ON THE DIRICHLET PROBLEM FOR A QUASILINEAR ELLIPTIC SECOND ORDER EQUATION WITH TRIPLE DEGENERACY AND SINGULARITY IN A DOMAIN WITH EDGE ON THE BOUNDARY

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Abstract. Boundedness and Hölder continuity of weak solutions have been proved for one quasilinear elliptic second order equation with triple degeneracy and singularity of the form

$$-\frac{d}{dx_i}\left(r^{\tau}|u|^q|\nabla u|^{m-2}u_{x_i}\right)+a_0r^{\tau-m}u|u|^{q+m-2}-\mu r^{\tau}u|u|^{q-2}|\nabla u|^m=f_0(x)-\frac{\partial f_i}{\partial x_i},$$
$$a_0\geq 0,\quad q\geq 0,\quad 0\leq\mu<1,\quad m>1,\quad \tau\geq m-2.$$

in the domain with edge on the boundary. There has been obtained the exact Hölder exponent near the edge.

In the present work we investigate the behaviour of weak solutions of the first boundary value problem for a quasilinear elliptic second order equation with triple degeneracy and singularity near the edge of the domain boundary. Namely, we shall derive the exact estimate of weak solutions in a neighborhood of edge of the domain boundary for the Dirichlet problem

$$\begin{cases} -\dfrac{d}{dx_i}\left(r^{\tau}|u|^q|\nabla u|^{m-2}u_{x_i}\right)+a_0r^{\tau-m}u|u|^{q+m-2}-\mu r^{\tau}u|u|^{q-2}|\nabla u|^m= \\ \qquad\qquad = f_0(x)-\frac{\partial f_i}{\partial x_i},\quad x\in G, & (1)\\ u(x)=0,\quad x\in\partial G; & (2)\\ a_0\geq 0,\ q\geq 0,\ 0\leq\mu<1,\ m>1,\ \tau\geq m-2 & (3)\end{cases}$$

(summation over repeated indices from 1 to n is understood), where G is $n-$ dimensional bounded domain the boundary of which contains an edge. Similar investigation has been lead by us early in [10] for a domain with conical boundary point. The special structure of the solution near an edge is of particular interest for physical applications (see [2, 8]). It can be used also to improve numerical algorithms (see [1, 3, 9]).

For $x=(x_1,\dots,x_n)$ let us define cylindrical coordinates $(\overline{x},r,\omega)$:

$$\overline{x}=(x_1,\dots,x_{n-2}),\quad r=\sqrt{x_{n-1}^2+x_n^2},\quad \omega=\operatorname{arcctg}\frac{x_{n-1}}{x_n}.$$

Let G be a domain in $\mathbb{R}^n$ bounded by $(n-1)-$ dimensional manifold ∂G that possesses the following properties:

1) ∂G contains smooth $(n-2)$-dimensional submanifold Γ without boundary, a neighborhood of every point of which is locally diffeomorphic to the dihedral cone $\mathbb{D} = \{(r,\omega)\big|0 < r < \infty, \quad \omega \in (-\omega_0/2, \omega_0/2)\} \times \mathbb{R}^{n-2}; \ 0 < \omega_0 < 2\pi$.
2) $\partial G \setminus \Gamma$ - smooth submanifold of $\mathbb{R}^n$.

Without loss of generality we can suppose that there exists the number $d > 0$ such that on Γ it can be selected the edge $\Gamma_0^d = \{(\overline{x}, 0, 0)\big| |\overline{x}| \le d\}$ with the centre in the origin. We define $G_0^d = G \cap \{(\overline{x}, r, \omega)\big| \overline{x} \in \mathbb{R}^{n-2}, \quad 0 < r < d, \quad \omega \in (-\omega_0/2, \omega_0/2)\}; \quad 0 < \omega_0 < 2\pi$. Thus we assume the $G_0^d \subset G$ and consequently the domain G to be a wedge in some neighborhood of the edge.

Let $L_p(G)$ be the usual Lebesgue's space. For real τ we define the space $V^k_{m,\tau}(G)$ as the closure of $C_0^\infty(\overline{G} \setminus \Gamma)$ with respect to the norm

$$\| u \|_{V^k_{m,\tau}(G)} = \left(\int\limits_G \sum_{|\beta|=0}^{k} r^{m(|\beta|-k)+\tau} |D^\beta u|^m dx \right)^{\frac{1}{m}}.$$

By $\mathfrak{N}^1_{m,\tau,q}(G)$ we will denote the set of functions $u(x) \in L_\infty(G)$, having first generalized derivatives with finite integral

$$\int\limits_G \left(r^\tau |u|^q |\nabla u|^m + r^{\tau-m} |u|^{q+m} \right) dx < \infty, \quad q \ge 0, \ \tau \ge m-2, \ m > 1.$$

and vanishing on ∂G in the sense of traces.

Definition. Function $u(x)$ is called a *weak solution* of (1)–(2), if $u(x) \in \mathfrak{N}^1_{m,\tau,q}(G)$ and satisfies

$$\int\limits_G \Big\{ r^\tau |u|^q |\nabla u|^{m-2} u_{x_i} \phi_{x_i} + a_0 r^{\tau-m} u|u|^{q+m-2}\phi -$$

$$- \mu r^\tau u|u|^{q-2} |\nabla u|^m \phi \Big\} dx = \int\limits_G \{ f_0(x)\phi + f_i \phi_{x_i} \} \, dx$$

for any function $\phi(x) \in L_\infty(G)$, vanishing on ∂G in the sense of traces and such that integrals at both sides are finite.

Assumptions: $f_0(x), f_1(x), \ldots, f_n(x)$ are measurable functions such that

$$f_0(x) \in L_p(G), \quad f_i(x) \in V^0_{\frac{mp}{m-1}, -\frac{\tau p}{m-1}}(G), \quad (i = 1, \ldots, n) \tag{4}$$

$$\frac{1}{p} < \frac{m}{n} - \frac{1}{t}, \quad \frac{1}{t} < \frac{m}{n} < 1 + \frac{1}{t} < m, \quad m - 2 \le \tau < min\left(m-1; \frac{2}{t}\right). \tag{5}$$

We prove the following assertions:

THEOREM 1. *Let* $u(x)$ *be a weak solution of* $(1)-(5)$. *Then there exists the constant* $M_0 > 0$ *depending only on* $mes\, G$, n, m, τ, μ, q, a_0, $\|f_0(x)\|_{L_p(G)}$, $\left\|r^{-\frac{\tau}{m}}|f(x)|\right\|_{L_{\frac{mp}{m-1}}(G)}$ *such that* $\|u\|_{L^\infty(G)} \le M_0$. *If in addition*

$$f_0(x) \in V^0_{p,-\tau p}(G),\ f_i(x) \in V^0_{p,-(1+\tau)p}(G) \cap V^0_{\frac{mp}{m-1},-\frac{\tau mp}{m-1}}(G), \quad (i = 1,\dots,n),$$

then $u(x)$ *is Hölder continuous in* G *with exponent* α, *depending only on the data of the assumptions and the domain* G.

For the proof of theorem 1 we use the u level sets technique (see §9 chapt. II [7]), well-known statement by Stampacchia (see Lemma B.1 [6]) and the appropriate imbedding theorems (see [4, 5]).

THEOREM 2. *Let* $u(x)$ *be a weak solution of* $(1)-(5)$ *with* $m \ge 2$. *Given*

$$f_0(x) \in L_p(G), \quad f_i(x) \in L^0_{\frac{mp}{m-1}}(G) \cap V^0_{p,-p}(G), \quad (i = 1,\dots,n)$$

and suppose in addition that there exists constant $k_1 \ge 0$ *such that*

$$\left|\frac{\partial f_i}{\partial x_i} - f_0(x)\right| \le k_1 r^\beta, \tag{6}$$

where

$$\beta > \lambda(q+m-1) - m + \tau; \tag{7}$$

$\lambda > 0$ *is a solution of the equation*

$$\int_0^{+\infty} \frac{\left[(m-1)y^2+\lambda^2\right](y^2+\lambda^2)^{\frac{m-4}{2}}dy}{(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m+\tau)(y^2+\lambda^2)^{\frac{m-2}{2}} - a_0} = \frac{\omega_0}{2}, \tag{8}$$

where $\omega_0 \in (0, 2\pi)$ *is the opening angle of the dihedral cone, such that:*

$$\lambda^m(q+m-1+\mu) + \lambda^{m-1}(2-m+\tau) > a_0. \tag{9}$$

Then given $\varepsilon > 0$ *there is a constant* $c_\varepsilon > 0$ *depending only on* ε, ω_0, n, m, μ, q, a_0, λ, k_1, M_0, $\|f_i(x)\|$, $i = 0, 1, \dots, n$ *in the corresponding spaces such that*

$$|u(x)| \le c_\varepsilon r^{\lambda-\varepsilon}.$$

The theorem 2 is proved by using the barrier functions technique (see [5]) and weak comparison principle (see chapt 10 [5]) together with the theorem 1.

The construction of barrier function. We construct the barrier function as

$$u = r^\lambda \Phi(\omega), \quad \omega \in \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right], \quad 0 < \omega_0 < 2\pi, \quad \lambda > 0 \tag{10}$$

for the homogeneous Dirichlet problem (1) - (3)

$$\begin{cases} \mathfrak{L}u \equiv \dfrac{d}{dx_i}\left(r^{\tau}|u|^q|\nabla u|^{m-2}u_{x_i}\right) = & (11)\\ \qquad\qquad = a_0 r^{\tau-m}u|u|^{q+m-2} - \mu r^{\tau}u|u|^{q-2}|\nabla u|^m, \quad x \in G_0, & \\ u(x) = 0, \quad x \in \Gamma_0 \cup \Gamma^+ \cup \Gamma^- & (12)\\ a_0 \geq 0,\ 0 \leq \mu < 1,\ q \geq 0,\ m \geq 2,\ \tau \geq m-2 & (13) \end{cases}$$

in $n-$ dimensional *infinite* dihedral cone G_0 with the edge $\Gamma_0 = \left\{(\overline{x},0,0)\big|\ |\overline{x}| \in \mathbb{R}^{n-2}\right\}$ that contains the origin and lateral faces $\Gamma^{\pm}$:

$$\Gamma^+ = \left\{(\overline{x},r,+\frac{\omega_0}{2})\big|\ \overline{x} \in \mathbb{R}^{n-2},\ 0 < r < \infty\right\};$$
$$\Gamma^- = \left\{(\overline{x},r,-\frac{\omega_0}{2})\big|\ \overline{x} \in \mathbb{R}^{n-2},\ 0 < r < \infty\right\}.$$

Then $\Phi(\omega)$ satisfies the equation:

$$\begin{cases} -\dfrac{d}{d\omega}\left[\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-2}{2}}|\Phi|^q\Phi'\right] + a_0\Phi|\Phi|^{q+m-2} - \mu\Phi|\Phi|^{q-2}\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m}{2}} = \\ \quad = \lambda[\lambda(q+m-1) - m + 2 + \tau]\Phi|\Phi|^q\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-2}{2}}, \quad \omega \in (-\omega_0/2, \omega_0/2), \\ \Phi(-\omega_0/2) = \Phi(\omega_0/2) = 0. \end{cases}$$

(14)

(15)

By setting $\Phi'/\Phi = y$ we arrive at:

$$\begin{cases} \left[(m-1)y^2 + \lambda^2\right](y^2+\lambda^2)^{\frac{m-4}{2}}y' + (m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + & (16)\\ \qquad +\lambda(2-m+\tau)(y^2+\lambda^2)^{\frac{m-2}{2}} = a_0,\ \omega \in (-\omega_0/2, \omega_0/2), & \\ y(0) = 0, \quad \lim_{\omega \to \frac{\omega_0}{2}-0} y(\omega) = -\infty. & (17) \end{cases}$$

We explain (17). In fact, from (14)-(15) it follows easily that $\Phi(-\omega) = \Phi(\omega), \omega \in [-\omega_0/2, \omega_0/2]$, and from this $y(-\omega) = -y(\omega), \omega \in [-\omega_0/2, \omega_0/2]$. Consequently, we have $y(0) = 0$. Further, from (16) we obtain:

$$\begin{aligned} -\left[(m-1)y^2 + \lambda^2\right](y^2+\lambda^2)^{\frac{m-4}{2}}y' &= \\ &= (m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m+\tau)(y^2+\lambda^2)^{\frac{m-2}{2}} - a_0 = \\ &= (y^2+\lambda^2)^{\frac{m-2}{2}}\left[(m-1+q+\mu)(y^2+\lambda^2) + \lambda(2-m+\tau)\right] - a_0 \geq \\ &\geq (y^2+\lambda^2)^{\frac{m-2}{2}}[\lambda^2(m-1+q+\mu) + \lambda(2-m+\tau)] - a_0 \geq \\ &\geq \lambda^m(m-1+q+\mu) + \lambda^{m-1}(2-m+\tau) - a_0 > 0 \quad (18) \end{aligned}$$

by (9) and (13). Thus it is proved that $y'(\omega) < 0, \omega \in [-\omega_0/2, \omega_0/2]$. Therefore $y(\omega)$ is decreasing function on $[-\omega_0/2, \omega_0/2]$. From this we conclude the last condition of (17).

Properties of the function $\Phi(\omega)$. We turn our attention to the properties of the function $\Phi(\omega)$. First of all, notice that the solutions of (14)-(15) are determined uniquely up to a scalar multiple provided that λ satisfies (9). We will consider the solution normed by the condition

$$\Phi(0) = 1. \tag{19}$$

We rewrite the (14) in the following form

$$\begin{aligned} &-\Phi\left[(m-1)\Phi'^2 + \lambda^2\Phi^2\right]\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-4}{2}}\Phi'' = \\ &= \Phi^2\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-4}{2}}\{\lambda[\lambda(m-1) - m + 2 + \tau]\left(\lambda^2\Phi^2 + \Phi'^2\right) + \\ &+ (m-2)\lambda^2\Phi'^2\} - a_0\Phi^m + (q+\mu)\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m}{2}}. \end{aligned} \tag{20}$$

Now, since $m \geq 2$, by virtue of (9) from the (20), it follows that

$$\begin{aligned} &-\Phi\left[(m-1)\Phi'^2 + \lambda^2\Phi^2\right]\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-4}{2}}\Phi'' \geq -a_0\Phi^m + \\ &+\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-2}{2}}\{(q+\mu)\left(\lambda^2\Phi^2 + \Phi'^2\right) + \lambda[\lambda(m-1) - m + 2 + \tau]\Phi^2\} \geq \\ &\geq \Phi^m\{(q+\mu+m-1)\lambda^m + (2-m+\tau)\lambda^{m-1} - a_0\} > 0 \end{aligned}$$

(here we take into account that by (9) $(q+\mu+m-1)\lambda^2 + (2-m+\tau)\lambda > 0$).

Summarizing the above we obtain the following:

$$\begin{gathered} \Phi(\omega) \geq 0 \quad \forall\omega \in [-\omega_0/2, \omega_0/2]; \qquad \Phi(-\omega_0/2) = \Phi(\omega_0/2) = 0; \\ \Phi(-\omega) = \Phi(\omega) \quad \forall\omega \in [-\omega_0/2, \omega_0/2]; \\ \Phi'(0) = 0; \quad \Phi''(\omega) < 0 \quad \forall\omega \in [-\omega_0/2, \omega_0/2]. \end{gathered} \tag{21}$$

COROLLARY 3.

$$\max_{[-\omega_0/2, \omega_0/2]} \Phi(\omega) = \Phi(0) = 1 \quad \Longrightarrow \quad 0 \leq \Phi(\omega) \leq 1 \quad \forall\omega \in [-\omega_0/2, \omega_0/2]. \tag{22}$$

Now we will solve the (14) - (17). Rewriting the (16) in the form $y' = g(y,)$ we observe that by (18), $g(y) \neq 0 \ \forall y \in \mathbb{R}$. Moreover, being rational functions with nonzero denominators $g(y)$ and $g'(y)$ are *continuous* functions. By the theory of ordinary differential equations the Cauchy's problem (16), (17) is uniquely solvable in the strip $\{(\omega, y)\} \subseteq \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right] \times (-\infty, +\infty)$. Integrating (14) - (17), we obtain

$$\begin{gathered} \Phi(\omega) = \exp\int_0^{\omega} y(\xi)d\xi, \\ \int_0^{-y} \frac{[(m-1)z^2 + \lambda^2](z^2+\lambda^2)^{\frac{m-4}{2}}dz}{(m-1+q+\mu)(z^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m+\tau)(z^2+\lambda^2)^{\frac{m-2}{2}} - a_0} = \omega. \end{gathered} \tag{23}$$

From this we get in particular (8). The expression (8) gives the equation for sharp finding of the exponent λ in (10). For the case $a_0 = 0$ this exponent is calculated explicitly in [10]; we denote this value by λ_0 :

$$\lambda_0 = \frac{\pi}{2\omega_0(m-1+q+\mu)}\left\{\frac{m(m-2)-2(m-2)t_0-t_0^2}{t_0+2(m-2)} + \right.$$
$$\left. + \frac{\sqrt{[t_0^2+2(m-2)t_0+m^2][t_0^2+2(m-2)t_0+(m-2)^2]}}{t_0+2(m-2)}\right\},$$

where $t_0 = \frac{\omega_0}{\pi}(2-m+\tau)$.

About the solutions of (8).

We set $$\mathfrak{F}(\lambda, a_0, \omega_0) \equiv -\frac{\omega_0}{2} + \tag{24}$$
$$+\int\limits_0^{+\infty} \frac{[(m-1)y^2+\lambda^2](y^2+\lambda^2)^{\frac{m-4}{2}}}{(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}}+\lambda(2-m+\tau)(y^2+\lambda^2)^{\frac{m-2}{2}}-a_0}dy.$$

Let us make a substitution $y = t\lambda$; $t \in (0, +\infty)$, we get:

$$\mathfrak{F}(\lambda, a_0, \omega_0) = -\frac{\omega_0}{2} + \int\limits_0^{+\infty} \Lambda(\lambda, a_0, t)dt, \tag{25}$$

$$\Lambda(\lambda, a_0, t) \equiv \frac{[(m-1)t^2+1](t^2+1)^{\frac{m-4}{2}}}{\lambda(m-1+q+\mu)(t^2+1)^{\frac{m}{2}}+(2-m+\tau)(t^2+1)^{\frac{m-2}{2}}-a_0/\lambda^{m-1}}. \tag{26}$$

Then the (8) takes the expression

$$\mathfrak{F}(\lambda, a_0, \omega_0) = 0. \tag{27}$$

According to what has been said above

$$\mathfrak{F}(\lambda_0, 0, \omega_0) = 0. \tag{28}$$

Direct calculations give:

$$\frac{\partial\Lambda}{\partial\lambda} < 0, \quad \frac{\partial\Lambda}{\partial a_0} > 0 \quad \forall t,\ \lambda,\ a_0. \tag{29}$$

Hence, we can apply the theorem of implicit functions: in a certain neighborhood of the point $(\lambda_0, 0)$ the (27) (and therefore and the (8)) determines $\lambda = \lambda(a_0, \omega_0)$ as single-valued *continuous* function of a_0, continuously depending on the parameter ω_0 and having continuous partial derivatives $\frac{\partial\lambda}{\partial a_0}, \frac{\partial\lambda}{\partial\omega_0}$. Applying the analytic continuation method, we obtain the solvability of the equation (8) for $\forall a_0$.

Now, we analyze the properties of λ as the function $\lambda(a_0, \omega_0)$. First, from (27) we get:

$$\frac{\partial \mathfrak{F}}{\partial \lambda}\frac{\partial \lambda}{\partial a_0} + \frac{\partial \mathfrak{F}}{\partial a_0} = 0, \quad \frac{\partial \mathfrak{F}}{\partial \lambda}\frac{\partial \lambda}{\partial \omega_0} + \frac{\partial \mathfrak{F}}{\partial \omega_0} = 0;$$

from this it follows that

$$\frac{\partial \lambda}{\partial a_0} = -\frac{\left(\frac{\partial \mathfrak{F}}{\partial a_0}\right)}{\left(\frac{\partial \mathfrak{F}}{\partial \lambda}\right)}; \quad \frac{\partial \lambda}{\partial \omega_0} = -\frac{\left(\frac{\partial \mathfrak{F}}{\partial \omega_0}\right)}{\left(\frac{\partial \mathfrak{F}}{\partial \lambda}\right)}. \tag{30}$$

But by virtue of (29) we have:

$$\frac{\partial \mathfrak{F}}{\partial a_0} = \int_0^{+\infty} \frac{\partial \Lambda}{\partial a_0} dy > 0, \ \frac{\partial \mathfrak{F}}{\partial \lambda} = \int_0^{+\infty} \frac{\partial \Lambda}{\partial \lambda} dy < 0, \ \frac{\partial \mathfrak{F}}{\partial \omega_0} = -\frac{1}{2} \quad \forall (\lambda, a_0). \tag{31}$$

From (30) - (31) we get:

$$\frac{\partial \lambda}{\partial a_0} > 0; \quad \frac{\partial \lambda}{\partial \omega_0} < 0 \qquad \forall a_0. \tag{32}$$

Thus we derive: *the function* $\lambda(a_0, \omega_0)$ ***increases*** *with respect to* a_0 *and* ***decreases*** *with respect to* ω_0 *.*
Now we prove following lemmas.

LEMMA 4.. *There takes place the inequality*

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^q |\Phi'|^m d\omega \le c(q, \mu, m, \lambda) \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+m} d\omega. \tag{33}$$

LEMMA 5. *Let inequalities (9), (13) hold and in addition*

$$q + \mu < 1. \tag{34}$$

Then we have

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi'|^m d\omega \le c(a_0, q, \mu, m, \lambda, \omega_0). \tag{35}$$

From (8) and (10) we get the function $w = r^\lambda \Phi(\omega)$ that will be *a barrier* of our boundary value problem $(1) - (3)$.

LEMMA 6. *Let* $\zeta(r) \in C_0^\infty[0, d]$. *Then the function* $\zeta(r)w(x) \in \mathfrak{N}^1_{m,\tau,q}(G_0^d)$. *If (9) and (34) hold, then* $\zeta(r)w(x) \in V^1_{m,\tau}(G_0^d) \cap L_\infty(G_0^d)$.

EXAMPLE. Let us take $m = 2$ and consider the Dirichlet problem

$$\begin{cases} \dfrac{d}{dx_i}\left(r^\tau |u|^q u_{x_i}\right) = a_0 r^{\tau-2} u|u|^q - \mu r^\tau u|u|^{q-2}|\nabla u|^2, \quad x \in G_0, \\ u(x) = 0, \quad x \in \Gamma_0 \cup \Gamma^+ \cup \Gamma^-, \\ a_0 \ge 0, \ 0 \le \mu < 1, \ q \ge 0, \ \tau \ge 0. \end{cases}$$

From (23), (8) we obtain

$$\lambda = \frac{\sqrt{\tau^2 + 4[(\pi/\omega_0)^2 + a_0(1+q+\mu)]} - \tau}{2(1+q+\mu)} \tag{36}$$

and

$$\Phi(\omega) = \left(\cos\frac{\pi\omega}{\omega_0}\right)^{\frac{1}{1+q+\mu}}, \quad \omega \in [-\omega_0/2, \omega_0/2]. \tag{37}$$

The solution of our problem is the function

$$w(r,\omega) = r^\lambda\left(\cos\frac{\pi\omega}{\omega_0}\right)^{\frac{1}{1+q+\mu}}, \qquad (r,\omega) \in G_0, \quad \omega \in [-\omega_0/2, \omega_0/2], \ 0 < \omega_0 < 2\pi, \tag{38}$$

where λ is defined by (36). The condition (9) for λ of Theorem 2 takes the form $(1+q+\mu)\lambda^2 + \lambda\tau > a_0$ and we see that this inequality is fulfilled. Now we calculate for (37):

$$\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi'(\omega)^2 d\omega = \frac{\pi}{(1+q+\mu)^2\omega_0}\frac{\Gamma(\frac{3}{2})\Gamma\left(\frac{1-q-\mu}{2(1+q+\mu)}\right)}{\Gamma\left(\frac{2+q+\mu}{1+q+\mu}\right)} \tag{39}$$

provided that $q+\mu < 1$. This integral is *nonconvergent*, if $q+\mu \geq 1$. At the same time for $\forall q \geq 0$ we have:

$$\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi(\omega)|^q \Phi'^2(\omega) d\omega = \frac{\pi}{(1+q+\mu)^2\omega_0}\frac{\Gamma(\frac{3}{2})\Gamma\left(\frac{1-\mu}{2(1+q+\mu)}\right)}{\Gamma\left(\frac{2+\frac{3}{2}q+\mu}{1+q+\mu}\right)}, \tag{40}$$

since $\mu < 1$. This is comletely in accordance with the above mentioned Lemmas 4-6. This demonstrates that $w(x) \in V^1_{2,\tau}(G^d_0)$, if $q+\mu < 1$, and $w(x) \notin V^1_{2,\tau}(G^d_0)$, if $q+\mu \geq 1$; for the latter case we have $w(x) \in \mathfrak{N}^1_{2,\tau,q}(G)$, $\forall\tau \geq 0$, $q > 0$.

More precise definition of the Hölder exponent. Now we shall prove the Theorem 2. For weak solutions of (1)-(5) we make more precise the value of $\alpha-$ Hölder exponent established in the Theorem 1. To this end we use the weak comparison principle (§10.4 [5]) and the barrier function constructed above. Before proving the theorem 2 we make some transformations and additional investigations. We make the change of function

$$u = v|v|^{t-1}; \qquad t = \frac{m-1}{q+m-1}. \tag{41}$$

As result the (1) - (2) takes the form:

$$\begin{cases} \mathfrak{M}_0 v(x) = F(x), & x \in G; \\ v(x) = 0, & x \in \partial G, \end{cases} \tag{42}$$

where

$$\mathfrak{M}_0 v(x) \equiv -\frac{d}{dx_i}\left(r^\tau |\nabla v|^{m-2} v_{x_i}\right) + \overline{a_0} r^{\tau-m} v|v|^{m-2} - \overline{\mu} r^\tau v^{-1} |\nabla v|^m; \tag{43}$$

$$\overline{a}_0 = t^{1-m} a_0; \quad \overline{\mu} = t\mu; \quad F(x) = t^{1-m}\left(f_0(x) - \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i}\right). \tag{44}$$

It is easy to verify that $v \in V^1_{m,\tau}(G) \cap L_\infty(G)$. By the definition of weak solution

$$Q(v,\phi) \equiv \int_G \left\{ r^\tau |\nabla v|^{m-2} v_{x_i}\phi_{x_i} + \overline{a}_0 r^{\tau-m} v|v|^{m-2}\phi - \overline{\mu} r^\tau v^{-1}|\nabla v|^m \phi - F(x)\phi \right\} dx =$$

$$= 0 \qquad \forall \phi(x) \in V^1_{m,\tau}(G) \cap L_\infty(G). \tag{45}$$

Now

$$\begin{cases} \overline{w} = r^{\overline{\lambda}} \overline{\Phi}(\omega), \\ \overline{\lambda} = \frac{1}{t}\lambda; \quad \overline{\Phi}(\omega) = \Phi^{\frac{1}{t}}(\omega) \end{cases} \tag{46}$$

plays the role of barrier function. Because of (15) - (17) and (8), it is easy to verify that $(\overline{\lambda}, \overline{\Phi}(\omega))$ is the solution to

$$\frac{d}{d\omega}\left[\left(\overline{\lambda}^2 \overline{\Phi}^2 + \overline{\Phi}'^2\right)^{\frac{m-2}{2}} \overline{\Phi}'\right] + \overline{\lambda}[\overline{\lambda}(m-1) - m + 2 + \tau]\overline{\Phi}\left(\overline{\lambda}^2\overline{\Phi}^2 + \overline{\Phi}'^2\right)^{\frac{m-2}{2}} =$$

$$= \overline{a}_0 \overline{\Phi}|\overline{\Phi}|^{m-2} - \overline{\mu}\frac{1}{\overline{\Phi}}\left(\overline{\lambda}^2\overline{\Phi}^2 + \overline{\Phi}'^2\right)^{\frac{m}{2}}, \quad \omega \in (-\omega_0/2, \omega_0/2), \tag{47}$$

$$\overline{\Phi}(-\omega_0/2) = \overline{\Phi}(\omega_0/2) = 0, \tag{48}$$

$$\int_0^{+\infty} \frac{\left[(m-1)y^2 + \overline{\lambda}^2\right](y^2 + \overline{\lambda}^2)^{\frac{m-4}{2}} dy}{(m-1+\overline{\mu})(y^2+\overline{\lambda}^2)^{\frac{m}{2}} + \overline{\lambda}(2-m+\tau)(y^2+\overline{\lambda}^2)^{\frac{m-2}{2}} - \overline{a}_0} = \frac{\omega_0}{2}. \tag{49}$$

It is obvious that the properties of (λ, Φ) established above remain valid also for $(\overline{\lambda}, \overline{\Phi}(\omega))$. In particular the (9) becomes of the form:

$$P_m(\overline{\lambda}) \equiv (m-1+\overline{\mu})\overline{\lambda}^m + (2-m+\tau)\overline{\lambda}^{m-1} - \overline{a}_0 > 0. \tag{50}$$

Now we consider the perturbation of the (47) - (49). Namely $\forall \varepsilon \in (0, 2\pi - \omega_0)$ we consider on the segment $[-\frac{\omega_0+\varepsilon}{2}, \frac{\omega_0+\varepsilon}{2}]$ the problem $(47)_\varepsilon - (49)_\varepsilon$ for $(\lambda_\varepsilon, \Phi_\varepsilon)$: the $(47)_\varepsilon - (49)_\varepsilon$ is obtained from the (47) - (49) by change in the last ω_0 by $\omega_0 + \varepsilon$ and $\overline{a}_0$ by $\overline{a}_0 - \varepsilon$. But in virtue of monotonicity-properties of the $\overline{\lambda}(\omega_0, \overline{a}_0)$ established above (see (32)), we obtain

$$0 < \lambda_\varepsilon < \overline{\lambda}, \quad \lim_{\varepsilon \to +0} \lambda_\varepsilon = \overline{\lambda}. \tag{51}$$

Now we establish lower bound for function $\Phi_\varepsilon(\omega)$.

LEMMA 7. *There exists* $\varepsilon^* > 0$ *such that*

$$\Phi_\varepsilon\left(\frac{\omega_0}{2}\right) \geq \frac{\varepsilon}{\omega_0+\varepsilon} \qquad \forall\varepsilon \in (0,\varepsilon^*). \tag{52}$$

Proof. We turn to the (50) : $P_m(\overline{\lambda}) > 0$. Since $P_m(\overline{\lambda})$ is a polinomial, by continuity, there exists a δ^*- neighborhood of $\overline{\lambda}$, in which (50) is satisfied as before, i.e. there exists $\delta^* > 0$ such that $P_m(\lambda) > 0$ for $\forall\lambda \;\big|\; |\lambda - \overline{\lambda}| < \delta^*$. We choose the number $\delta^* > 0$ in the such way; in particular the inequality

$$P_m(\overline{\lambda} - \delta) > 0 \quad \forall\delta \in (0, \delta^*) \tag{53}$$

holds. We recall that $\overline{\lambda}$ solves the (49). By (51), now for every $\delta \in (0,\delta^*)$ we can put $\lambda_\varepsilon = \overline{\lambda} - \delta$ and solve $(49)_\varepsilon$ together with this λ_ε with respect to ε; let $\varepsilon(\delta) > 0$ be obtained solution. Since (51) is true, $\lim_{\delta\to+0}\varepsilon(\delta) = +0$. Thus we have the sequence of problems $(47)_\varepsilon - (49)_\varepsilon$ with respect to

$$(\lambda_\varepsilon, \Phi_\varepsilon(\omega)) \quad \forall\varepsilon \;\big|\; 0 < \varepsilon < \min(\varepsilon(\delta); 2\pi - \omega_0) = \varepsilon^*(\delta), \quad \forall\delta \in (0,\delta^*). \tag{54}$$

We consider $\Phi_\varepsilon(\omega)$ with $\forall\varepsilon$ from (54). In the same way as (21) we verify that

$$\Phi_\varepsilon''(\omega) < 0 \quad \forall\omega \in \left[-\frac{\omega_0+\varepsilon}{2}, \frac{\omega_0+\varepsilon}{2}\right].$$

But this inequality implies that the function $\Phi_\varepsilon(\omega)$ is convex up on $[-\frac{\omega_0+\varepsilon}{2}, \frac{\omega_0+\varepsilon}{2}]$, i.e.

$$\begin{gathered}\Phi_\varepsilon(\alpha_1\omega_1 + \alpha_2\omega_2) \geq \alpha_1\Phi_\varepsilon(\omega_1) + \alpha_2\Phi_\varepsilon(\omega_2)\\ \forall\omega_1,\omega_2 \in \left[-\frac{\omega_0+\varepsilon}{2}, \frac{\omega_0+\varepsilon}{2}\right]; \quad \alpha_1 \geq 0,\ \alpha_2 \geq 0 \;|\alpha_1 + \alpha_2 = 1.\end{gathered} \tag{55}$$

We put $\alpha_1 = \frac{\omega_0}{\varepsilon+\omega_0}$, $\alpha_2 = \frac{\varepsilon}{\varepsilon+\omega_0}$; $\omega_1 = \frac{\varepsilon+\omega_0}{2}$, $\omega_2 = 0$. By the $(48)_\varepsilon$, we obtain $\Phi_\varepsilon\left(\frac{\omega_0}{2}\right) \geq \frac{\varepsilon}{\omega_0+\varepsilon}\Phi_\varepsilon(0) = \frac{\varepsilon}{\omega_0+\varepsilon}$, q.e.d. Lemma is proved.

COROLLARY 8..

$$\begin{gathered}\frac{\varepsilon}{\omega_0+\varepsilon} \leq \Phi_\varepsilon(\omega) \leq 1 \quad \forall\omega \in [-\omega_0/2, \omega_0/2];\\ \forall\varepsilon \in (0,\varepsilon^*).\end{gathered} \tag{56}$$

Proof of the Theorem 2. Let $(\lambda_\varepsilon, \Phi_\varepsilon(\omega))$ be the solution of $(47)_\varepsilon - (49)_\varepsilon$, $\varepsilon \in (0,\varepsilon^*)$, where ε^* is defined by (54). We shall introduce

$$w_\varepsilon(\overline{x}, r, \omega) = r^{\lambda_\varepsilon}\Phi_\varepsilon(\omega), \quad \forall\overline{x},\ r \in [0,d],\ \omega \in [-\omega_0/2, \omega_0/2].$$

Let us apply to the problem (42) - (44) weak comparison principle (see, for example, chapter 10 [5]), on comparing the solution of this problem with the function w_ε in the domain G_0^d. We denote: $\Omega_d = G \cap \{(\overline{x}, r, \omega) | \overline{x} \in \mathbb{R}^{n-2}, \quad r = d, \quad \omega \in [-\omega_0/2, \omega_0/2]\} \subset \partial G_0^d$. Elementary calculations show that

$$w_\varepsilon\big|_{\partial G_0^d \setminus \Omega_d} \geq 0 = v(x), \quad x \in \partial G_0^d \setminus \Omega_d; \tag{57}$$

$$w_\varepsilon\big|_{\Omega_d} \geq \frac{\varepsilon}{\omega_0 + \varepsilon} d^{\overline{\lambda}} \tag{58}$$

(by virtue of (51) and (56)).Finally, for (43) - (44) we obtain:

$$\mathfrak{M}_0 w_\varepsilon(\overline{x}, r, \omega) = r^{(m-1)\lambda_\varepsilon - m + \tau} \Bigg\{ -\frac{d}{d\omega}\left[\left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + {\Phi'}_\varepsilon^2\right)^{\frac{m-2}{2}} \Phi'_\varepsilon \right] -$$

$$- \lambda_\varepsilon[\lambda_\varepsilon(m-1) - m + 2 + \tau]\Phi_\varepsilon \left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + {\Phi'}_\varepsilon^2\right)^{\frac{m-2}{2}} + \overline{a}_0 \Phi_\varepsilon^{m-1} -$$

$$- \overline{\mu} \frac{1}{\Phi_\varepsilon} \left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + {\Phi'}_\varepsilon^2\right)^{\frac{m}{2}} \Bigg\} = \varepsilon r^{(m-1)\lambda_\varepsilon - m + \tau} \Phi_\varepsilon^{m-1}$$

by virtue of $(47)_\varepsilon$. Taking into account (56), we get

$$\mathfrak{M}_0 w_\varepsilon(\overline{x}, r, \omega) \geq \frac{\varepsilon^m}{(\omega_0 + \varepsilon)^{m-1}} r^{(m-1)\lambda_\varepsilon - m + \tau}. \tag{59}$$

Now let $\phi(x) \in V_{m,\tau}^1(G) \cap L_\infty(G)$ in (45) be such that $\phi(x) \geq 0 \quad \forall x \in \overline{G}; \quad \phi(x) = 0 \quad \forall x \in G \setminus G_0^d$. Then $\forall A > 0$ from (45) it follows that:

$$Q(Aw_\varepsilon, \phi) = \int\limits_{G_0^d} \phi(x)\Big(\mathfrak{M}_0(Aw_\varepsilon) - F(x)\Big)dx =$$

$$= \int\limits_{G_0^d} \phi(x)\Big(A^{m-1}\mathfrak{M}_0 w_\varepsilon(x) - F(x)\Big)dx \underset{\text{by (6), (44), (59)}}{\geq}$$

$$\geq \int\limits_{G_0^d} \phi(x)\Bigg\{\left(\frac{A}{\varepsilon + \omega_0}\right)^{m-1} \varepsilon^m r^{(m-1)\lambda_\varepsilon - m + \tau} - k_1 t^{1-m} r^\beta \Bigg\} dx \underset{\text{by (7), (41), (46), (51)}}{\geq}$$

$$\geq \left\{ \left(\frac{A}{\varepsilon + \omega_0}\right)^{m-1} \varepsilon^m - k_1 \left(\frac{m-1+q}{m-1}\right)^{m-1} \right\} \int\limits_{G_0^d} \phi(x) r^{(m-1)\lambda_\varepsilon - m + \tau} dx \geq 0, \tag{60}$$

if $A > 0$ is chosen sufficiently large:

$$A \geq \frac{(m-1+q)(\varepsilon + \omega_0)}{\varepsilon(m-1)} \left(\frac{k_1}{\varepsilon}\right)^{\frac{1}{m-1}}. \tag{61}$$

Further, by the Theorem 1 $v(x)\Big|_{\Omega_d} \leq c_0 d^{\alpha/t}$, therefore by (58)

$$Aw_\varepsilon\Big|_{\Omega_d} \geq \frac{A\varepsilon}{\omega_0 + \varepsilon} d^{\overline{\lambda}} \geq v(x)\Big|_{\Omega_d}, \tag{62}$$

if $A > 0$ is chosen possibly much more:

$$A \geq \frac{c_0(\varepsilon + \omega_0)}{\varepsilon} d^{\frac{(\alpha-\lambda)(q+m-1)}{m-1}}. \tag{63}$$

Thus, if $A > 0$ is chosen according to (61), (63), then from (45), (60), (62), (57) and (42) we obtain:

$$\begin{cases} Q(Aw_\varepsilon, \phi) \geq 0, \quad Q(v, \phi) = 0 \quad \text{in} \quad G_0^d; \\ Aw_\varepsilon \Big|_{\partial G_0^d} \geq v \Big|_{\partial G_0^d}. \end{cases}$$

Moreover, it is easy to verify that rest conditions of the weak comparison principle are fulfilled; by this principle we get: $v(x) \leq Aw_\varepsilon(x), \quad \forall x \in \overline{G_0^d}$. Similarly we prove that $v(x) \geq -Aw_\varepsilon(x), \quad \forall x \in \overline{G_0^d}$. Thus, finally, we have

$$|v(x)| \leq Aw_\varepsilon(x) \leq Ar^{\lambda_\varepsilon}, \quad \forall x \in \overline{G_0^d}. \tag{64}$$

Resubstituting the old variables, by (41), (46) we obtain from (64) the required bound. Theorem 2 is proved.

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