# HIGH ORDER SYMMETRIES OF VARIATIONS AND NONLINEAR QUASI-CONTRACTIVE ESTIMATES APPROACH TO THE PARABOLIC REGULARITY PROBLEMS

© ALEXANDER VAL.ANTONIOUK AND ALEXANDRA VICT.ANTONIOUK

Kiev, Ukraine

The correct definition of differential operators and different mathematical objects naturally requires the construction of functional spaces of their action and the study of associated regularity problems. There is Cauchy-Liouville-Picard regularity scheme, created initially for the study of regular dependence of solutions to differential equations with respect to the initial data and different kind parameters [1]-[4].

Let us briefly recall it in the adaptation to the simplest case of first order differential equation on line

$$y_t(x) = x - \int_0^t F(y_s(x))ds, \tag{1}$$

where nonlinear drift $F$ has all bounded derivatives

$$\exists K_n \qquad \sup_{x\in \mathbb{R}^1} |F^{(n)}(x)| \le K_n$$

in particular it is Lipschitz

$$\forall x, y \in \mathbb{R}^1 \quad |F(x) - F(y)| \le K_1 |x - y|.$$

## 1. Cauchy-Liouville-Picard regularity scheme.

The application of fixed point techniques leads to the existence of solution to (1) via iteration of

$$y_t^{n+1}(x) = x - \int_0^t F(y_s^n)ds.$$

After introduction of the implicit function

$$\{\Phi(x,y)\}_t = y_t - x + \int_0^t F(y_s)ds = 0, \tag{2}$$

the following estimate $||\Phi'_y - 1|| \le tK_1 \le \varepsilon$ gives for small $t$ the first order differentiability with respect to the initial data $\exists \frac{\partial}{\partial x} y_t(x) = -[\Phi'_y]\Phi'_x$. In a similar way the high order differentiability on initial data $\exists \frac{\partial^n}{\partial x^n} y_t(x)$ for small $t > 0$ can be derived from theorem on differentiability of implicit function. Finally, solution $y_t(x)$ is regular on initial data for all $t \geq 0$ due to the semigroup property of flow $y_{t+s}(x) = y_t(y_s(x))$.

For the associated semigroup $(P_t f)(x) = f(y_t(x))$ the Cauchy-Liouville-Picard scheme leads to the quasicontractive estimates on any order regularity $||P_t||_{L(C_b^n)} \leq \exp(tM_{K_1,\dots,K_n})$ in the standard spaces of continuously differentiable functions with bounded derivatives.

This scheme permits natural generalizations to the infinite dimensional space $X$. In particular, the development of infinite dimensional analogies of the implicit function techniques inspired the interpretation of variations as Frechet derivatives $y_t^{(n)}(x) \in B_n = L(X, B_{n-1})$, $B_0 = X$ and the study of semigroup in spaces of Frechet differentiable functions. In a similar way the introduction of the concept of full metric space and Browder index theorems were closely related with the fixed point results [1]-[4].

## 2. Nonlinear symmetries of variations.

What happens, however, for the essentially nonlinear equations? In this case the derivatives $F^{(n)}$ become unbounded, so the fixed point arguments and implicit function results become inapplicable. The traditional way to get round this difficulty uses some Lipschitz approximations of the initial equation and applies certain a priori estimates and imbedding theorems with the idea to recover the regularity of initial problem.

Unfortunately, there are counter-examples to the Cauchy-Liouville-Picard scheme [5], that demonstrate the inapplicability of the Frechet topologies in the essentially nonlinear case. Moreover, the modifications that account the influence of nonlinearity parameters must be entered.

In this report we develop further the direct methods for the study of nonlinear differential equations. Some steps were already reported at conferences [6]-[9] in applications to different finite and infinite dimensional problems.

We deal with the parabolic Cauchy problem

$$\frac{\partial}{\partial t} u(t,x) + Hu(t,x) = 0 \tag{3}$$

in unbounded domain $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, generated by operator

$$H = -\frac{1}{2} \operatorname{Tr} B(x) B^*(x) \frac{\partial^2}{\partial x^2} + < F(x), \frac{\partial}{\partial x} >$$

with unbounded globally non-Lipschitz coefficients. In particular, to the problem (3) corresponds a stochastic analogue of (1), the diffusion equation

$$y_t^x = x + \int_0^t B(y_s^x) dW_s - \int_0^t F(y_s^x) ds, \tag{4}$$

that generates the solutions to (3) via Kolmogorov representation

$$u(t,x) = (\exp(-tH)f)(x) = \mathbb{E}\, f(y_t^x),$$

$\mathbb{E}$ denotes the expectation with respect to the Wiener measure.

Below we obtain any order regularity of semigroup $\exp(-tH)$ in terms of nonlinear behaviour of drift $F$ and diffusion $B$. Conditions on coefficients are

1. Coercitivity and dissipativity: $\forall M\ \exists C_M, K_M^1, K_M^2$ such that

$$< F(x) - F(y), x - y > - M||B(x) - B(y)||_{HS}^2 \geq -C_M|x-y|^2 \qquad (5)$$

$$< F(x), x > - M||B(x)||_{HS}^2 \geq -K_M^1 - K_M^2|x|^2$$

where $||\cdot||_{HS}$ denotes Hilbert-Schmidt norm of matrix. By [10]-[12] this condition is necessary for the existence and uniqueness of solutions to (3)-(4). To achieve $C^\infty$ properties of process $y_t^x$ we also need the following assumption: $\forall M_1, M_2\ \exists K\ \forall x, h \in I\!R^n$

$$< h, F'(x)[h] > - M_1||B'(x)[h]\,||_{HS}^2 - M_2 \frac{|< h, B'(x)[h]B^*(x)x >|}{1+|x|^2} \geq -K|h|^2 \qquad (6)$$

where $H'(x)[h]$ means derivative in direction $h$ at point $x$.

2. Nonlinearity parameters: $\exists k_F,\ k_B\ ,\ k_B \leq k_F/2$ such that $\forall j\ \exists K_j$

$$||F^{(j)}(x) - F^{(j)}(y)|| \leq K_j|x-y|(1+|x|+|y|)^{k_F-1} \qquad (7)$$

$$||B^{(j)}(x) - B^{(j)}(y)|| \leq K_j|x-y|(1+|x|+|y|)^{k_B-1}$$

Last conditions generalize the Lipschitz behaviour to the polynomial class, restriction $k_B \leq k_F/2$ is natural in account of the coercitivity condition. In particular, (7) implies

$$||F^{(j)}(x)|| \leq K_j'(1+|x|)^{k_F}, \qquad ||B^{(j)}(x)|| \leq K_j'(1+|x|)^{k_B}. \qquad (7')$$

The idea lies in the backgrounds of variational calculus. Let us consider nonlinear functional $F(y)$ on state space $Y$ and denote by $d$ the derivative operation. Take $n^{th}$ order variation

$$d^nF(y) = d^{n-1}(F'(y)dy) = F'(y)d^ny + \sum_{j_1+..+j_s=n, s=2,n-1} F^{(s)}(y)d^{j_1}y..d^{j_s}y + F^{(n)}(y)[dy]^n \qquad (8)$$

The *main observation* is that the right hand side of (8) contains simultaneously $n^{th}$ variation $d^ny$ and first variation in $n^{th}$ degree $[dy]^n$. Similar symmetry is also reflected in the intermediate terms

$$d^{j_1}y...d^{j_s}y \ \leftrightarrow\ [dy]^{j_1}...[dy]^{j_s} \ \leftrightarrow\ [dy]^n \quad \text{because} \quad j_1+...+j_s = n \qquad (9)$$

and, moreover, is present for any order variation $\{d^jF\}_{j\geq 1}$.

This symmetry has immediate consequences for the nonlinear evolutional equations like (1), (4). For $\tau = \{j_1, ..., j_a\} \in \{1, ..., n\}^a$ consider the high order derivative $\partial_\tau = \dfrac{\partial}{\partial x_{j_a}} ... \dfrac{\partial}{\partial x_{j_1}}$ of diffusion process $y_t^x$

$$y_t^\tau = \partial_\tau y_t^x = \partial_\tau\,[\text{r.h.s. of (4)}] = in.d. + \int_0^t \partial_\tau B(y_t^x)dW_t - \int_0^t \partial_\tau F(y_t^x)dt =$$

$$= in.d. + \int_0^t \sum_{\gamma_1\cup..\cup\gamma_a=\tau,\ a\geq 1} B^{(a)}(y_t^x)[y_t^{\gamma_1},..,y_t^{\gamma_a}]dW_t - \int_0^t \sum_{\gamma_1\cup..\cup\gamma_a=\tau,\ a\geq 1} F^{(a)}(y_t^x)[y_t^{\gamma_1},..,y_t^{\gamma_a}]dt \tag{10}$$

with summations of similar to (8) structure, $H^{(a)}(x)[h_1,...,h_a]$ means $a^{th}$ order directional derivative at point $x\in \mathbb{R}^n$.

For equation (10) variation $y^\tau$ in the l.h.s. is proportional to the first variation in $|\tau|^{th}$ power in the r.h.s., or, in other words,

$$y^{(1)} \sim [y^{(n)}]^{1/n} \tag{11}$$

Let us introduce the homogeneous with respect to this symmetry expression

$$\rho_\tau(y,t) = \sum_{\gamma\subset\tau} \mathbb{E}\, p_{|\gamma|}(|y_t^x|^2)\cdot ||y_t^\gamma||^{m/|\gamma|} \tag{12}$$

that, due to $y_t^\gamma = \partial_\gamma y_t^x$, reflects the regularity of solutions with respect to the initial data. We demonstrate below that the knowledge of symmetry (11) is sufficient for the study of regularity in the essentially non-Lipschitz case. In particular, there is a hierarchy of weights $\{p\}$, determined by parameter $k_F$ that leads to quasi-contractive a priori estimate on regularity [5]-[9].

### 3. Nonlinear estimate on variations.

THEOREM. Suppose conditions 1.,2. hold. Let polynomial weights $p_j\geq 1$ fulfill hierarchy

$$\forall j_1+...+j_a=i \quad [p_i(z^2)]^i\,(1+|z|)^{mk_F} \leq [p_{j_1}(z^2)]^{j_1}...[p_{j_a}(z^2)]^{j_a}, \quad z\in \mathbb{R}_+ \quad (13).$$

Then

$$\exists M:\ \ \rho_\tau(y,t) = \sum_{\gamma\subset\tau} \mathbb{E} p_{|\gamma|}(|y_t^x|)\,||y_t^\gamma||^{m/|\gamma|} \leq e^{Mt}\rho_\tau(y,0)$$

.

**Proof.** The detail proof can be found in [13]-[16], here we outline how works symmetry (11). Applying Ito formula to each term in (12) and separating the coercitive part we have estimate [13,Lemma 3]

$$g_\gamma(t) = \mathbb{E} p_{|\gamma|}(|y_t^x|^2)||y_t^\gamma||^{m/|\gamma|} \leq g_\gamma(0) - \int_0^t \mathbb{E}\,[Hp_{|\gamma|}(|\cdot|^2)](y_t^x)\,||y_t^\gamma||^{m/|\gamma|}dt + \tag{15}$$

$$+\frac{m}{|\gamma|}\int_0^t \mathbb{E} p_{|\gamma|}(|y_t^x|^2)\,||y_t^\gamma||^{m/|\gamma|-2}\{-<y_t^\gamma, F'(y_t^x)[y_t^\gamma]> + (\frac{m}{|\gamma|}-1)||B'(y_t^x)[y_t^\gamma]\,||_{HS}^2 +$$

$$+2\frac{p'_{|\gamma|}}{p_{|\gamma|}} < B'(y_t^x)[y_t^\gamma]B^*(y_t^x)y_t^x, y_t^\gamma >\}dt + \tag{16}$$

$$+\frac{m}{|\gamma|}\int_0^t \mathbb{E}\, p_{|\gamma|}(|y_t^x|^2)\,||y_t^\gamma||^{m/|\gamma|-2}\{- < \Psi_\gamma^F, y_t^\gamma > + (\frac{m}{|\gamma|}-1)||\Psi_\gamma^B||_{HS}^2 +$$

$$+2\frac{p'_{|\gamma|}}{p_{|\gamma|}} < \Psi^B_\gamma B^*(y^x_t)y^x_t, y^\gamma_t >\}dt \tag{17}$$

where

$$\Psi^H_\gamma = \sum_{\alpha_1\cup\ldots\cup\alpha_a=\gamma,\ a\geq 2} H^{(a)}(y^x_t)[y^{\alpha_1}_t, \ldots, y^{\alpha_a}_t]. \tag{18}$$

For any polynomial $p$ term with generator (15) may be estimated by dissipativity assumption (5)

$$\exists M_p \quad [Hp(|\cdot|^2)](x) \geq -M_p p(|x|^2) \quad \forall x \in I\!R^n$$

The application of condition (6) and polynomiality of $p$: $|p'(z)|(1+|z|) \leq Kp(z)$ to (16) gives that (16) $\leq const \int_0^t g_\gamma(t)dt$ .

Finally, symmetries (11) are used for the treatment of expressions in (17). Let us demonstrate the estimation of first term. Using Hölder inequality and non-Lipschitzness (7) we obtain

$$(17)_1 \leq C\int_0^t g_\gamma(t)dt + C' \sum_{\alpha_1\cup..\cup\alpha_a=\gamma, a\geq 2} \int_0^t I\!E p_{|\gamma|}(|y^x_t|^2)\,||F^{(a)}(y^x_t)[y^{\alpha_1}_t, .., y^{\alpha_a}_t]\,||^{m/|\gamma|}dt \leq$$

$$\leq C\int_0^t g_\gamma(t)dt + C' \sum_{\alpha_1\cup..\cup\alpha_a=\gamma, a\geq 2} \int_0^t I\!E\, p_{|\gamma|}(|y^x_t|^2)(1+|y^x_t|)^{k_F m/|\gamma|} \prod_{j=1}^a ||y^{\alpha_j}_t||^{m/|\gamma|}dt \tag{19}$$

Using $|\alpha_1| + \ldots + |\alpha_a| = |\gamma|$ we apply hierarchy (13) and Hölder inequality with $q_i = |\gamma|/|\alpha_i|$, $\sum q_i = 1$ to get

$$(19)_{\alpha_1,\ldots,\alpha_a} \leq \int_0^t I\!E \prod_{j=1}^a (p_{|\alpha_j|}(|y^x_t|^2)\,||y^{\alpha_j}_t||^{m/|\alpha_j|})^{|\alpha_j|/|\gamma|}dt \leq const \sum_{\alpha\subset\gamma,\ |\alpha|<|\gamma|} \int_0^t g_\alpha(t)dt$$

where, due to $a \geq 2$ , in the last summation arise only the lower order terms.

The second term in (17) is estimated in a similar way with the help of (7). Using that

$$\frac{p'_{|\gamma|}}{p_{|\gamma|}}||\Psi^B_\gamma B^*(y^x_t)y^x_t|| \leq \frac{|y^x_t|}{1+|y^x_t|^2}||\Psi^B_\gamma||_{HS}\cdot||B(0) + \int_0^1 d\lambda \frac{d}{d\lambda}B(\lambda y^x_t)|| \leq$$

$$\leq (||B(0)|| + \sup_{\Lambda\in[0,1]} ||B'(\lambda y^x_t)||)\cdot||\Psi^B_\gamma||_{HS} \leq C(1+|y^x_t|)^{k_B}\,||\Psi^B_\gamma||$$

and $2k_B \leq k_F$ we estimate the third term in (17).

Induction on $|\gamma|$ in expression $\rho_\tau$ (12) finishes the proof.

## 4. High order differentiability of $y^x_t$ with respect to the initial data.

In a similar way nonlinear symmetries (11) permits to construct variational processes $y^x_t$ as solutions to the non-autonomous inhomogeneous with respect to $y^\tau_t$ problem (10) and prove their continuous in mean dependence on $x \in I\!R^n$ [13].

In particular, under conditions (5)-(7) we have $C^\infty$-differentiability in mean of process $\xi_t^x$ [13,Th.7]. This property again exploits symmetries (11): Using representation (10) we have

$$\frac{y_t^\tau(x+\varepsilon e_k)-y_t^\tau(x)}{\varepsilon}-y_t^{\tau\cup\{e_k\}}(x)=\int_0^t\Theta_\tau^B(\varepsilon)dW_t-\int_0^t\Theta_\tau^F(\varepsilon)dt \tag{20}$$

with

$$\Theta_\tau^H(\varepsilon)=\sum_{\gamma_1\cup\ldots\cup\gamma_a=\tau,\ a\geq1}\frac{1}{\varepsilon}H^{(a)}(y_t^{x+\delta e_k})[y_t^{\gamma_1}(x+\delta e_k),...,y_t^{\gamma_a}(x+\delta e_k)]\Big|_{\delta=0}^{\delta=\varepsilon}-$$

$$-\sum_{\alpha_1\cup\ldots\cup\alpha_b=\tau\cup\{k\},\ b\geq1}H^{(b)}(y_t^x)[y_t^{\alpha_1}(x),...,y_t^{\alpha_b}(x)]$$

By $D(\vec{y})-D(\vec{x})=D'(\vec{x})[\vec{y}-\vec{x}]+\int_0^1(D'(\vec{x}+\ell(\vec{y}-\vec{x}))-D'(\vec{x}))[\vec{y}-\vec{x}]d\ell$, function $\Theta$ can be transformed further to the form

$$\Theta_\tau^H(\varepsilon)=\sum_{\ldots}H^{(c)}(y_t^x)[\frac{y_t^{x+\varepsilon e_k}-y_t^x}{\varepsilon}-y_t^{\{k\}}(x),y_t^{\beta_1}(x),...,y_t^{\beta_a}(x)]+$$

$$+\sum_{\ldots}H^{(d)}(y_t^x)[y_t^{\delta_1}(x),...,\frac{y_t^{\delta_j}(x+\varepsilon e_k)-y_t^{\delta_j}(x)}{\varepsilon}-y_t^{\delta_j\cup\{k\}}(x),...,y_t^{\delta_b}(x)]+\text{ rest terms} \tag{21}$$

Therefore, during estimation of differences in (20), one comes to the same structure differences of lower order in (21). Furthermore, symmetry (11) works again for terms (21). In particular, it is also present in the rest terms in (21), resulting in the $C^\infty$ properties of $y_t^x$ with respect to the initial data.

## 5. Regular properties of parabolic problem (3).

It can be proved that the strong continuity in time of solutions $\exp(-tH)f$ fails to hold in the topologies of continuous differentiability. Thus analytic techniques do not work, nevertheless one may use Kolmogorov representation $(\exp(-tH)f)(x)=\mathbb{E}f(y_t^x)$. By direct differentiation

$$\partial_\tau(\exp(-tH)f)(x)=\partial_\tau\mathbb{E}\,f(y_t^x)=\mathbb{E}\,\partial_\tau f(y_t^x)=\text{ by (3) }=$$

$$=\sum_{\gamma_1\cup\ldots\cup\gamma_a=\tau,\ a\geq1}\mathbb{E}\,f^{(a)}(y_t^x)[y_t^{\gamma_1},...,y_t^{\gamma_a}]$$

Here variations represent some kind of convolutional kernels that relate differentiability of initial data $f(0,x)$ with its evolution $f(t,x)=(\exp(-tH)f)(x)$. Nonlinear estimate on $\rho_\tau$ makes the rest and one has result about regular properties.

THEOREM. Under conditions (5)-(7) the diffusion process $y_t^x$ is $C^\infty$-differentiable in mean with respect to the initial data $x\in\mathbb{R}^n$.

For a family of polynomial weights $\{q_j\in C^\infty(\mathbb{R}^1)\}_{j\geq0}$, hierarchied by

$$\forall j\geq0\quad q_j(z)(1+|z|)^{k_F}\leq q_{j+1}(z), \tag{22}$$

and any $m \in \mathbb{N}$ the semigroup $e^{-tH}$ preserves the space of continuously differentiable functions $C_{\bar{q}}^m(\mathbb{R}^n)$

$$\forall t \geq 0 \;\; e^{-tH} : C_{\bar{q}}^m(\mathbb{R}^n) \to C_{\bar{q}}^m(\mathbb{R}^n), \qquad \exists C_m, K_m \;\; \|e^{-tH}\|_{L(C_{\bar{q}}^m)} \leq C_m e^{tK_m},$$

topologized with norm

$$\|f\|_{C_{\bar{q}}^m}^2 = \sup_{|\tau| \leq m} \sup_{x \in \mathbb{R}^n} \frac{|\partial_\tau f(x)|^2}{q_{|\tau|}(|x|^2)}. \tag{23}$$

**Proof** may be found in [13],[14],[16].

**6. The raise of regularity of initial function via stochastic calculus of variations.**

In this part we discuss a new class of nonlinear estimates on variations, that lead to the raise of smoothness properties for solutions of (3)

$$\forall t > 0 \quad \exp(-tH) : C^n \to C^{n+m} \quad \forall n, m \geq 0$$

We use the dependence of diffusion process $y_t^x = y_t^x(\omega)$ (4) on the random parameter $\omega$, defined on the Wiener space $\omega = \{\omega_t\}_{t \geq 0} \in \Omega = C_0(\mathbb{R}_+, \mathbb{R}^n)$, index $0$ means that $\omega(0) = 0$.

The introduction of differential structure on Wiener space is a subject of Malliavin calculus [17]-[19]. For example, the continuous function $F : \Omega \to \mathbb{R}^1$ is stochastically differentiable iff for any bounded adapted continuous process $u_t(\omega)$ exists a limit

$$D_u F(\omega) = \lim_{\varepsilon \to 0} \frac{F(\omega_\varepsilon) - F(\omega)}{\varepsilon}, \quad \omega_\varepsilon(t) = \omega(t) + \varepsilon \int_0^t u_s(\omega) ds \tag{24}$$

in any $L^p(\Omega, P)$ space, $p \in (1, \infty)$, $P$ denotes Wiener measure.

The stochastic can be calculated for different objects, for example

$$D_u f(F_1, ..., F_m) = \sum_{j=1}^m \partial_j f(F_1, ..., F_m) D_u F_j \qquad D_u \int_0^t \alpha_s ds = \int_0^t (D_u \alpha_s) ds \tag{25}$$

$$D_u \int_0^t \sum_{j=1}^n \alpha_s^j dW_s^j = \int_0^t \sum_{j=1}^n (D_u \alpha_s^j) dW_s^j + \sum_{j=1}^n \int_0^t \alpha_s^j u_s^j ds$$

so one can work with the stochastic derivatives of diffusion processes $D_u y_t^x$, $D_u y_t^\tau$ by writing corresponding equations.

The importance of stochastic derivative may be expressed in the integration by parts formula for Wiener measure: for stochastically differentiable functions $F, G$

$$\mathbb{E}\, D_u F \cdot G = -\mathbb{E}\, F \cdot D_u G + \mathbb{E}\, FG \cdot \int_0^\infty u_s dW_s \tag{26}$$

therefore Wiener integral represents a dual stochastic gradient, acting on unit $D_u^* 1 = \int_0^\infty u_s dW_s$.

From (25)-(26) may be obtained different representations of semigroup derivative [17]-[20], like

$$\frac{\partial}{\partial x}(\exp(-tH)f)(x) = \mathbb{E}\, f(y_t^x) \cdot \{\frac{y_t^{(1)}}{D_u y_t^x} \int_0^t u_t dW_t - D_u(\frac{y_t^{(1)}}{D_u y_t^x})\} \tag{27}$$

or similar with the Malliavin determinant, that lead to the raise of smoothness properties: for integrable expression in brackets $\{...\}$ from continuity of $f$ follows $C^1$ differentiability of semigroup flow $\exp(-tH)f$ in the l.h.s. for any $t > 0$, i.e. the raise of smoothness.

Unfortunately, because $(1/x)' = -1/x^2$ changes the sign, the coercitivity and dissipativity assumptions are violated for the fraction $1/D_u y_t^x$ and representations (27) become singular in the essentially nonlinear case. Therefore the research in the Malliavin calculus was mainly concentrated on the Lipschitz classes of diffusions, that naturally arise, for example, on the compact manifolds.

The solution of singularities was found in [8], [20], it exploits that field $u$ in (27) is a free parameter and one can find $\tilde{u}$ such that fraction $\dfrac{y_t^{(1)}}{D_u y_t^x}$ transforms to factor $1/t$. In particular, it becomes possible to provide the high order analogies to (27):

THEOREM [20]. Let $\Psi_t v = B^{-1}(y_t^x)\partial_v y_t^x$ for directional derivative $\partial_v = < v, \dfrac{\partial}{\partial x} >$, $v \in \mathbb{R}^n$. Then

$$D_{\Psi_t v} f(\xi_t^x) = t\partial_v f(\xi_t^x),$$

for $f \in C^1_{pol}(\mathbb{R}^n)$ of polynomial behaviour with derivatives. Moreover, the high order raise of smoothness representations hold

$$\partial_{v_1}...\partial_{v_j}[\exp(-tH)f](x) = \frac{1}{t^j}\mathbb{E} f(y_t^x)\Gamma_{v_1}...\Gamma_{v_j} 1 \tag{28}$$

Operator $\Gamma_v$ is defined on differentiable functions on $\mathbb{R}^n \times \Omega$ by formula

$$\Gamma_v h(t,x,\omega) = \{t\partial_v - D_{\Psi_t v} + \int_0^t \Psi_t v \cdot dW_t\} h(t,x,\omega). \tag{29}$$

From representations (28)-(29) follows that the verification of the differentiability property requires estimates on the mixed derivatives $\{\partial_v^x, D_{\Psi_t v}\}$ of process $y_t^x$.

The nonlinear quasi-contractive estimates on the pure derivatives with respect to the initial data $\partial_v^x$ of process $y_t^x$ were already obtained in previous sections. Because by chain rule (25) the stochastic variation $D_{\Psi_t v}$ represents derivation operation, the nonlinear symmetries (9), (11) arise again. Direct calculations [15] lead to the new symmetries

$$\mathbb{D}^\beta y_t^\alpha \approx t^{|\beta|}[y^{(1)}]^{|\beta|} y_t^\alpha \approx t^{|\beta|}[y^{(1)}]^{|\alpha|+|\beta|}$$

where we introduced notation $\mathbb{D}^j = D_{\Psi_t e_j}$ and $\mathbb{D}^\beta = \mathbb{D}^{j_a}...\mathbb{D}^{j_1}$ for $\beta = \{j_1, ..., j_a\}$.

Similarly, a certain hierarchy of weights leads to the nonlinear estimate on mixed derivatives $\mathbf{D}_k = \{\partial_k, \mathbb{D}^k\}$

$$\rho_\tau(y,t) = \sum_{\alpha\cup\beta\subset\tau} \mathbb{E}\, p_{\alpha,\beta}(|y_t^x|^2)\,||\frac{\mathbf{D}_{k_b}...\mathbf{D}_{k_1} y_t^x}{t^{|\beta|}}||^{m/|\alpha|+|\beta|} \le e^{Mt}\rho_\tau(y,0),$$

for $\alpha \cup \beta = \{k_1, ..., k_b\}$. Via representations (28) this estimate implies the raise of smoothness property:

THEOREM. Under conditions (5)-(7) the process $y_t^x$ and its variations $y_t^\gamma$ are stochastically differentiable.

Suppose that diffusion coefficient is uniformly invertable $\exists K: \forall x \in I\!R^n \, ||B^{-1}(x)||_{HS} \le K$. Then for any continuous function $f \in C_{pol}(I\!R^n)$ of no more than polynomial behaviour the evolution $\exp(-tH)f \in C^\infty_{pol}(I\!R^n)$ represents $C^\infty$-smooth function of no more than polynomial behaviour with derivatives.

Moreover, for any polynomial $q_0 \in C_{pol}(I\!R_+)$ such that

$$\lim_{R\to\infty} \sup_{||x||| \ge R} \frac{|f(x)|}{q_0(|x|^2)} = 0$$

the smoothing estimate holds

$$||\exp(-tH)f||_{C^m_{\vec{q}}(I\!R^n)} \le \frac{K_m e^{tM_m}}{t^{m/2}} ||f||_{C_{q_0}(I\!R^n)} \qquad (30).$$

The spaces $C^m_{\vec{q}}$ are constructed like in (23) with hierarchy (22), generated by function $q_0$.

**Proof** may be found in [13], [16]. We remark finally that restriction of uniform invertability of $B$ can be weakened to the polynomial estimate $||B^{-1}(x)||_{HS} \le K(1 + |x|)^d$. In this case the hierarchies of topologies $C^m_{\vec{q}}(I\!R^n)$ in the raise of smoothness estimates should be generated by factors $(1 + |x|)^{k_F + d}$.

## REFERENCES


1. Hadamard J., *Lectures on Cauchy's problem in Linear Partial Differential Equations*, Yale Univ. Press, New Haven, 1923. Reprinted by Dover, New York, 1952.
2. Kamke E., *Differentialgleichungen reeler functionen*, Mathematik in Monographen und Lehrbüchern, v.7, Academische Verlag mbH, Leipzig 1930.
3. Klein F., *Vorlesungen über die Entwicklung der Mathematik in 19 Jahrhundert*, Grundlehren der Math. Wiss., Springer Verlag, Berlin 1926.
4. Simmons G.F., *Differential equations with applications and historical notes*, Internat. Series in Pure and Appl. Math., McGraw-Hill Book Comp., 1972.
5. Antoniouk A.Val., Antoniouk A.Vict., *Nonlinear Effects in $C^\infty$-smooth properties of Infinite dimensional evolutions. Classical Gibbs models*, submitted to Springer-Verlag, 197 pp.
6. Antoniouk A.Val., Antoniouk A.Vict., *Quasi-contractive smooth nonlinear calculus for flows, generated by ordinary and stochastic differential equations,*, Bielefeld "Classic and quantum evolutions", 1995.
7. Antoniouk A.Val., Antoniouk A.Vict., *Nonlinear calculus of variations and properties of infinite dimensional semigroups, associated with nonlinear stochastic differential equations*, Barcelona "Stochastic Analysis" 1997.
8. Antoniouk A.Val., Antoniouk A.Vict., *Nonlinear calculus of variations in applications to the $C^\infty$ - properties of general non-Lipschitz diffusions*, "Munich Stochastic days" 1998.
9. Antoniouk A.Val., Antoniouk A.Vict., *High order symmetries of variational calculus and Cauchy-Liouville-Picard regularity scheme*, Lviv "NPDE" 1999.
10. Krylov N.V., Rosovskii B.L., *Cauchy problem for linear stochastic PDE*, Izvestia Acad. Nauk USSR **6** (1977), 1329-1347.

11. Krylov N.V., Rosovskii B.L., *On the evolutionary stochastic equations*, Ser. "Contemporary problems of Mathematics", VINITI, Moscow **14** (1979), 71-146.
12. Pardoux E., *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics **3** (1979), 127-167.
13. Antoniouk A.Val., Antoniouk A.Vict., *Nonlinear estimates on regularity of non-Lipschitz diffusions*, Collection of papers, Kiev Inst.of Math. (1999), 1-83.
14. Antoniouk A.Val., Antoniouk A.Vict., *$C^\infty$ regularity on initial data for nonlinear diffusion equations via quasi-contractive nonlinear estimates*, submitted to Theor.Probab. and Rel.Fields.
15. Antoniouk A.Val., Antoniouk A.Vict., *Nonlinear diffusions and their variations as stochastically differentiable functionals on Wiener space*, submitted to Theor.Probab. and Rel.Fields.
16. Antoniouk A.Val., Antoniouk A.Vict., *Smooth and smoothing properties of Feller semigroups, generated by essentially non-Lipschitz diffusions*, submitted to Theor.Probab. and Rel.Fields.
17. Bismut J.-M., *Large deviations and the Malliavin calculus*, Progress in Mathematics, v. 45, Birkhäuser, 1984.
18. Malliavin P., *Stochastic Analysis*, Springer Verlag, 1997.
19. Nualart D., *The Malliavin calculus and related topics*, Springer-Verlag, 1995.
20. Antoniouk A.Val., Antoniouk A.Vict., *High order formulas for derivatives of nonlinear diffusion semigroups*, Ukrainian Math.Journ., 2001, N 1, 117-122 pp.

DEPT. OF NONLINEAR ANALYSIS,
KIEV INST. OF MATHEMATICS NASU,
TERESCHENKIVSKA 3,
01601 MSP KIEV-4, UKRAINE
*E-mail address*: antoniouk@imath.kiev.ua