ON TOPOLOGICAL APPROACH TO FULLY NONLINEAR PARABOLIC PROBLEMS

© Igor V. Skrypnik, Igor B. Romanenko

Donetsk, Kyiv; Ukraine

1. Problem formulation

1.1. Functonal spaces.

Let n, m, $\{m_j\}_{j=1}^m$ be positive integers such that $0 \le m_j \le 2m-1$ and p, T be positive real numbers. In what follows Ω denotes a bounded open set in R^n with sufficiently smooth boundary $\partial\Omega$. We shall use notations $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$.

Let $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n)$ be multi-index with non-negative integer components. For $x \in R^n$ we denote $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$.

For $u : Q_T \to R$ and positive integer $k \ge 0$ we shall use notations

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u, \quad D^k u = \{D^\alpha u : |\alpha| = k\}.$$

Noting $M(k)$ we'll mean amount of all different multiindexes of order that is less or equal k.

We fixed further some notations and definitions of norms in anisotropic Hölder and Sobolev spaces that are analogous to corresponding spaces in the monograph [1].

For positive integer b and positive non-integer k the space $C^{(bk,k)}(\overline{Q_T})$ is defined to be the Banach space of all functions u, that have continuous derivatives $\left(\frac{\partial}{\partial t}\right)^s D^\alpha u(x,t)$, $|\alpha| + bs \le bk$ with $(x,t) \in \overline{Q_T}$ and the finite norm

$$|u|_{Q_T}^{(bk,k)} = \sum_{|\alpha|+bs \le [bk]} \left| \left(\frac{\partial}{\partial t}\right)^s D^\alpha u(x,t) \right|_{Q_T}^{(0)} + |u|_{b,Q_T}^{(bk)},$$

where $|u|_{Q_T}^{(0)} = \max\{|u(x,t)| : (x,t) \in \overline{Q_T}\}$, $[k]$ is greatest integer function of k and

$$|u|_{b,Q_T}^{(bk)} = \sum_{|\alpha|+bs=[bk]} \left| \left(\frac{\partial}{\partial t}\right)^s D^\alpha u(x,t) \right|_{x,Q_T}^{(bk-[bk])} +$$

$$+ \sum_{0<bk-|\alpha|-bs<b} \left| \left(\frac{\partial}{\partial t}\right)^s D^\alpha u(x,t) \right|_{t,Q_T}^{\left(\frac{bk-|\alpha|-bs}{b}\right)},$$

$$|u|_{x,Q_T}^{(l)} = \sup\left\{ \frac{|u(x,t)-u(y,t)|}{|x-y|^l} : x,y \in \Omega, x \ne y, t \in (0,T) \right\},$$

$$|u|_{t,Q_T}^{(l)} = \sup\left\{ \frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^l} : x \in \Omega, t,\tau \in (0,T), t \ne \tau \right\},$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

where $l \in (0,1)$.

For $p>1$ and positive integers b, k by $W_p^{(bk,k)}(Q_T)$ we denote the Banach space of all functions u that have generalized derivatives $\left(\frac{\partial}{\partial t}\right)^s D^\alpha u \in L_p(Q_T)$, $|\alpha|+bs \le bk$. The norm of the space $W_p^{(bk,k)}(Q_T)$ will be defined in the following way:

$$\|u\|_{p,Q_T}^{(bk,k)} = \left\{ \sum_{|\alpha|+bs\le bk} \left\| \left(\frac{\partial}{\partial t}\right)^s D^\alpha u \right\|_{p,Q_T}^p \right\}^{\frac{1}{p}}, \text{ where}$$

$$\|u\|_{p,Q_T} = \left\{ \int_{Q_T} |u|^p \, dxdt \right\}^{\frac{1}{p}}.$$

In the case of positive non-integer k such that bk is not integer we denote $W_p^{(bk,k)}(Q_T)$ to be Banach space of all functions u that have generalized derivatives $\left(\frac{\partial}{\partial t}\right)^s D^\alpha u \in L_p(Q_T)$, $|\alpha|+bs\le bk$ and finite norm

$$\|u\|_{p,Q_T}^{(bk,k)} = \left\{ \sum_{|\alpha|+bs\le [bk]} \left\| \left(\frac{\partial}{\partial t}\right)^s D^\alpha u \right\|_{p,Q_T}^p + \left(\|u\|_{b,p,Q_T}^{(bk)}\right)^p \right\}^{1/p},$$

$$\|u\|_{b,p,Q_T}^{(bk)} = \left\{ \sum_{|\alpha|+bs=[bk]} \left(\left\| \left(\frac{\partial}{\partial t}\right)^s D^\alpha u \right\|_{x,p,Q_T}^{(bk-[bk])} \right)^p + \right.$$

$$\left. + \sum_{0<bk-|\alpha|-bs<b} \left(\left\| \left(\frac{\partial}{\partial t}\right)^s D^\alpha u \right\|_{t,p,Q_T}^{\left(\frac{bk-|\alpha|-bs}{b}\right)} \right)^p \right\}^{1/p},$$

$$\|u\|_{x,p,Q_T}^{(l)} = \left\{ \int_0^T dt \int\int_{\Omega^2} \frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+pl}} dxdy \right\}^{1/p}, \quad 0<l<1,$$

$$\|u\|_{t,p,Q_T}^{(l)} = \left\{ \int_\Omega dx \int\int_{[0,T]^2} \frac{|u(x,t)-u(x,\tau)|^p}{|t-\tau|^{1+pl}} dtd\tau \right\}^{1/p}, \quad 0<l<1.$$

Let S be the $n-1$ – dimentional surface in R^n and $l_0 \ge 0$. We shall say that S belongs to class C^{l_0} if there exists a finite collection of open sets $\{U_i\}_{i=1}^I$ and $d>0$ such that:

S_1) $S \subset \bigcup_{i=1}^{I} U_i$;

S_2) for each i there exists $\xi^{(i)} \in S \bigcap U_i$ such that the set $S\bigcap U_i$ in local Cartesian system $\{y\}$ with origin at $\xi^{(i)}$ is given by the equation $y_n = h_i(y')$, $y' \in D(d)$, where $y' = (y_1, y_2, \dots y_{n-1})$, $D(d) = (-d,d)^{n-1}$;

S_3) for each i $h_i \in C^{l_0}(D(d))$.

Let b be positive integer, $k>0$, $p>1$, and $\partial\Omega \in C^{l_0}$, where $l_0 \ge \max\{bk, 1\}$. We denote $D_T(d) = D(d)\times(0,T)$. For $u: S_T \to R$ we shall use the notations $u^{(i)}(y',t) = u(\phi_i(y', h_i(y')), t)$, where $(y',t) \in D_T(d)$, $i=\overline{1,I}$ and $\phi_i(y)$ is the transformation from local coordinate system $\{y\}$ to system $\{x\}$.

We define the space $C^{(bk,k)}(S_T)$ as the set of all functions $u : S_T \to R$ such that $u^{(i)} \in C^{(bk,k)}(D_T(d))$, $i = \overline{1,I}$ with the norm

$$|u|_{S_T}^{(bk,k)} = \max\{|u^{(i)}|_{D_T(d)}^{(bk,k)}, i = \overline{1,I}\}.$$

We define the space $W_p^{(bk,k)}(S_T)$ as the set of all functions $u : S_T \to R$ such that $u^{(i)} \in W_p^{(bk,k)}(D_T(d)), i = \overline{1,I}$ with the norm

$$\|u\|_{p,S_T}^{(bk,k)} = \left\{ \sum_{i=1}^{I} \left(\|u^{(i)}\|_{p,D_T(d)}^{(bk,k)} \right)^p \right\}^{\frac{1}{p}}.$$

It is a simple task to check that norms corresponding to different covers of $\partial\Omega$ by sets U_i are equivalent.

We denote for positive integer k

$$W_p^{(bk,k),0}(Q_T) := \left\{ u \in W_p^{(bk,k)}(Q_T) : \frac{\partial^s u}{\partial t^s} = 0,\ 0 \le s \le k-1 \right\}.$$

1.2. Main problem and assumptions.

We consider boundary value problem

$$\Phi[u] \equiv \frac{\partial u}{\partial t} - F(x, t, u, D^1 u, \dots D^{2m} u) = f(x,t),\ (x,t) \in Q_T, \tag{1.1}$$

$$\Psi_j[u] \equiv G_j(x, t, u, \dots D^{m_j} u) = g_j(x,t),\ (x,t) \in S_T,\ j = \overline{1,m}, \tag{1.2}$$

$$u(x,0) = h(x), \qquad x \in \Omega \tag{1.3}$$

Solution of problem (1.1) – (1.3) will be considered in the space $W_p^{(4m,2)}(Q_T)$. We assume that numbers p, n, m, m_j satisfy inequalities

$$\begin{aligned} &p \ge 2, \quad p > \frac{2m+n}{2m}, \\ &p \ne \frac{2m+1}{2m-m_j}, \quad m_j \le 2m-1, \quad j = \overline{1,m} \end{aligned} \tag{1.4}$$

and the boundary $\partial\Omega$ of the domain Ω satisfies the condition

$$\partial\Omega \in C^{4m}. \tag{1.5}$$

We define

$$F_\alpha(x,t,\xi) := \frac{\partial}{\partial \xi_\alpha} F(x,t,\xi), |\alpha| \le 2m, \xi = \{\xi_\alpha \in R : |\alpha| \le 2m\},$$

$$G_{j\beta}(x,t,\zeta_j) := \frac{\partial}{\partial \zeta_\beta} G_j(x,t,\zeta_j), \quad |\beta| \le m_j, \quad \zeta_j = \{\zeta_\beta \in R : |\beta| \le m_j\}, \quad j = \overline{1,m}.$$

and suppose that the following conditions for the functions F G_j are fulfilled:

F_1) functions $F(x,t,\xi)$ have all mixed continuous derivatives with respect to ξ up to the order 2m+1, $F(x,t,0)\equiv 0$;

F_2) there exists continuous function $\nu : R^+ \to R^+$ such that for each $\xi \in R^{M(2m)}$, $\eta \in R^n$ the inequality

$$(-1)^{m+1} \sum_{|\alpha|=2m} F_\alpha(x,t,\xi)\eta^\alpha \geq \nu(|\xi|)|\eta|^{2m}$$

holds;

G_1) for each $j=1,\dots m$ the function $G_j(x,t,\zeta_j)$ has all mixed continuous derivatives with respect to variables ζ up to the order $4m-m_j+1$, $G_j(x,t,0)\equiv 0$;

For $(x,t)\in S_T$, $\xi \in R^{M(2m)}$, $\zeta_j=\{\xi_\beta : |\beta|\leq m_j\}$ (here $j=1,\dots m$) we define

$$L(x,t,\xi,\delta+\tau\eta,q) := q-(-1)^m \sum_{|\alpha|=2m} F_\alpha(x,t,\xi)(\delta+\tau\eta)^\alpha,$$

$$B_j(x,t,\zeta_j,\delta+\tau\eta) := \sum_{|\beta|=m_j} G_{j\beta}(x,t,\zeta_j)(\delta+\tau\eta)^\beta, \quad j=\overline{1,m},$$

where η is the unit vector in the direction of outward normal to $\partial\Omega$ at the point x, δ means an arbitrary vector from the plane tangent to $\partial\Omega$ at the point x, τ is complex variable and q is real number.

If $q \geq -\tilde{\nu}|\delta|^{2m}$, $0<\tilde{\nu}<\nu(|\xi|)$ and $|q|+|\delta|>0$, then $L(x,t,\xi,\delta+\tau\eta,q)$ as a polynomial of τ has m roots τ_s^+ with positive real part, other roots are with negative real part [1]. We denote

$$L^+(x,t,\xi,\delta,\tau,q) := \prod_{s=1}^{m}(\tau-\tau_s^+)$$

and assume that the following condition (Lopatynsky condition) is fulfilled:

G_2) for each $(x,t)\in S_T$, $\xi\in R^{M(2m)}$ and δ, that belongs to the tangent plane to $\partial\Omega$ at the point x, inequalities $q\geq -\tilde{\nu}|\delta|^{2m}$, $0<\tilde{\nu}<\nu(|\xi|)$ and $|q|+|\delta|>0$ imply linear independence of B_j by module of L^+.

We assume that the following inclusions for the functions from the right side of (1.1) - (1.3) are fullfilled:

$$f\in W_p^{(2m,1)}(Q_T), \quad g_j \in W_p^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp})}(S_T), \quad j=\overline{1,m},$$
$$h\in W_p^{(4m-\frac{2m}{p})}(\Omega). \tag{1.6}$$

We also assume that compatibility conditions for problem (1.1) – (1.3) are fullfilled. We shall use notation

$$u^{(0)}(x) := u(x,0), \quad u^{(1)}(x) := \frac{\partial u}{\partial t}(x,0)$$

From (1.1), (1.3) we can determine that

$$u^{(0)}(x) := h(x),$$
$$u^{(1)}(x) := \frac{\partial u}{\partial t}(x,0) = f(x,0) + F(x,0,h,D^1h,\dots D^{2m}h)$$

Formulating compatibility conditions for (1.1) – (1.3) we shall use the following equalities (here $j = 1,\dots m$):

$$G_j(x,0,u^{(0)},\dots D^{m_j}u^{(0)}) = g_j(x,0), \tag{1.7}$$

$$\frac{\partial}{\partial t}G_j(x,0,u^{(0)},\dots D^{m_j}u^{(0)}) + \\ + \sum_{|\beta|\le m_j} G_{j\beta}(x,0,u^{(0)},\dots D^{m_j}u^{(0)})D^{\beta}u^{(1)}(x,0) = \frac{\partial}{\partial t}g_j(x,0). \tag{1.8}$$

We say that compatibility conditions for (1.1) – (1.2) are fullfilled if

C) for each $j = 1,\dots m$ condition (1.7) is satisfied and condition (1.8) is fulfilled for such j that $p > \frac{2m+1}{2m-m_j}$.

2. Reduction to operator equation

2.1. Reduction to the problem with zero initial conditions.

In a standard way (see [2]) we can construct a function $v \in W_p^{(4m,2)}(Q_T)$ that satisfies conditions

$$v(x,0) = u^{(0)}(x), \quad \frac{\partial v}{\partial t}(x,0) = u^{(1)}(x) \quad x \in \Omega.$$

Now we introduce a new function $u_1 = u - v$. If u is the solution of problem (1.1) – (1.3) then u_1 is the solution of boundary value problem

$$\frac{\partial u_1}{\partial t} - F^{(1)}(x,t,u_1,D^1u_1,\dots D^{2m}u_1) = f^{(1)}(x,t), \quad (x,t) \in Q_T, \tag{2.1}$$

$$G_j^{(1)}(x,t,u_1,\dots D^{m_j}u_1) = g_j^{(1)}(x,t),\ (x,t) \in S_T,\ j = \overline{1,m}, \tag{2.2}$$

$$u_1(x,0) = 0, \qquad x \in \Omega, \tag{2.3}$$

where

$$F^{(1)}(x,t,u_1,D^1u_1,\dots D^{2m}u_1) = \\ F(x,t,u_1+v,D^1(u_1+v),\dots D^{2m}(u_1+v)) - \\ - F(x,t,v,D^1v,\dots D^{2m}v),$$

$$f^{(1)}(x,t) = f(x,t) - \widetilde{\Phi}[v],$$
$$G_j^{(1)}(x,t,u_1,\dots D^{m_j}u_1) = G_j(x,t,u_1+v,\dots D^{m_j}u_1+v) - \widetilde{\Psi_j}[v],$$

$$g_j^{(1)}(x,t) = g_j(x,t) - \widetilde{B_j}[v],\ j = \overline{1,m}.$$

Additionaly to F_1), F_2), G_1), G_2), we introduce conditions

F_3) function $F(x,t,\xi)$ has all continuous derivatives with respect to variables ξ_β up to the order $2m+1$, $F(x,t,0) \equiv 0$;

F_4) operators

$$F_\alpha(\cdot,\cdot,u,D^1u,\dots D^{2m}u) : W_p^{(4m,2)}(Q_T) \to W_p^{(2m,1)}(Q_T)$$

are bounded and continuous;

G_3) for each j function $G_j(x,t,\zeta_j)$ has all mixed continuous derivatives with respect to η_β up to the order $4m - m_j + 1$, $G_j(x,t,0) \equiv 0$;

G_4) operators

$$G_{j\beta}(\cdot,\cdot,u,\dots D^{m_j}u) : W_p^{(4m-\frac{1}{p},2-\frac{1}{2mp}),0}(S_T) \to W_p^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp})}(S_T)$$

are bounded and continuous.

LEMMA 2.1. *Assume that conditions (1.4) – (1.7),* F_1) , F_2), G_1), G_2) *for problem (1.1) – (1.3) are satisfied and* $u \in W_p^{(4m,2)}(Q_T)$ *is the solution of (1.1) – (1.3). Then*

i) *function* $F^{(1)}(x,t,\xi)$ *satisfies conditions* F_2) *(with some function* $\nu^{(1)}$ *that, possibly, differs from* ν), F_3), F_4), *and functions* $G_j^{(1)}(x,t,\zeta_j)$ *satisfy conditions* G_2), G_3), G_4);

ii) *the following inclusions for* $u_1(x,t), f^{(1)}(x,t), g_j^{(1)}(x,t)$ *are fulfilled:*

$$u_1 \in W_p^{(4m,2),0}(Q_T), \qquad f^{(1)} \in W_p^{(2m,1),0}(Q_T),$$
$$g_j^{(1)} \in W_p^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp}),0}(S_T), \quad j = \overline{1,m}$$

LEMMA 2.2. *Assume that conditions of lemma 2.1 are fulfilled and* $u_1 \in W_p^{(4m,2),0}(Q_T)$ *is the solution of (2.1) – (2.3). Then function* $u(x,t) = u_1(x,t) + v(x,t)$ *is the solution of (1.1) – (1.3).*

2.2. Definition of operator.

Thus, instead of problem (1.1) – (1.3) we can analyze equivalent problem

$$\widetilde{\Phi}[u] \equiv \frac{\partial u}{\partial t} - F(x,t,u,D^1u,\dots D^{2m}u) = f(x,t), \quad (x,t) \in Q_T, \tag{2.4}$$

$$\widetilde{\Psi_j}[u] \equiv G_j(x,t,u,\dots D^{m_j}u) = g_j(x,t),\ (x,t) \in S_T,\ j = \overline{1,m}, \tag{2.5}$$

$$u \in W_p^{(4m,2),0}(Q_T), \tag{2.6}$$

where F satisfies conditions F_2), F_3), F_4), functions G_j satisfy conditons G_2), G_3), G_4), and for the functions f, g_j the following inclusions are fullfilled:

$$f \in W_p^{(2m,1),0}(Q_T),$$
$$g_j \in W_p^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp}),0}(S_T), \quad j = \overline{1,m} \tag{2.7}$$

Nonlinear operator corresponding to (2.4) – (2.6) will be defined by the following equality

$$\langle Au, \phi\rangle := \frac{1}{p}\frac{d}{ds}\left[\left(\left\|\widetilde{\Phi}[u+s\phi]-f\right\|_{p,Q_T}^{(2m,1)}\right)^p + \right.$$
$$\left.\left. + \sum_{j=1}^{m}\left(\left\|\widetilde{\Psi}_j[u+s\phi]-g_j\right\|_{p,S_T}^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp})}\right)^p\right]\right|_{s=0} \tag{2.8}$$

In (2.8) $\{u,\phi\} \in W_p^{(4m,2),0}(Q_T)$ and notification $\langle Au,\phi\rangle$ means value of functional Au on function ϕ.

The following theorem formulates the main properties the operator A, defined by (2.8):

THEOREM 2.3. *Assume that conditions* (1.4), (1.5), (2.7) F_2) – F_4), G_2) – G_4) *for problem* (2.4) – (2.6) *are fulfilled. Then*

i) *for each* $u \in W_p^{(4m,2),0}(Q_T)$ Au *appears to be linear and continuous functional on* $W_p^{(4m,2),0}(Q_T)$.

ii) *operator* A *is bounded, continuous and satisfies the* $(S)_+$ *condition on the space* $W_p^{(4m,2),0}(Q_T)$.

REMARK 2.1. *We recall the definition of the condition* $(S)_+$ *for the operator* A *acting from the Banach space* X *into the dual space* X^* *(see, for example,* [7]*). Analogous condition in* [6] *was called the condition* α).

We say that an operator $A : X \to X^*$ *satisfies the condition* $(S)_+$ *if for arbitrary sequence* $\{u_j\} \subset X$ *which converges weakly to some* $u_0 \in X$ *and satisfies condition*

$$\limsup_{j\to\infty}\langle Au_j, u_j - u_0\rangle \le 0$$

we have that u_j *converges strongly to* u_0.

2.3. Reduction to operator equation.

Now we can introduce an operator equation

$$Au = 0, \qquad u \in W_p^{(4m,2),0}(Q_T), \tag{2.9}$$

where the operator A is defined by (2.8). The following theorem shows connection between the equation (2.9) and boundary value problem (2.4) – (2.6).

THEOREM 2.12. *Assume that problem* (2.4) – (2.6) *satisfies conditions (1.4), (1.5), (2.7),* F_2) – F_4), G_2) – G_4). *Function* $u \in W_p^{(4m,2),0}(Q_T)$ *is a solution for the problem* (2.4) – (2.6) *if and only if it is the solution for the equation* (2.9).

2.4. Topological characteristic of parabolic problem.

Using notion of operator degree for $(S)_+$ operators (see [6,7]) we can introduce topological characteristic for problem (2.4) – (2.5). Namely for an arbitrary bounded domain D in $W_p^{(4m,2),0}(Q_T)$ we define an integer number $Deg(A,\overline{D},0)$ (see section 2, chapter 2 from [7]) if the following condition is satisfied

$$Au \neq 0, \qquad u \in \partial D. \tag{2.10}$$

Some results of application of this characteristic to the study of solvability of initial boundary value problem (2.4) – (2.6) will be given in the section 3.

3. Some applications

Having reduced problem (2.4) – (2.6) to operator equation with operator satisfying $(S)_+$ condition we can investigate solvability of operator equation (2.9) instead of studying solvablitity of problem (2.4) – (2.6). On this way we can apply topological methods developed in [6,7]. The results of the such way of studying are given here without proofs.

3.1. Uniqueness of solution.

THEOREM 3.1. *Let conditons (1.4), (1.5), (2.7), F_2) – F_4), G_2) – G_4) for the problem* (2.4) – (2.6) *be fulfilled. Then problem* (2.4) – (2.6) *can have at most one solution.*

COROLLARY 3.2. *Assume that conditions (1.4) – (1.7), F_1) , F_2), G_1), G_2) for problem* (1.1) – (1.3) *are fulfilled. Then problem* (1.1) – (1.3) *can have at most one solution.*

3.2. Local existence of solution.

THEOREM 3.3. *Assume that conditions (1.4), (1.5), (2.7), F_2) – F_4), G_2) – G_4) for the problem* (2.4) – (2.6) *are satisfied and K is some positive number. Then there exists positive T_0, dependent on K, but independent of functions from the right side of problem* (2.4) – (2.6), *such the problem* (2.4) – (2.6) *has a solution $u \in W_p^{(4m,2),0}(Q_T)$ for $0 < T < T_0$ if the following inequalities hold:*

$$\|f\|_{p,Q_T}^{(2m,1)} \leq K, \quad \|g_j\|_{p,S_T}^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp})} \leq K, \quad j=\overline{1,m}.$$

COROLLARY 3.4. *Assume that conditions (1.4) – (1.7), F_1) , F_2), G_1), G_2) for the problem* (1.1) – (1.3) *are fulfilled and K is some positive number. Then there exists positive T_0 that depends on K, such that the problem* (1.1) – (1.3) *has a solution $u \in W_p^{(4m,2)}(Q_T)$ for $0 < T < T_0$ if the following inequalities hold:*

$$\|f\|_{p,Q_T}^{(2m,1)} \leq K, \quad \|g_j\|_{p,S_T}^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp})} \leq K, \quad j=\overline{1,m},$$
$$\|h\|_{p,\Omega}^{4m-\frac{2m}{p}} \leq K.$$

3.3. Conditional solvability of initial boundary value problems.

We include initial boundary value problem (2.4) – (2.6) in one-parametrical family of problems

$$\widetilde{\Phi_\tau}[u] = \frac{\partial u}{\partial t} - F_\tau(x,t,u,D^1u,\dots D^{2m}u) = \tau f(x,t), \quad (x,t) \in Q_T, \tag{2.11}$$

$$\widetilde{\Psi_{j,\tau}}[u] = G_{j,\tau}(x,t,u,\dots D^{m_j}u) = \tau g_j(x,t), \ (x,t) \in S_T, \ j=\overline{1,m}, \tag{2.12}$$

$$u \in W_p^{(4m,2),0}(Q_T), \tag{2.13}$$

where $F_\tau(x,t,\xi) := F(\tau,x,t,\xi)$, $\tau \in [0,1]$, $(x,t) \in Q_T$, $\xi \in R^{M(2m)}$, $G_{j,\tau}(x,t,\zeta_j) := G_j(\tau,x,t,\zeta_j)$, $\tau \in [0,1]$, $(x,t) \in S_T$, $\zeta_j \in R^{M(m_j)}$, $j=\overline{1,m}$.

We assume that $F(x,t,\xi) = F_1(x,t,\xi)$, $G_j(x,t,\zeta_j) = G_{j,1}(x,t,\zeta_j)$, where functions $F(x,t,\xi)$, $G_j(x,t,\zeta_j)$ figure in the left side of equations in problem (2.4) – (2.6).

THEOREM 3.5. *Let functions $F_\tau(x,t,\xi)$ together with all their derivatives by ξ_β up to the order $2m+1$ be continuous for $\tau \in [0,1]$, $(x,t) \in Q_T$, $\xi \in R^{M(2m)}$ and $F_\tau(x,t,0) \equiv 0$ and we assume that for every $\tau \in [0,1]$ function $F_\tau(x,t,\xi)$ satisfies conditions F_2), F_4). Let function $G_{j,\tau}(x,t,\zeta_j)$, $j \in 1,\dots m$ and all its derivatives up to the order $4m-m_j+1$ by ζ_β be continuous for $\tau \in [0,1]$, $(x,t) \in S_T$, $\zeta_j \in R^{M(m_j)}$ and $G_{j,\tau}(x,t,0) \equiv 0$ and functions $G_{j,\tau}$ satisfy condiitions G_2), G_4), for every $\tau \in [0,1]$. Assume that conditions (1.4), (1.5) are fullfilled and for each $\tau \in [0,1]$ inclusions (2.7) are valid. We suppose that there exists a number $R = R(f, g_1, \dots g_m)$ independent of τ and such that problem (2.11) – (2.13) for each $\tau \in [0,1]$ has no solutions outside the ball*

$$\{u \in W_p^{(4m,2),0}(Q_T) : \|u_\tau\|_{p,Q_T}^{(4m,2)} \le R\}.$$

Then problem (2.4) – (2.6) has the unique solution $u \in W_p^{(4m,2),0}(Q_T)$.

3.4. Theorem of domain preservation.

THEOREM 3.6. *Assume that initial boundary value problem (2.4) – (2.6) satisfies conditions of Theorem 2.3 and D is the open set in $W_p^{(4m,2),0}(Q_T)$. Then the set*

$$R(D) := \Big\{ \Big(\frac{\partial u}{\partial t} - F(\cdot,\cdot,u,D^1 u,\dots D^{2m} u), G_1(\cdot,\cdot,u,\dots D^{m_1} u),\dots \\ G_m(\cdot,\cdot,u,\dots D^{m_m} u) \Big) : u \in D \Big\}$$

will be open in space

$$W_p^{(2m,2\{m_j\}),0}(Q_T,S_T) := W_p^{(2m,1),0}(Q_T) \times \left(\prod_{j=1}^{m} W^{(4m-m_j-\frac{1}{p},2-\frac{m_j}{2m}-\frac{1}{2mp}),0}(S_T) \right).$$

COROLLARY 3.7. *Assume that initial boundary value problem (2.4) – (2.6) satisfies conditions of Theorem 2.3. Then the set*

$$R^{(0)} := \{(f,g_1,\dots g_m) \in W_p^{(2m,\{m_j\}),0}(Q_T,S_T) : \\ \text{problem (2.4) – (2.6) has a solution } u \in W_p^{(4m,2),0}(Q_T)\}$$

is open in $W_p^{(2m,\{m_j\}),0}(Q_T,S_T)$.

COROLLARY 3.8. *Assume that initial boundary value problem (2.4) – (2.6) satisfies conditions of Theorem 2.3 and $U : R^{(0)} \to W_p^{(4m,2),0}(Q_T)$ is operator which maps vector of functions $(f,g_1,\dots g_m) \in R^{(0)}$ (where $R^{(0)}$ is defined in Corollary 3.7) to the solution of problem (2.4) – (2.6). Then operator U is continuous on $R^{(0)}$.*

3.5. Convergence of the Galerkin Approximants.

Let $\{v_k\}_{k=1}^{\infty}$ be the complete system of functions in $W_p^{(4m,2),0}(Q_T)$. Assume that initial boundary value problem (2.4) – (2.6) satisfies conditions of theorem 2.3. We call " $\mathfrak{K}$ - approximate solution" of boundary value problem (2.4) – (2.6) the function $u_{\mathfrak{K}}$ such that

$$u_{\mathfrak{K}} = \sum_{k=1}^{\mathfrak{K}} c_k^{(\mathfrak{K})} v_k(x,t)$$

and

$$\langle A u_{\mathfrak{K}}, v_k \rangle = 0, \ k = \overline{1, \mathfrak{K}},$$

where $c_k^{(\mathfrak{K})}$ are real numbers and operator is defined by (2.8).

We say that problem (2.4) – (2.6) has a " bounded sequence of $\mathfrak{K}$ - approximate solutions" if there exists number $\mathfrak{K}_0$ such that for $\mathfrak{K} \geq \mathfrak{K}_0$ problem (2.4) – (2.6) has $\mathfrak{K}$ - approximate solution and the sequence $\{u\}_{\mathfrak{K}=\mathfrak{K}_0}^{\infty}$ is bounded.

THEOREM 3.9. *Assume that conditions (1.4), (1.5), (2.7), F_2) – F_4), G_2) – G_4) for problem (2.4) – (2.6) are fulfilled. The problem* (2.4) – (2.6) *has a solution* $u_0 \in W_p^{(4m,2),0}(Q_T)$ *if and only if it has bounded sequence of* $\mathfrak{K}$ *- approximate solutions* $\{u_{\mathfrak{K}}\}_{\mathfrak{K}=\mathfrak{K}_0}^{\infty}$. *The sequence* $u_{\mathfrak{K}}$ *strongly converges to* u_0 *in* $W_p^{(4m,2),0}(Q_T)$.

REFERENCES

1. Il'in V.P., *The properties of some classes of differentiable functions of several variables defined in an n-dimentional region*, Trudy Math. Inst. Steklov, **66** (1962), 227 – 363.
2. Kartsatos A.G., Skrypnik I.V., *A global approach to fully nonlinear parabolic problems*, Trans. Amer. Math. Soc. **352** (2000), 4603 – 4640.
3. Ladyzhenskaya O.A., Solonnikov V.A., Uraltseva N.N., *Linear and quasilinear equations of parabolic type*, Nauka, Moskow, 1967.
4. Romanenko I.B., *Reduction of fully nonlinear boundary parabolic problems to operator equations*, Nonlinear oscillations **3** (2000), 400 – 413.
5. Romanenko I., *On some embedding inequalities for anisotropic Sobolev spaces*, Visnyk Kyivs'kogo Universitety **5** (2000), 39 – 46.
6. Skrypnik I.V., *Nonlinear Higher Order Elliptic Equations*, Naukova Dumka, Kyiv, 1973.
7. Skrypnik I.V., *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Amer. Math. Soc. Transl., Ser.II, 139, AMS, Rhode Island, 1994.
8. Skrypnik I.V., Romanenko I.B., *A priori estimates for solutions of linear parabolic problems with Sobolev coefficients*, Ukrainian Mathematical Journal **51** (1999), 1534 – 1548.
9. Solonnikov V.A., *On boundary value problems for general linear parabolic systems of higher order*, Trudy Math. Inst. Steklov, **83** (1965), 1 – 162.

SKRYPNIK I.V.,
INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS,
R.LUXEMBURG STR., 74,
DONETSK, UKRAINE
E-mail address: skrypnik@iamm.ac.donetsk.ua

ROMANENKO I.B.,
MECH. AND MATH. FACULTY,
TARAS SHEVCHENKO UNIVERSITY,
VOLODYMYRS'KA STR., 64,
KYIV, UKRAINE
E-mail address: igrom@ambernet.kiev.ua