpha \in (\bar{\alpha}, 0)$ the only global solution of class P_α to the inequality:

$$\partial_t u \geq \sum_{i=1}^{n} \partial_{x_i} \Big[\mid \nabla u \mid^{p-2} \partial_{x_i} u \Big] + u^q \tag{3.13}$$

in S is trivial.

It can be observed that both conditions (3.10) and (3.12) reduce to the Fujita condition (3.7) when $m = 1$, respectively when $p = 2$. The same condition (3.7) determines the critical exponent for inequality (3.8) for any $\theta \geq 0$ (see Theorem 3.3 above).

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