UNIVERSAL BOUNDS FOR GLOBAL SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS

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1. Introduction.

In this survey we consider parabolic problems for which blow-up in finite time occurs for some initial data but global positive solutions may also exist. We present results on universal L^∞-bounds for global positive solutions. These bounds will be of the form

$$u(x,t) \le C(\tau), \qquad x \in \Omega,\ t \ge \tau > 0,$$

where $C(\tau) > 0$ does not depend on initial data.

The first bound of this kind (see Section 2) was established in [FSW] for a semilinear parabolic equation on a bounded domain using a weighted Lebesgue space approach. An improvement of the result from [FSW] was given in [Q2] (see Section 3) for space-dimensions two and three. The method of [Q2] relies on scaling, energy and Hardy's inequality. Universal bounds for an equation in selfsimilar variables (in the whole space $\mathbb{R}^N$) were derived in [MS] (see Section 4) employing convolution Lebesgue spaces. For a degenerate parabolic equation, universal bounds were obtained in [S] (see Section 5) by energy estimates, interpolation inequalities and regularizing properties. The smoothing effect, scaling and energy estimates are used in [QS1] to establish universal bounds for the heat equation with nonlinear boundary conditions (see Section 6). The very interesting question of the blow-up rate of the constant $C(\tau)$ as $\tau \to 0$ has been addressed in [QS2] (see Section 7).

2. Semilinear equation on a bounded domain.

Consider the problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \quad x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), x \in \Omega, \end{cases} \tag{2.1}$$

with $p > 1$ and $u_0 \in L^\infty(\Omega)$, Ω is a bounded domain in $\mathbb{R}^N$.

The study of boundedness of global solutions of this problem was initiated in [NST]. It was shown there that if Ω is convex, $u_0 \ge 0$ and $p < (N+2)/N$ then every global solution is uniformly bounded by a constant which depends on u_0 in a complicated way. (In particular, this constant depends on the shape of $u(\cdot, t_0)$ near $\partial\Omega$ for some

$t_0 > 0$). Let $p_s = (N+2)/(N-2)$ for $N > 2$ ($p_s = \infty$ if $N \leq 2$). Ni, Sacks and Tavantzis also proved in [NST] that global unbounded weak solutions exist for $p \geq p_s$, $N > 2$.

Slightly later, Cazenave and Lions [CL] derived a uniform a priori bound (depending on $\sup_\Omega |u_0|$) for global solutions if $(3N-4)p < (3N+8)$ and they proved that global solutions are bounded (without giving any information on the bound) when $p < p_s$. An a priori estimate for positive global solutions was established by Giga [G] when $p < p_s$. More recently, Quittner [Q1] has shown that an a priori bound holds for all global solutions provided $p < p_s$. The a priori bounds in [G] and [Q1] depend on $\sup_\Omega |u_0|$.

The question whether the global unbounded weak solutions found in [NST] are classical for all $t > 0$ was answered by Galaktionov and Vázquez [GV] in the radial case on a ball. The answer is positive if $p = p_s$ and negative for $p_s < p$ ($< 1 + 6/(N-10)$ if $N > 10$), u_0 radially decreasing. In fact, global classical solutions are bounded in the latter case.

It is easy to see that an a priori bound of the form

$$u(\cdot,t) \leq C(\sup_\Omega u_0, p, \Omega), \qquad t \geq 0,$$

cannot hold for global positive solutions of (2.1) when $p \geq p_s$ and Ω is starshaped. Indeed, such an estimate would imply the existence of a positive steady state.

One of the main aims of [FSW] was to establish an a priori bound for global solutions of (2.1) which is **universal**, that is, **independent of** u_0:

THEOREM 2.1. *Assume* $p > 1$, $(N-1)p < N+1$ *and let* $\tau > 0$. *There exists a constant* $C(\Omega, p, \tau) > 0$, *independent of* u, *such that for all nonnegative global solutions* u *of* (2.1), *it holds*

$$\sup_\Omega u(\cdot,t) \leq C(\Omega,p,\tau) \qquad \textit{for} \quad t \geq \tau. \tag{2.2}$$

In other words, Theorem 2.1 shows that there exists a global absorbing bounded set (after a positive time) for all global nonnegative trajectories of (2.1).

It is clear that (2.2) cannot hold for $\tau = 0$ since there are initial data u_0 arbitrarily large in the L^∞-norm and such that the corresponding solutions are global. It is also obvious that there is no universal bound like (2.2) for global solutions which change sign because sign-changing stationary solutions can be arbitrarily large in the L^∞-norm.

To prove (2.2) we use the smoothing effect for solutions of (2.1) in weighted Lebesgue spaces. Next we briefly describe both linear and local nonlinear theories in these spaces.

Let Ω be any bounded domain of $\mathbb{R}^N$. We denote by $(e^{t\Delta})_{t\geq 0}$ the Dirichlet heat semigroup on $L^2(\Omega)$. We denote by $\lambda_1 > 0$ the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and by $\varphi_1 = \varphi_1(x) > 0$ the corresponding eigenfunction, normalized by $\int_\Omega \varphi_1 = 1$. We also define the function $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

For any Borel measure μ on Ω, the spaces $L^q_\mu(\Omega)$ are defined in the usual way for $1 \leq q \leq \infty$. In particular, we will consider the spaces $L^q_{\varphi_1}(\Omega)$ and $L^q_\delta(\Omega)$, corresponding respectively to $\mu = \varphi_1(x)\,dx$ and $\mu = \delta(x)\,dx$.

It is clear that $L^\infty_{\varphi_1}(\Omega) = L^\infty_\delta(\vee mega) = L^\infty(\Omega)$. For $1 \leq q < \infty$, the spaces $L^q_{\varphi_1}(\Omega)$ and $L^q_\delta(\Omega)$, are endowed respectively with the norms

$$\|\phi\|_{q,\varphi_1} = \left(\int_\Omega |\phi(x)|^q \varphi_1(x)\,dx\right)^{1/q}$$

and

$$\|\phi\|_{q,\delta} = \left(\int_\Omega |\phi(x)|^q \delta(x)\,dx\right)^{1/q}.$$

Since φ_1 and δ are bounded functions on Ω, we have $L^q_{\varphi_1}(\Omega) \subset L^1_{\varphi_1}(\Omega)$ and $L^q_\delta(\Omega) \subset L^1_\delta(\Omega)$ for all $1 \le q \le \infty$.

When Ω has a smooth, say, C^2 boundary, it is well-known that there exist constants c_1, $c_2 > 0$ such that

$$c_1\delta(x) \le \varphi_1(x) \le c_2\delta(x), \quad x \in \Omega.$$

It then follows that $L^q_{\varphi_1}(\Omega) = L^q_\delta(\Omega)$ and that the two norms are equivalent.

The main result in the linear theory developped in [FSW] is the following theorem.

THEOREM 2.2. ([FSW]) *Let $1 \le q \le r \le \infty$ and $\alpha = \frac{N+1}{2}(\frac{1}{q} - \frac{1}{r})$. There exists $C = C(\Omega) > 0$ such that, for all $\phi \in L^q_\delta(\Omega)$, it holds*

$$\|e^{t\Delta}\phi\|_{r,\delta} \le Ct^{-\alpha}\|\phi\|_{q,\delta}, \quad t > 0.$$

The estimate from Theorem 2.2 is optimal.

THEOREM 2.3. ([FSW]) *Let $1 \le q < r \le \infty$ and $\alpha = \frac{N+1}{2}(\frac{1}{q} - \frac{1}{r})$. Let Ω be a smoothly bounded domain, and assume that there exists $x_0 \in \partial\Omega$ such that $\partial\Omega$ coincides locally around x_0 with a hyperplane. Then for all $\epsilon > 0$, there exist $\phi \in L^q_\delta(\Omega)$ and C, $\tau > 0$, such that*

$$\|e^{t\Delta}\phi\|_{r,\delta} \ge Ct^{-\alpha+\epsilon}, \quad 0 < t < \tau.$$

Next we present the main results of the local nonlinear theory. In what follows, we assume that Ω is a (C^2) smooth bounded domain of $\mathbb{R}^N$, and that

$$q_c = \frac{(N+1)(p-1)}{2}, \quad p > 1.$$

The problem (2.1) will be studied under the form of the (formally equivalent) integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}(|u(s)|^{p-1}u(s))\,ds. \tag{2.3}$$

THEOREM 2.4. ([FSW]) *Let $q > q_c$ and $q \ge 1$.*
(i) For every $M > 0$, there exist $T = T(M) > 0$ and $K = K(M) > 0$ such that if $u_0 \in L^q_\delta(\Omega)$ with $\|u_0\|_{q,\delta} \le M$, then there is a solution $u \in C([0,T]; L^q_\delta(\Omega))$ of (2.3) satisfying

$$u \in C((0,T]; L^r_\delta(\Omega)), \quad q < r \le \infty,$$

$$t^{\frac{N+1}{2}(\frac{1}{q}-\frac{1}{r})}\|u(t)\|_{r,\delta} \le K, \quad 0 < t \le T, \quad q \le r \le \infty. \tag{2.4}$$

This solution is unique in the class

$$C([0,T]; L^q_\delta(\Omega)) \cap L^\infty_{\text{loc}}((0,T); L^{pq}_\delta(\Omega)).$$

(ii) If the maximal existence time $T^* = T^*(u_0)$ of the solution is finite, then

$$\lim_{t\to T^*} \|u(t)\|_{r,\delta} = \infty, \quad q \le r \le \infty.$$

More precisely, we have the lower estimates

$$\|u(t)\|_{r,\delta} \ge C(T^* - t)^{\frac{N+1}{2r} - \frac{1}{p-1}}, \qquad 0 \le t < T^*, \quad q \le r \le \infty.$$

THEOREM 2.5. ([FSW]) *Let $q = q_c$ and $q > 1$.*
(i) For every $u_0 \in L^q_\delta(\Omega)$, there exist $T = T(u_0) > 0$, $K = K(u_0) > 0$ and a solution $u \in C([0,T]; L^q_\delta(\Omega))$ of (2.3) satisfying

$$u \in C((0,T]; L^r_\delta(\Omega)), \quad q < r \le \infty,$$

$$t^{\frac{N+1}{2}(\frac{1}{q} - \frac{1}{r})}\|u(t)\|_{r,\delta} \le K, \quad 0 < t \le T, \quad q \le r \le \infty.$$

This solution is unique in the class

$$C([0,T]; L^q_\delta(\Omega)) \cap L^\infty_{\text{loc}}((0,T); L^r_\delta(\Omega)),$$

where $1 \le r/p < q < r$.
(ii) If the maximal existence time $T^*(u_0)$ of the solution is finite, then

$$\lim_{t\to T^*} \|u(t)\|_{r,\delta} = \infty, \quad q < r \le \infty.$$

(iii) If $\|u_0\|_{q,\delta}$ is sufficiently small, then $T^*(u_0) = \infty$, and $\lim_{t\to\infty} \|u(t)\|_{q,\delta} = 0$.

The following result shows that the restriction $q \ge q_c$ is actually optimal for local existence, at least for some domains Ω.

THEOREM 2.6. ([FSW]) *Let $1 \le q < q_c$ (hence $p > 1 + 2/(N+1)$), and assume that Ω satisfies the assumptions of Theorem 2.2. Then there exist initial data $u_0 \in L^q_\delta$, $u_0 \ge 0$, such that no local solution u of (2.3) exists with $u \ge 0$.*

Proof of Theorem 2.1. In what follows, C denotes various positive constants depending only on the indicated arguments.

We start from the classical eigenfunction's estimate of Kaplan [K]. Multiplying the first equation in (2.1) by φ_1 and integrating by parts, we obtain

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 + \lambda_1 \int_\Omega u(t)\varphi_1 = \int_\Omega u^p(t)\varphi_1. \tag{2.5}$$

By Jensen's inequality, it follows that

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 \ge \Big(\int_\Omega u(t)\varphi_1\Big)^p - \lambda_1 \int_\Omega u(t)\varphi_1.$$

Since u exists globally, one then necessarily has

$$\int_\Omega u(t)\varphi_1 \le C(\Omega,p) \equiv \lambda_1^{1/(p-1)},\ t\ge 0. \tag{2.6}$$

Now integrating (2.5) in time over $(0,\tau/2)$, and using (2.6), we obtain

$$\int_0^{\tau/2}\int_\Omega u^p\varphi_1 = \int_\Omega u(\tau/2)\varphi_1 - \int_\Omega u(0)\varphi_1 + \lambda_1\int_0^{\tau/2}\int_\Omega u(t)\varphi_1$$
$$\le C(\Omega,p,\tau) \equiv (1+\lambda_1\tau/2)\lambda_1^{1/(p-1)}.$$

In particular, there exists some $\tau_1\in(0,\tau/2)$ such that

$$\int_\Omega u^p(\tau_1)\varphi_1 \le \frac{2}{\tau}\int_0^{\tau/2}\int_\Omega u^p\varphi_1 \le C(\Omega,p,\tau),$$

or in other words:

$$\|u(\tau_1)\|_{p,\delta} \le C(\Omega,p,\tau).$$

Since by assumption $p<(N+1)/(N-1)$, p is thus supercritical, that is, $p>\frac{(N+1)(p-1)}{2}$. We then deduce from the smoothing property (2.4) that

$$\|u(\tau_2)\|_\infty \le C(\Omega,p,\tau),$$

for some $\tau_2\in(\tau_1,\tau)$. Since $(N+1)/(N-1)\le(N+2)/(N-2)$, it is known from the result of Giga [G] (see also [Q1]) that $\|u(t)\|_\infty$ is bounded on $[\tau_2,\infty)$ by a constant depending only on $\|u(\tau_2)\|_\infty$ (and on Ω and p). The conclusion follows. □

3. Semilinear equation on a bounded domain, $N\le 3$.

THEOREM 3.1. ([Q2]) *Let $N\le 3$, $p<p_s$ and $\tau>0$. Then there exists $C=C(\Omega,p,\tau)>0$ such that any global positive solution u of (2.1) satisfies*

$$\sup_\Omega u(\cdot,t)\le C(\Omega,p,\tau) \qquad \textit{for}\quad t\ge\tau.$$

Sketch of the proof. Due to the a priori bound from [G] (or [Q1]), it is sufficient to show that for any $\tau>0$ there is $C=C(\Omega,p,\tau)>0$ such that for any global positive solution u it holds that $\|u(t)\|_{W^{1,2}(\Omega)}\le C$ for some $t\in[0,\tau]$.

Suppose the contrary. Then there exists a sequence u_k of global positive solutions and $\tau>0$ such that $\|u(t)\|_{W^{1,2}(\Omega)}>k$ for any $t\in[0,\tau]$. For $t_k\in(0,\tau)$ denote

$$M_k=\max_\Omega u_k(t_k)=u_k(x_k,t_k), \qquad \nu_k=M_k^{-(p-1)/2},$$
$$\Omega_k=\frac{1}{\nu_k}(\Omega-x_k), \qquad B_R^k=\{x\in\Omega_k:|x|<R\},$$
$$w_k(x)=\frac{1}{M_k}u_k(x_k+\nu_k x,t_k)$$
$$\tilde w_k(x)=M_k^{-p}(u_k)_t(x_k+\nu_k x,t_k).$$

Using energy estimates and Hardy's inequality it is possible to show that there is a sequence $\{t_k\} \subset (0,\tau)$ such that

$$\int_{B_R^k} |\tilde{w}_k(x)|^2 dx \to 0 \quad \text{as} \quad k \to \infty \tag{3.1}$$

for any $R > 0$. The function w_k satisfies $0 \le w_k \le 1 = w_k(0)$ and

$$\begin{aligned} \Delta w_k + w_k^p - \tilde{w}_k = 0, &\qquad x \in \Omega_k, \\ w_k = 0, &\qquad x \in \partial\Omega_k. \end{aligned}$$

Since $\{w_k\}$ is uniformly Hölder continuous on $B_{R/2}^k$ and (3.1) holds (the assumption $N \le 3$ was used to prove these two facts), one can pass to the limit and obtain a positive solution of one of the following two problems:

$$\begin{aligned} &\Delta w + w^p = 0 \quad \text{in} \quad \mathbb{R}^N, \\ &\Delta w + w^p = 0 \quad \text{in} \quad \mathbb{R}_+^N, \qquad w = 0 \quad \text{on} \quad \partial\mathbb{R}_+^N. \end{aligned}$$

But it is well known that neither of these problems has a positive solution if $p < p_s$. □

It is an open problem whether (2.2) holds for global positive solutions of (2.1) if $p < p_s$ and $N > 3$.

4. Semilinear equation in $\mathbb{R}^N$ in backward selfsimilar variables.

Consider now the problem

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

with $p > 1$. For $a \in \mathbb{R}^N$ and $T > 0$ we rescale the solution u which exists for $t \in [0,T)$ by setting

$$y = \frac{x-a}{\sqrt{T-t}}, \qquad s = -\log(T-t),$$

$$w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t).$$

Then $w = w_a$ is defined in $\mathbb{R}^N \times (s_0, \infty)$, $s_0 = -\log T$, and it satisfies

$$\begin{cases} w_s = \Delta w - \frac{1}{2} y \cdot \nabla w + |w|^{p-1}w - \frac{1}{p-1} w, & y \in \mathbb{R}^N, \quad s > s_0, \\ w(y,s_0) = T^{\frac{1}{p-1}} u_0(a + \sqrt{T}y), & y \in \mathbb{R}^N. \end{cases} \tag{4.2}$$

Boundedness of global solutions of (4.2) is then equivalent to the boundedness of the function $(T-t)^{\frac{1}{p-1}} \sup_{\mathbb{R}^N} |u(\cdot,t)|$. This fact was employed in [GK] in order to obtain results on the blow-up rate of u by showing that global solutions of (4.2) are bounded. The next theorem says that under some assumptions one has a universal bound away from $s = s_0$.

THEOREM 4.1. ([MS]) *Let* $\tau > 0$, $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$ *and assume one of the following:*

(i) $p < 1 + \frac{2}{N}$,

(ii) u_0 *is radially symmetric and nonincreasing in* $r = |x|$ *and*

$$(N-2)p < N+2 \qquad \text{if} \quad N \leq 3,$$
$$(N-2)p < N \qquad \text{if} \quad N > 3.$$

Then there is $C = C(\tau, p, N) > 0$ *such that for any global solution* w *of* (4.2) *it holds that*

$$w(\cdot, s) \leq C \qquad \text{for} \quad s \geq s_0 + \tau.$$

We present the main idea of the proof in the case (i). Multiplying the first equation in (4.2) by

$$\rho(y) = (4\pi)^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}}$$

and integrating we obtain

$$\rho * w(s) = \int_{\mathbb{R}^N} w(y,s)\,\rho(y)\,dy \leq (p-1)^{-\frac{1}{p-1}}, \qquad s \geq s_0. \tag{4.3}$$

This leads to the study of (4.2) in convolution Lebesgue spaces

$$L^q_{\rho,*} = \left\{ f \in L^q_{loc}(\mathbb{R}^N) \,:\, \|f\|^*_{q,\rho} = \left(\sup_{a \in \mathbb{R}^N} \int_{\mathbb{R}^N} |f(x)|^q \,\rho(a-x)\,dx \right)^{\frac{1}{q}} < \infty \right\}.$$

The result in the case (i) is a direct consequence of the bound (4.3) and of the smoothing property of (4.2) in these spaces. We remark that such a smoothing property does not hold in weighted Lebesgue spaces L^q_ρ.

5. Degenerate equation on a bounded domain.

In this section we discuss a universal bound for global weak solutions of the problem

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{5.1}$$

with $1 < m < p$ and $u_0^m \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$, Ω is a bounded domain in $\mathbb{R}^N$.

THEOREM 5.1. ([S]) *Assume that* $p < m\left(1 + \frac{2}{N}\right)$ *if* $N > 1$. *Then for any* $\tau > 0$ *there is* $C = C(\tau, \Omega, p, m)$ *such that any global solution of* (5.1) *satisfies*

$$u(\cdot, t) \leq C \qquad \text{for} \quad t \geq \tau.$$

To prove this result, first an a priori bound which depends on initial data is derived using energy and interpolation inequalities in a similar spirit as in [CL]. This a priori bound holds for

$$p < m + \frac{10m+2}{3N-4} \qquad \text{if} \quad N > 1.$$

Kaplan's eigenfunction method and smoothing properties of (5.1) yield then the universal bound.

6. Linear equation with a nonlinear boundary condition.

In [QS1], the following problem is considered:

$$\begin{cases} u_t = \Delta u - u, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = u^p, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{6.1}$$

where $p > 1$ and Ω is a bounded domain in $\mathbb{R}^N$.

THEOREM 6.1. ([QS1]) *Assume that either $p < 1 + \frac{1}{N}$ or $(N-2)p < N$, $N \leq 3$. Then for every $\tau > 0$ there exists $C = C(\Omega, p, \tau)$ such that any global solution u of (6.1) satisfies*

$$u(\cdot, t) \leq C \qquad \text{for} \quad t \geq \tau.$$

Again, an a priori bound which depends on $\sup_\Omega u_0$ is established first in [QS1]. This is achieved by a scaling argument in the spirit of [G]. The next step is a universal bound in $L^1(\Omega)$ for $t \geq 0$. The result in the case $p < 1 + 1/N$ follows then from an $L^1 - L^\infty$-estimate. A modification of the method from [Q2] (cf. Section 3) yields the conclusion when $(N-2)p < N$, $N \leq 3$.

7. Blow-up rate of the universal constant.

In this final section we discuss the initial blow-up rate or, in other words, the behavior of the universal constant C as $\tau \to 0$.

THEOREM 7.1. ([B], [MS]) *Assume $p > 1$, $0 < T < \infty$ and let u be a positive solution of*

$$u_t = \Delta u + u^p, \quad 0 < t < T, \quad x \in \mathbb{R}^N,$$

with $u(\cdot, t) \in L^\infty(\mathbb{R}^N)$ for $0 < t < T$. Assume further that one of the following holds:
(i) $(N-1)^2 p < N(N+2)$,
(ii) $u(\cdot, t)$ is radially symmetric and nonincreasing in $r = |x|$ and $(N-2)p < N+2$, $N \leq 3$.
Then there is $C = C(p, N) > 0$ such that

$$u(x,t) \leq C t^{-\frac{1}{p-1}}, \qquad x \in \mathbb{R}^N, \ 0 < t < \frac{T}{2}.$$

Under the assumption (i), this result was established in [B] using Bernstein type gradient estimates, Aronson-Serrin Harnack inequalities and multiplier arguments. The result in [B] is in fact local in the sense that

$$u(x,t) \le Ct^{-\frac{1}{p-1}}, \qquad x \in \omega,\ 0 < t < \frac{T}{2},$$

holds for positive solutions of

$$u_t = \Delta u + u^p, \quad 0 < t < T, \quad x \in \Omega,$$

where $\omega \subset\subset \Omega \subset \mathbb{R}^N$. The second part of Theorem 7.1 has been proved in [MS] by modifying the arguments from [Q2] (see Section 3).

THEOREM 7.2. ([QS2]) *Let the assumptions of Theorem 2.1 or Theorem 3.1 be satisfied. Then there are $c = c(\Omega, p) > 0$ and $\alpha = \alpha(p, N) > 0$ such that the constant C in Theorem 2.1 or 3.1 is of the form $C(\Omega, p, \tau) = c\max(\tau^{-\alpha}, 1)$. If $p < 1 + \frac{2}{N+1}$ then one can take $\alpha = (N+1)/2$.*

THEOREM 7.3. ([QS2]) *Assume that $(N-2)p < N$ or $p < p_s$, $N = 3$. Let u be a global solution of*

$$u_t = \Delta u + u^p - u, \quad x \in \Omega, \quad t > 0,$$
$$\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0,$$
$$u(x,0) = u_0(x) \ge 0, \quad x \in \Omega,$$

with $p > 1$ and $u_0 \in L^\infty(\Omega)$, Ω is a bounded domain in $\mathbb{R}^N$. Then there are $c = c(\Omega, p) > 0$ and $\alpha = \alpha(p, N) > 0$ such that $u(\cdot, t) \le c\max(t^{-\alpha}, 1)$. If $p < 1 + \frac{2}{N}$ then one can choose $\alpha = N/2$.

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