LOCAL SOLVABILITY OF FULLY NONLINEAR PARABOLIC PROBLEMS OF HIGHER ORDER

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The paper is devoted to reduction of fully nonlinear parabolic problems of high order to operator equations involving operator satisfying (S_+) condition. The topological methods could be used to investigate solvability of such operator equations. The theorems of uniqueness and local existence for solution of boundary value problem, proved by topological approach are formulated in the paper. The results formulated are generalizations of analogous facts proved in [1].

Let $n \geq 2$, m, $\{m_j\}_{j=1}^m$ be positive integers, $0 \leq m_j \leq 2m-1$, $j = \overline{1,m}$ and $T > 0$, $p \geq 2$ be real numbers. For bounded domain $\Omega \in R^n$ with boundary $\partial\Omega$ we denote $Q_T := \Omega \times (0,T)$, $S_T := \partial\Omega \times (0,T)$ and consider the following boundary value problem:

$$\frac{\partial u}{\partial t} - F(x,t,u,D^1u,\dots D^{2m}u) = f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

$$G_j(x,t,u,\dots D^{m_j}u) = g_j(x,t), \quad (x,t) \in S_T, \quad j = \overline{1,m}, \tag{2}$$

$$u \in W_p^{(4m,2),0}(Q_T). \tag{3}$$

Functions F, G_j in problem (1) - (3) are supposed to be fully non-linear. Notification $D^k u$ means all possible partial derivatives $D^\alpha u$ of order k by variables $x \in \Omega$ where α is multiindice.

For positive integer k anysotropic space $W_p^{(2mk,k)}(Q_T)$ is the Banach space of all real functions $u(x,t)$ on Q_T that have generalized derivatives $\left(\frac{\partial}{\partial t}\right)^s D^\alpha u \in L_p(Q_T)$, where $|\alpha| + 2ms \leq 2mk$, with the norm

$$\|u\|_{p,Q_T}^{(2mk,k)} := \left(\sum_{|\alpha|+2ms\leq 2mk} \left\| \left(\frac{\partial}{\partial t}\right)^s D^\alpha u \right\|_{p,Q_T}^p \right)^{\frac{1}{p}},$$

$$\|u\|_{p,Q_T} := \left(\int_{Q_T} |u(x,t)|^p dx dt \right)^{\frac{1}{p}}.$$

For the boundary $\partial\Omega$ we assume the inclusion

$$\partial\Omega \in C^{4m} \tag{4}$$

is satisfied. In other words, it is possible to choose a finite collection $\{U_i\}_{i=1}^I$ of open sets and $d > 0$ with properties

S_1) $\partial\Omega \subset \bigcup_{i=1}^{I} U_i$;

S_2) for each $i = 1 \ldots I$ there exists $\xi^{(i)} \in \partial\Omega \bigcap U_i$ such that in the local Cartesian coordinate system $\{y\}$ with origin at $\xi^{(i)}$ the set $\partial\Omega \bigcap U_i$ is given by the equation

$$y_n = h_i(y'),\ y' = (y_1, y_2 \ldots y_{n-1}) \in D(d) := (-d, d)^{n-1};$$

S_3) for each $i = 1 \ldots I \quad h_i \in C^{4m}(D(d))$.

Let us denote by $\phi_i(y)$ the transformation from the local coordinate system $\{y\}$ to coordinate system $\{x\}$ where $i \in 1 \ldots I$. Then we can introduce the notifications $u^{(i)}(y', t) = u(\phi_i(y', h_i(y')), t),\ (y', t) \in D_T(d) := D(d) \times (0, T),\ i = \overline{1, I}$.

For non-integer $k > 0$ such that $2mk$ is not integer too we define the space $W_p^{(2mk,k)}(S_T)$ as the space of all real functions $u(x,t)$ on S_T such that for every $i = 1 \ldots I$ the functions $u^{(i)}(y', t)$ have generalized derivatives $\left(\frac{\partial}{\partial t}\right)^s D_{y'}^{\alpha} u^{(i)} \in L_p(S_T)$, $|\alpha| + 2ms < 2mk$ (where $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_{n-1})$ and $D_{y'}^{\alpha}$ means differentiation by variables y') and finite norm

$$\|u\|_{p,S_T}^{(2mk,k)} := \sum_{i=1}^{I} \|u^{(i)}\|_{p,D_T(d)}^{(2mk,k)},$$

$$\|v\|_{p,D_T(d)}^{(2mk,k)} := \left(\sum_{|\alpha|+2ms<2mk} \left\| \left(\frac{\partial}{\partial t}\right)^s D_{y'}^{\alpha} v \right\|_{p,D_T(d)}^p + \right.$$

$$+ \sum_{|\alpha|+2ms=[2mk]} \int_0^T dt \int_{D(d)} \int_{D(d)} \frac{\left| \left(\frac{\partial}{\partial t}\right)^s D_{y'}^{\alpha} v(y', t) - \left(\frac{\partial}{\partial t}\right)^s D_{z'}^{\alpha} v(z', t) \right|^p}{|y' - z'|^{(2mk - [2mk])p + n - 1}} dy' dz' +$$

$$\left. + \sum_{0 < 2mk - |\alpha| - 2ms < 2m} \int_{D(d)} dy' \int_0^T \int_0^T \frac{\left| \left(\frac{\partial}{\partial t}\right)^s D_{y'}^{\alpha} u(y', t) - \left(\frac{\partial}{\partial \tau}\right)^s D_{y'}^{\alpha} v(y', \tau) \right|^p}{|t - \tau|^{\frac{2mk - |\alpha| - 2ms}{2m} p + 1}} dt d\tau \right)^{\frac{1}{p}}$$

It's not difficult to show that norms of the space $W_p^{(2mk,k)}(S_T)$ corresponding to different covers $\{U_i\}$ are actually equivalent.

We define $W_p^{(2mk,k),0}(Q_T)$ as the subspace of $W_p^{(2mk,k),0}(Q_T)$ consisting of all functions u , which satisfy zero initial conditions

$$\lim_{t \to 0} \frac{\partial^l}{\partial t^l} u(x, t) = 0, \quad x \in \Omega \quad 0 \le l \le k - \frac{1}{p}.$$

The spaces $W_p^{(2mk,k),0}(S_T)$ are defined analogously.

We'll assume that numbers $n, m, \{m_j\}_{j=1}^m, p$ satisfy inequalities

$$p > \frac{2m+n}{2m}, \quad p \neq \frac{2m+1}{2m - m_j}, \quad j = \overline{1, m} \tag{5}$$

and the following inclusions for functions on the right side of the problem hold

$$f \in W_p^{(2m,1),0}(Q_T),\ g_j \in W_p^{\left(4m-m_j-1,2-\frac{m_j}{2m}-\frac{1}{2mp}\right),0}(S_T),\ j=\overline{1,m}. \tag{6}$$

Let us define $M(k)$ to be an amount of all different multiindices α such that $|\alpha| \le k$. We'll use the following notations:

$$F_\alpha(x,t,\xi) := \frac{\partial}{\partial \xi_\alpha} F(x,t,\xi), \quad \xi = \{\xi_\alpha \in R : |\alpha| \le 2m\} \in R^{M(2m)},$$

$$G_{j\beta}(x,t,\zeta) := \frac{\partial}{\partial \zeta_\beta} G_j(x,t,\zeta), \quad \zeta = \{\zeta_\beta \in R : |\beta| \le m_j\} \in R^{M(m_j)}, \quad j=\overline{1,m}$$

We assume that functions on the left side of the problem (1) – (3) satisfy the following conditions:

F1) Function $F(x,t,\xi)$ has all continuous partial derivatives by ξ_α up to the order $2m+1$, $F(\cdot,\cdot,0) \equiv 0$;

F2) There exists such $\nu \in C(\overline{R^+}, R^+)$ that for each $\xi \in R^{M(2m)}, \eta \in R^n$ the inequality

$$(-1)^{m+1} \sum_{|\alpha|=2m} F_\alpha(x,t,\xi)\eta^\alpha \ge \nu(|\xi|)|\eta|^{2m};$$

holds;

F3) Operators

$$\widetilde{F}_\alpha(u) := F_\alpha(\cdot,\cdot,u,D^1u,\dots D^{2m}u) : W_p^{(4m,2),0}(Q_T) \to W_p^{(2m,1)}(Q_T)$$

are bounded and continuous;

G1) For each $j = 1\dots m$ function $G_j(x,t,\zeta)$ has all continuous partial derivatives by ζ_β up to the order $4m-m_j+1$, $G_j(\cdot,\cdot,0) \equiv 0$;

Let $x \in \partial\Omega,\ t \in (0,T)$, $\xi \in R^{M(2m)}$, $\zeta = \{\xi_\alpha : |\alpha| \le m_j\}$, η be the unit vector in the direction of outward normal to $\partial\Omega$ at point x and δ belong to tangent plane to $\partial\Omega$ at point x. For complex τ and real q we define

$$L(x,t,\xi,\delta+\tau\eta,q) := q - (-1)^m \sum_{|\alpha|=2m} F_\alpha(x,t,\xi)(\delta+\tau\eta)^\alpha,$$

$$B_j(x,t,\zeta,\delta+\tau\eta) := \sum_{|\beta|=m_j} G_{j\beta}(x,t,\zeta)(\delta+\tau\eta)^\beta, \quad j=\overline{1,m}.$$

If $q \ge -\tilde{\nu}|\delta|^{2m},\ 0 < \tilde{\nu} < \nu(|\xi|)$ and $|q|+|\delta| > 0$, then $L(x,t,\xi,\delta+\tau\eta,q)$ has m roots τ_s^+ with positive real parts as polinimial of τ (the other roots are with negative real parts) [3]. Let

$$L^+(x,t,\xi,\delta,\tau,q) := \prod_{s=1}^{m} (\tau - \tau_s^+).$$

We assume that the following condition is satisfied (Lopatinsky condition):

G2) For each $(x,t) \in S_T$, $\xi \in R^{M(2m)}$ and δ, from the tangent plane to $\partial\Omega$ at point x polynomials B_j of τ are linear independent by the module of polynomial L^+ of τ provided that inequalities $q \geq -\tilde{\nu}|\delta|^{2m}$, $0 < \tilde{\nu} < \nu(|\xi|)$ hold;

G3) Operators

$$\widetilde{G}_{j\beta}(u) := G_{j\beta}(\cdot,\cdot,u,\dots D^{m_j}u) : W_p^{(4m-\frac{1}{p},2-\frac{1}{2mp}),0}(S_T) \to W_p^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp})}(S_T)$$

are bounded and continuous;

We also assume that compatibility conditions for problem (1) – (3) are satisfied. It means that

C) For each $j = 1,\dots m$ $g_j(x,0) = 0$ and the equality

$$\sum_{|\beta|\leq m_j} G_{j\beta}(x,0,0)D^\beta f(x,0) = \lim_{t\to 0}\frac{\partial}{\partial t}g_j(x,t),$$

is fulfilled for such j that $p > \frac{2m+1}{2m-m_j}$.

Now we start with reduction of the boundary value problem (1) – (3) to operator equation. To this end we define auxiliary operators

$$L(u)\phi := \sum_{|\alpha|\leq 2m} F_\alpha(x,t,u,D^1u,\dots D^{2m}u)D^\alpha\phi, \quad (x,t)\in Q_T,$$

$$B_j(u)\phi := \sum_{|\beta|\leq m_j} G_{j\beta}(x,t,u,\dots D^{m_j}u)D^\beta\phi,\ (x,t)\in S_T,\ j=\overline{1,m}$$

for $\{u,\phi\} \subset W_p^{(4m,2),0}(Q_T)$.

Let $\psi_p[s] := s|s|^{p-2}$ and $\mu_j := 4m - m_j - 1$. Operator $A : W_p^{(4m,2),0}(Q_T) \to \left[W_p^{(4m,2),0}(Q_T)\right]^*$ corresponding to problem (1) – (3) will be defined by the following equalities:

$$\begin{aligned} Au :=& A_E\left(u, \frac{\partial u}{\partial t} - F(x,t,u,D^1u,\dots D^{2m}u) - f(x,t)\right) + \\ &+ \sum_{j=1}^{m} A_{B_j}\left(u, G_j(x,t,u,\dots D^{m_j}u) - g_j(x,t)\right), \end{aligned} \tag{7}$$

where

$$\begin{aligned} < A_E(u,v),\phi > :=& \sum_{|\alpha|+2ms\leq 2m}\int_{Q_T}\psi_p\left[\left(\frac{\partial}{\partial t}\right)^s D^\alpha v(x,t)\right]\cdot \\ &\cdot\left(\frac{\partial}{\partial t}\right)^s D^\alpha\left[\frac{\partial\phi}{\partial t} - L(u)\phi(x,t)\right]dxdt, \end{aligned} \tag{8}$$

and for fixed $j \in 1,\dots m$

$$< A_{B_j}(u,w),\phi > := \sum_{k=1}^{3} < A_{B_j}^{(k)}(u,w),\phi >, \tag{9}$$

$$<A^{(1)}_{B_j}(u,w),\phi>:=\sum_{i=1}^{I}\sum_{|\beta|+2ms\le\mu(j)}\int_{D_T(d)}\cdot$$
$$\cdot\psi_p\left[\left(\frac{\partial}{\partial t}\right)^s D^{\beta}_{y'}w^{(i)}(y',t)\right]\left(\frac{\partial}{\partial t}\right)^s D^{\beta}_{y'}[B_j(u)\phi]^{(i)}(y',t)dy'dt, \tag{10}$$

$$<A^{(2)}_{B_j}(u,w),\phi>:=\sum_{i=1}^{I}\sum_{|\beta|+2ms=\mu(j)}\int_0^T dt\int\int_{[D(d)]^2}\cdot$$
$$\cdot\psi_p\left[\left(\frac{\partial}{\partial t}\right)^s D^{\beta}_{y'}w^{(i)}(y',t)-\left(\frac{\partial}{\partial t}\right)^s D^{\beta}_{z'}w^{(i)}(z',t)\right]\cdot$$
$$\cdot\left\{\left(\frac{\partial}{\partial t}\right)^s D^{\beta}_{y'}[B_j(u)\phi]^{(i)}(y',t)-\left(\frac{\partial}{\partial t}\right)^s D^{\beta}_{z'}[B_j(u)\phi]^{(i)}(z',t)\right\}\cdot \tag{11}$$
$$\cdot\frac{dy'dz'}{|y'-z'|^{n+p-2}},$$

$$<A^{(3)}_{B_j}(u,w),\phi>:=\sum_{i=1}^{I}\sum_{l=2m-m_j}^{\mu(j)}\sum_{|\beta|+2ms=l}\int_{D(d)}dy'\cdot$$
$$\cdot\int_0^T\int_0^T\psi_p\left[\left(\frac{\partial}{\partial t}\right)^s D^{\beta}_{y'}w^{(i)}(y',t)-\left(\frac{\partial}{\partial \tau}\right)^s D^{\beta}_{y'}w^{(i)}(y',\tau)\right]\cdot$$
$$\cdot\left\{\left(\frac{\partial}{\partial t}\right)^s D^{\beta}_{y'}[B_j(u)\phi]^{(i)}(y',t)-\right. \tag{12}$$
$$\left.-\left(\frac{\partial}{\partial \tau}\right)^s D^{\beta}_{y'}[B_j(u)\phi]^{(i)}(y',\tau)\right\}\frac{dtd\tau}{|t-\tau|^{p(l)}},$$

where $p(l)=1+(2-\frac{m_j+l}{2m}-\frac{1}{2mp})p,\ l=\overline{2m-m_j,\mu(j)}$.

The following theorem tells us about properties of operator, defined by formulas (7)–(12).

THEOREM 1. *Assume that conditions* (4)–(6), F1)– F3), G1)– G3) *and* C) *are satisfied. Then*

(1) *Operator* A, *defined by* (7)–(12), *acts from the space* $W_p^{(4m,2),0}(Q_T)$ *to adjoint one;*
(2) *For* $p\ge 2$ *operator* A *is bounded, continuous and satisfies* (S_+) *condition on* $W_p^{(4m,2),0}(Q_T)$.

According to [2], the definition of (S_+) condition is the following:

(S_+) CONDITION. Let X be reflexive Banach space and $H: X\to X^*$. We say that H satisfies (S_+) condition on X, if for each sequence $\{u_k\}_{k=1}^{\infty}\subset X$, from the $u_k\rightharpoonup u_0\in X$, $\overline{\lim}_{k\to\infty}<Hu_k,u_k-u_0>\le 0$ it follows that $u_k\to u_0$.

We can investigate the operator equation
$$Au=0,\quad u\in W_p^{(4m,2),0}(Q_T) \tag{13}$$
together with problem (1) – (3). Next theorem tells us about the connection between problem (1) – (3) and equation (13).

THEOREM 2. *Assume that conditions of theorem* 1 *are satisfied. Then function* u *is the solutuion of the problem* (1)–(3) *if and only if it is the solution of the equation* (13).

Now we can investigate equation (13) instead of the problem (1) – (3). On this way we can use topological methods, developed in [1]. As a result, the following uniqueness and local existence theorems for the problem (1) – (3) are proved:

THEOREM 3. *Assume that conditions of theorem* 1 *are satisfied. Then the problem* (1)–(3) *can have at most one solution.*

THEOREM 4. *Assume that conditions of theorem* 1 *are satisfied and* $p \geq 2$. *Then for every* $K > 0$ *there exists* $\tau > 0$ *dependent only on* K, Ω, F, G_j *such that there exists a solution* $u \in W_p^{(4m,2),0}(Q_\tau)$ *of the problem* (1)–(3) *in cylinder* Q_τ *provided that the inequalities*

$$\|f\|_{p,Q_\tau}^{(2m,1)} \leq K, \quad \|g_j\|_{p,S_\tau}^{\left(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp}\right)} \leq K$$

hold.

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