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SOME RESULTS ON MP-INJECTIVITY AND MGP-INJECTIVITY OF RINGS AND MODULES

ДЕЯКІ РЕЗУЛЬТАТИ ПРО MP-ІН'ЄКТИВНІСТЬ ТА MGP-ІН'ЄКТИВНІСТЬ КІЛЕЦЬ ТА МОДУЛІВ

We study MP-injective rings and MGP-injective rings satisfying some additional conditions. Using the concepts of MP-injectivity and MGP-injectivity of rings and modules, we present some new characterizations of QF-rings, semisimple Artinian rings, strongly regular rings, and simple Artinian rings.

Вивчаються MP-ін'єктивні та MGP-ін'єктивні кільця, що задовольняють деякі додаткові умови. Із застосуванням понять MP-ін'єктивності та MGP-ін'єктивності кілець та модулів наведено нові характеризації QF-кілець, напівпростих кілець Артіна, сильно регулярних кілець та простих кілець Артіна.

1. Introduction. Throughout this article, R is an associative ring with an identity. For a subset X of R, the right and left annihilators of X are denoted by $r(X)$ and $l(X)$, respectively. To facilitate, $\mathbf{r}(a)$ is called a special right annihilator of R for each $a \in R$. The Jacobson radical of R is denoted by $J = J(R)$, the right singular ideal of R is denoted by $Z_r = Z(R_R)$. The right socle of R is denoted by $S_r = \mathrm{Soc}(R_R)$. Let M be an R-module and N be a submodule of M, following [1], we write $N \subseteq^{\mathrm{ess}} M$ to indicate that N is an essential submodule of M. Concepts which have not been explained can be found in [1] and [2].

Recall that a ring R is *right P-injective* [3] if every R-homomorphism from a principal right ideal of R to R extends to an endomorphism of R. A ring R is *right generalized principally injective* (briefly *right GP-injective*) [4] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R. GP-injective rings are studied in papers [4 – 8]. In [8], GP-injective rings are called *YJ-injective* rings.

In [2], the concepts of right P-injective rings and right GP-injective rings are generalized to *right MP-injective* rings and *right MGP-injective* rings, respectively, and some interesting results on these rings are obtained. Following [2], a right R-module N is *MP-injective* if, for every R-monomorphism from a principal right ideal of R to N extends to a homomorphism of R to N, the ring R is right MP-injective if R_R is MP-injective; a right R-module N is MGP-injective if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any R-monomorphism from a^nR to N extends to a homomorphism of R to N, the ring R is right MGP-injective if R_R is MGP-injective. In this paper, we shall study some new properties of MP-injective rings and MGP-injective rings, and give some new characterizations of QF-rings rings, semisimple artinian rings, von Neumann regular rings, strongly regular rings and simple artinian rings by MP-injectivity and MGP-injectivity of rings and modules.

2. Results. Recall that a ring R is QF if it is right or left self-injective and right or left artinian, a ring R is *semiregular* if $R/J(R)$ is von Neumann regular and idempotents can be lifted modulo $J(R)$, a ring R is right *CF* if every cyclic right R-module embeds in a free module, a ring R is *right mininjective* if every R-homomorphism from a minimal right ideal of R to R extends to an endomorphism of R. These concepts can be found in [1]. It is well known that right CF-rings are left P-injective [1] (Lemma 7.2 (1)); and a ring R is QF if and only if R is right artinian and right and left mininjective [9] (Corollary 4.8). According to [10], a ring R is *right* 2-*simple injective* if every R-homomorphism from a 2-generated right ideal of R to R with simple image extends to an endomorphism of R.

Theorem 2.1. *Let R be a right MGP-injective ring. Then the following statements are equivalent*:

(1) *R is a QF-ring*;

(2) *R is a right* 2-*simple injective ring with ACC on right annihilators*;

(3) *R is right CF-ring and the ascending chain* $\mathbf{r}(a_1) \subseteq \mathbf{r}(a_2a_1) \subseteq \mathbf{r}(a_3a_2a_1) \subseteq \ldots$ *terminates for every sequence* $\{a_1, a_2, \ldots\} \subseteq R$;

(4) *R is a semiregular right CF-ring.*

Proof. (1) ⇒ (2). Since a QF-ring is right self-injective and right noetherian, so (1) implies (2).

(2) ⇒ (1). Suppose (2) holds. Then since R is a right MGP-injective ring with ACC on right annihilators, by [2] (Corollary 3.12(1)), R is semiprimary. Noting that R is right 2-simple injective, by [10] (Theorem 17(17)), R is a QF-ring.

(1) ⇒ (3). Assume (1). Then since every injective module over a QF-ring is projective, so every right R-module embeds in a free module, and hence R is a right CF-ring. Note that a QF-ring is right noetherian, the last assertion of (3) is clear.

(3) ⇒ (4). By [2] (Theorem 3.11), R is right perfect, so that it is semiregular.

(4) ⇒ (1). Note that right MGP-injectivity implies that $J(R) = Z_r$ by [2] (Theorem 3.4(2)), so R is right artinian by [11] (Corollary 2.9). Since R is right and left mininjective, by [9] (Corollary 4.8), R is QF.

Theorem 2.1 is proved.

***Corollary* 2.1** ([12], Corollary 3)**.** *The following statements are equivalent for a ring R*:

(1) *R is a QF-ring*;

(2) *R is a right* 2-*injective ring with ACC on right annihilators.*

Lemma 2.1. *Let R be a left noetherian ring. If I is an ideal of R and* $\mathbf{r}(I) \subseteq^{\text{ess}} R_R$, *then* I *is nilpotent.*

Proof. Since R is left noetherian and $\mathbf{r}(I^i)$ is an ideal for each positive integer i, there exists $k \geq 1$ such that $\mathbf{r}(I^k) = \mathbf{r}(I^{k+1}) = \ldots$. If I is not nilpotent, choose $\mathbf{l}(x)$ maximal in $\{\mathbf{l}(y) \mid I^k y \neq 0\}$. Then $I^{2k}x \neq 0$ because $\mathbf{r}(I^{2k}) = \mathbf{r}(I^k)$, so there exists $a \in I^k$ such that $I^k ax \neq 0$. Since $\mathbf{r}(I) \subseteq \mathbf{r}(I^k)$ and $\mathbf{r}(I) \subseteq^{\text{ess}} R_R$, we have that $\mathbf{r}(I^k) \subseteq^{\text{ess}} R_R$. Thus $axR \cap \mathbf{r}(I^k) \neq 0$, say $0 \neq axb \in \mathbf{r}(I^k)$, then, $I^k xb \neq 0$ and $I^k a \subseteq \mathbf{l}(xb)$ but $I^k a \nsubseteq \mathbf{l}(x)$, which contradicts the maximality of $\mathbf{l}(x)$. Therefore I is nilpotent.

Lemma 2.1 is proved.

Theorem 2.2. *Let R be a left noetherian right MGP-injective ring. Then*:

(1) $\mathbf{r}(J) \subseteq^{\text{ess}} R_R$;

(2) *J is nilpotent*;

(3) $\mathbf{r}(J) \subseteq^{\text{ess}} {}_RR$;

(4) $\mathbf{lr}(J) = J$.

Proof. (1). Let $0 \neq x \in R$. Since R is left noetherian, the non-empty set $\mathcal{F} = \{\mathbf{l}((xa)^k) \mid a \in R,\, k > 0 \text{ such that } (xa)^k \neq 0\}$ has a maximal element, say $\mathbf{l}((xy)^n)$.

We claim that $J(xy)^n = 0$. If not, then there exists $t \in J$ such that $t(xy)^n \neq 0$. Since R is right MGP-injective, there exists a positive integer m such that $(t(xy)^n)^m \neq 0$ and $b \in R(t(xy)^n)^m$ for every $b \in R$ with $\mathbf{r}((t(xy)^n)^m) = \mathbf{r}(b)$. Write $(t(xy)^n)^m = s(xy)^n$, where $s = (t(xy)^n)^{m-1}t \in J$. We proceed with the following two cases.

Case 1: $\mathbf{r}((xy)^n) = \mathbf{r}(s(xy)^n)$. Then $(xy)^n = cs(xy)^n$, i.e., $(1 - cs)(xy)^n = 0$. Since $s \in J$, $1 - cs$ is invertible. So we have $(xy)^n = 0$. This is a contradiction.

Case 2: $\mathbf{r}((xy)^n) \neq \mathbf{r}(s(xy)^n)$. Then there exists $u \in \mathbf{r}(s(xy)^n)$ but $u \notin \mathbf{r}((xy)^n)$. Thus, $s(xy)^n u = 0$ and $(xy)^n u \neq 0$. This shows that $s \in \mathbf{l}((xy)^n u)$ and $\mathbf{l}((xy)^n u) \in \mathcal{F}$. Noting that $s \notin \mathbf{l}((xy)^n)$, so the inclusion $\mathbf{l}((xy)^n) \subset \mathbf{l}((xy)^n u)$ is strict. This contracts the maximality of $\mathbf{l}((xy)^n)$ in $\mathcal{F}$.

Thus, $J(xy)^n = 0$, and so $0 \neq (xy)^n \in xR \cap \mathbf{r}(J)$, proving (1).

(2). By (1) and Lemma 2.1.

(3). If $0 \neq c \in R$, we must show that $Rc \cap \mathbf{r}(J) \neq 0$. This is clear if $Jc = 0$. Otherwise, since J is nilpotent by (2), there exists $m \geq 1$ such that $J^m c \neq 0$ but $J^{m+1}c = 0$. Then $0 \neq J^m c \subseteq Rc \cap \mathbf{r}(J)$, as required.

(4). By (1) and [2] (Theorem 3.4), $\mathbf{lr}(J) \subseteq Z_r = J$, so that $\mathbf{lr}(J) = J$.

Theorem 2.2 is proved.

Theorem 2.3. *Let R be a left noetherian right MGP-injective ring. Then the following statements are equivalent*:

(1) *R is right Kasch*;

(2) *R is left* C_2;

(3) *R is left* GC_2;

(4) *R is semilocal*;

(5) *R is left artinian*;

(6) *the ascending chain* $\mathbf{r}(a_1) \subseteq \mathbf{r}(a_2a_1) \subseteq \mathbf{r}(a_3a_2a_1) \subseteq \ldots$ *terminates for every sequence* $\{a_1, a_2, \ldots\} \subseteq R$.

Proof. (1) $\Rightarrow$ (2). By [1] (Proposition 1.46).

(2) $\Rightarrow$ (3); and (5)$\Rightarrow$ (6) are obvious.

(3) $\Rightarrow$ (4). Since left noetherian ring is left finite dimensional, and left finite dimensional left GC_2 ring is semilocal [13] (Lemma 1.1), so (4) follows from (3).

(4) $\Rightarrow$ (5). Since R is left noetherian right MGP-injective, by Theorem 2.2(2), J is nilpotent. And so R is left noetherian and semiprimary by hypothesis, as required.

(5) $\Rightarrow$ (1). Assume (5). Then R is semiperfect right mininjective ring and $S_r \subseteq^{\text{ess}} R_R$. So that R is a right minfull ring. By [1] (Theorem 3.12), R is right Kasch.

(6) $\Rightarrow$ (4). By [2] (Theorem 3.11).

Theorem 2.3 is proved.

***Corollary* 2.2.** *Let R be a left noetherian right MGP-injective right finite dimensional ring. Then R is left artinian.*

Proof. Since R is right MGP-injective, by [2] (Theorem 3.4(1)), R is right GC_2. But right finite dimensional right GC_2 ring is semilocal, so R is left artinian by Theorem 2.3.

***Corollary* 2.3.** *The following statements are equivalent for a ring R*:

(1) *R is a QF-ring*;

(2) *R is left artinian and right* 2*-injective*;

(3) *R is left noetherian right* 2*-injective and right Kasch*;

(4) *R is a left noetherian right* 2*-injective semilocal ring*;

(5) *R is left noetherian right* 2*-injective and left* C_2;

(6) *R is left noetherian right* 2*-injective and left* GC_2;

(7) *R is left noetherian right* 2*-injective and the ascending chain* $\mathbf{r}(a_1) \subseteq \mathbf{r}(a_2a_1) \subseteq \subseteq \mathbf{r}(a_3a_2a_1) \subseteq \ldots$ *terminates for every sequence* $\{a_1, a_2, \ldots\} \subseteq R$;

(8) *R is left noetherian right* 2*-injective and right finite dimensional.*

Proof. By Theorem 2.3 (2) through (7) are equivalent. (1) ⇒ (8) is clear. (8) ⇒ (2) by Corollary 2.2. (2) ⇒ (1) by [10] (Theorem 17).

Lemma 2.2. *Let M be a right R-module and* $N_R \subseteq^{\mathrm{ess}} M_R$. *Then* $(N : x) \subseteq^{\mathrm{ess}} R_R$ *for all* $x \in M$, *where* $(N : x) = \{a \in R \mid xa \in N\}$.

Proof. Let $x \in M$. For each $0 \neq a \in R$, if $xa = 0$, then $a \in (N : x)$, thus $0 \neq aR = (N : x) \cap aR$. If $xa \neq 0$, then since $N \subseteq^{\mathrm{ess}} M$, $N \cap xaR \neq 0$, so that there exists $0 \neq xar \in N$, and thus $0 \neq ar \in (N : x) \cap aR$. Hence, $(N : x) \subseteq^{\mathrm{ess}} R_R$.

Lemma 2.2 is proved.

Theorem 2.4. *The following conditions are equivalent for a ring* R:

(1) *R is a semisimple artinian ring*;

(2) *R is right Kasch and every simple right R-module is MGP-injective*;

(3) *R is right Kasch and every simple right R-module is mininjective.*

Proof. It is obvious that (1) ⇒ (2) ⇒ (3).

(3) ⇒ (1). For any right R-module A, let $E(A)$ be the injective hull of A. If $A \neq \neq E(A)$, then there exists $x \in E(A) - A$. By Lemma 2.2, we have $(A : x) \subseteq^{\mathrm{ess}} R_R$. Clearly, $(A : x) \neq R$. Thus there exists a maximal right ideal M of R such that $(A : x) \subseteq M$. Clearly, $M \subseteq^{\mathrm{ess}} R_R$. Since R is right Kasch, there exists $0 \neq a \in R$ such that $M = \mathbf{r}(a)$. Now we define $f\colon aR \to R/\mathbf{r}(a); ay \mapsto y + \mathbf{r}(a)$, then f is a right R-homomorphism. Since aR is a minimal right ideal and $R/\mathbf{r}(a)$ is a simple right R-module, by hypothesis, there is $b \in R$ such that $1 + \mathbf{r}(a) = f(a) = ba + \mathbf{r}(a)$, which yields that $1 - ba \in \mathbf{r}(a)$, and so $a = aba$. Let $e = ba$, then $0 \neq e = e^2$. It follows that $M = \mathbf{r}(e) = (1 - e)R$, and then $M \cap eR = 0$ but $eR \neq 0$, which contradicts that $M \subseteq^{\mathrm{ess}} R_R$. Hence, $A = E(A)$, i.e., A is injective. Therefore R is a semisimple artinian ring.

Theorem 2.4 is proved.

The following Lemma 2.3 (1) and (2) are well-known results, we give their proof here for completeness.

Lemma 2.3. *Let R be a prime ring, then*:

(1) *if I is a nonzero ideal of R, then I is essential in R both as a left ideal and as a right ideal*;

(2) *if R is a semisimple artinian ring, then it is a simple artinian ring*;

(3) *if R satisfies the ascending chain condition for special right annihilators, then* $Z_r = 0$.

Proof. (1). If K is a right ideal of R satisfies $K \cap I = 0$, then $KI \subseteq K \cap I = 0$. Since R is a prime ring, $K = 0$, and so I is an essential right ideal of R. Similarly, I is an essential left ideal of R.

(2). Let I be a nonzero ideal of R. Then since R is a semisimple artinian ring, there exists a right ideal T of R such that $I \oplus T = R$. By (1), $T = 0$, and thus $I = R$. This proves that R is a simple artinian ring.

(3). Since R satisfies the ascending chain conditions for special right annihilators, the set $\{\mathbf{r}(x) \mid 0 \neq x \in R\}$ has a maximal element $\mathbf{r}(a)$. If $Z_r \neq 0$, then $aZ_ra \neq 0$ because R is a prime ring (otherwise, if $aZ_ra = 0$, then $aZ_r(aR) = 0$, and so $aZ_r = 0$, i.e., $(Ra)Z_r = 0$, which implies that $Z_r = 0$, a contradiction). Thus there is $b \in Z_r$ such that $aba \neq 0$. It follows from the maximality of $\mathbf{r}(a)$ that $\mathbf{r}(a) = \mathbf{r}(aba)$. Since $aba \in Z_r$, we have $\mathbf{r}(a) = \mathbf{r}(aba) \subseteq^{\text{ess}} R_R$, and whence $\mathbf{r}(a) \cap baR \neq 0$. So that there exists $c \in R$ such that $bac \neq 0$ and $abac = 0$, which implies that $c \in \mathbf{r}(aba) = \mathbf{r}(a)$. Thus $ac = 0$, and then $bac = 0$ which contradicts $bac \neq 0$. Therefore $Z_r = 0$.

Lemma 2.3 is proved.

Theorem 2.5. *The following statements are equivalent for a ring R*:

(1) *R is a simple artinian ring*;

(2) *R is a right MGP-injective prime ring such that the ascending chain $\mathbf{r}(a_1) \subseteq \subseteq \mathbf{r}(a_2a_1) \subseteq \mathbf{r}(a_3a_2a_1) \subseteq \ldots$ terminates for every sequence $\{a_1, a_2, \ldots\} \subseteq R$*;

(3) *R is a prime ring such that $0 \neq S_r$ is MP-injective, and R satisfies the ascending chain condition for special right annihilators.*

Proof. It is obvious that (1) ⇒ (2) and (3).

(2) ⇒ (1). By [2] (Theorem 3.17) and Lemma 2.3(2).

(3) ⇒ (1). We first prove that R is a semisimple artinian ring. If not, then $S_r \neq R$. Since R satisfies the ascending chain conditions for special right annihilators, the set $\{\mathbf{r}(x) \mid x \in R - S_r\}$ has a maximal element $\mathbf{r}(a)$. By Lemma 2.3(3), there exists a nonzero right ideal T of R such that $\mathbf{r}(a) \oplus T \subseteq^{\text{ess}} R_R$. By Lemma 2.3(1), $T \cap S_r \neq 0$, so that there exists $0 \neq b \in T \cap S_r$. Now we define $f\colon abR \to S_r$; $abx \mapsto bx$, then f is a right R-monomorphism. Since S_r is MP-injective, then there is $y \in S_r$ such that $b = f(ab) = yab$, which implies that $(a - aya)b = 0$, i.e., $b \in \mathbf{r}(a - aya)$. Since $a \notin S_r$ and $y \in S_r$, $a - aya \notin S_r$. By the maximality of $\mathbf{r}(a)$, we have $\mathbf{r}(a) = \mathbf{r}(a - aya)$. It follows that $ab = 0$, and so $b = yab = 0$, which contradicts $b \neq 0$. Therefore, $S_r = R$, i.e., R is a semisimple artinian ring. Since R is a prime ring, by Lemma 2.3(2), R is a simple artinian ring.

Theorem 2.5 is proved.

Recall that a ring R is a *right SF ring* if every simple right R-module is flat, a ring R is a *right quasi-duo ring* if every maximal right ideal of R is an ideal, a ring R is a quasi-duo ring if it is left or right quasi-duo. These concepts can be found in [14].

Proposition 2.1. *If every maximal essential right ideal of R is MP-injective, then R is a right SF ring.*

Proof. Let S be a simple right R-module, then there exists a maximal right ideal M of R such that $S \cong R/M$. If M is essential right ideal, then by hypothesis, M is MP-injective. So for any $a \in R$, if $y = xa \in Ra \cap M$, then since the inclusion mapping $yR \to M$ extends to a right R-homomorphism $f\colon R \to M$, so that $y = = f(y) = f(1)y = (f(1)x)a \in Ma$. Hence $Ra \cap M = Ma$, this shows that M is a pure submodule of R, and therefore R/M is flat.

Proposition 2.1 is proved.

Definition 2.1. *Let R be a ring. A right R-module N is WMGP-injective if, for any $a \in R$, there exists a positive integer n such that any R-monomorphism from a^nR to N extends to a homomorphism of R to N. The ring R is right WMGP-injective if R_R is WMGP-injective.*

***Example* 2.1.** *Let* $R = \left\{ \begin{bmatrix} a & v \\ 0 & a \end{bmatrix} \middle| \, a \in F, \ v \in V \right\}$ *be the trivial extension of the field F by the two-dimensional vector space V over F. Then R is a commutative WMGP-injective ring that is not MGP-injective.*

Proof. Let $V = uF \oplus wF$. For any $x \in R$, write $x = \begin{bmatrix} a & v \\ 0 & a \end{bmatrix}$. If $a \neq 0$, then x is invertible, so $xR = R$, and thus any R-homomorphism from xR to R extends to an endomorphism of R. If $a = 0$, then $x^2 = 0$, and so any R-homomorphism from x^2R to R extends to an endomorphism of R. Hence, R is WMGP-injective. Let $x_0 = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}$, $y_0 = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}$, then ${x_0}^2 = 0$, $\mathbf{r}(x_0) = \mathbf{r}(y_0) = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$, but $Rx_0 = \begin{bmatrix} 0 & Fu \\ 0 & 0 \end{bmatrix}$ and $Ry_0 = \begin{bmatrix} 0 & Fw \\ 0 & 0 \end{bmatrix}$. So $Ry_0 \nsubseteq Rx_0$. This shows that the R-monomorphism from x_0R to R via $x_0r \mapsto y_0r$ can not be extended to an endomorphism of R. whence R is not MGP-injective.

Proposition 2.1 is proved.

Next, we give some new characterizations of strongly regular rings.

Theorem 2.6. *The following conditions are equivalent for a ring R:*

(1) *R is a strongly regular ring*;

(2) *every maximal right ideal of R is MGP-injective and* $\mathbf{l}(a)$ *is an ideal for each* $a \in R$;

(3) *R is a reduced ring and every maximal essential right ideal of R is MGP-injective*;

(4) *R is a reduced ring and every maximal essential right ideal of R is WMGP-injective or a right annihilator*;

(5) *R is a quasi-duo ring, and every maximal essential right ideal of R is MP-injective.*

Proof. $(1) \Rightarrow (2)$. Since R is a strongly regular ring, by [15] (Proposition 12.3), R is von Neumann regular and every left ideal is two-sided, so (2) holds.

$(2) \Rightarrow (3)$. We need only to prove that R is reduced. Let $a \in R$ with $a^2 = 0$, we claim that $a = 0$. Otherwise, if $a \neq 0$, then $a \in \mathbf{l}(a) \neq R$. By (2), $\mathbf{l}(a)$ is an ideal, so there exists a maximal right ideal M such that $\mathbf{l}(a) \subseteq M$. Since M is MGP-injective, the inclusion mapping $aR \to M$ extends to a homomorphism from R to M, and so there exists $b \in M$ such that $a = ba$. Thus $1 - b \in \mathbf{l}(a) \subseteq M$, and then $1 \in M$, a contradiction. Therefore $a = 0$, and hence R is reduced.

$(3) \Rightarrow (4)$. Since right MGP-injective module is right WMGP-injective, so (3) implies (4).

$(4) \Rightarrow (1)$. For any $a \in R$, we claim that $aR + \mathbf{r}(a) = R$. In fact, if $aR + \mathbf{r}(a) \neq R$, then there exists a maximal right ideal M of R such that $aR + \mathbf{r}(a) \subseteq M$. We claim that M is an essential right ideal. Otherwise, there there exists $0 \neq b \in R$ such that $bR \cap M = 0$. Since M is a maximal right ideal, $bR \oplus M = R$, and so $M = eR$ for some $e^2 = e \in R$. Clearly, $a = ea$, i.e., $1 - e \in \mathbf{l}(a)$. Since aR is reduced, $\mathbf{l}(a) \subseteq \mathbf{r}(a)$, so that $1 - e \in \mathbf{r}(a) \subseteq M$, and hence $1 \in M$, a contradiction. Therefore M is a maximal essential right ideal. By hypothesis, M is WMGP-injective or a right annihilator.

Case 1: If M is WMGP-injective. Then there exists a positive integer n such that any R-monomorphism from a^nR to M extends to a homomorphism of R to M. Now we define $f\colon a^nR \to M$ by $f(a^nx) = ax$, where $x \in R$, noting that R is reduced, by [2] (Lemma 3.20), f is well-defined, and f is a right R-homomorphism, and so there exist $u \in M$ such that $a = ua^n$. Thus, $1 - ua^{n-1} \in \mathbf{l}(a) \subseteq \mathbf{r}(a) \subseteq M$, it follows that $1 \in M$, a contradiction.

Case 2: If M is a right annihilator. Then there exists $0 \neq c \in R$ such that $M = \mathbf{r}(c)$. Thus, $c \in \mathbf{lr}(c) = \mathbf{l}(M) \subseteq \mathbf{l}(a) \subseteq \mathbf{r}(a) \subseteq M = \mathbf{r}(c)$, so that $c^2 = 0$. Since R is reduced, $c = 0$, a contradiction too.

Therefore, these contradictions show that $aR + \mathbf{r}(a) = R$. Write $1 = as + t$, where $s \in R$, $t \in \mathbf{r}(a)$, then $a = a^2s + at = a^2s$. Consequently, R is strongly regular.

$(5) \Rightarrow (1)$. By Proposition 2.1 and [14] (Theorem 4.10).

Theorem 2.6 is proved.

Theorem 2.7. *If R is a right MGP-injective ring, then it is a classical quotient ring, and so every right* (*left*) *R-module is divisible.*

Proof. Let $\mathbf{l}(a) = \mathbf{r}(a) = 0$. Then $\mathbf{l}(a^k) = \mathbf{r}(a^k) = 0$ for every positive integer k. By the right MGP-injectivity of R, there exists a positive integer n such that $b \in Ra^n$ for every $b \in R$ with $\mathbf{r}(a^n) = \mathbf{r}(b)$, in particular, $1 = ca^n$ for some $c \in R$. Thus, $a^nca^n = a^n$, noticing that $\mathbf{l}(a^n) = \mathbf{r}(a^n) = 0$, we have $a^nc = ca^n = 1$. Hence R is a classical quotient ring, and so every right (left) R-module is divisible.

Theorem 2.7 is proved.

Proposition 2.2. *If every maximal essential right ideal of R is WMGP-injective or a right annihilator, then R is a classical quotient ring.*

Proof. Let a be a nonzero divisor of R, i.e., $\mathbf{l}(a) = \mathbf{r}(a) = 0$. Then there exists a right ideal K such that $aR \oplus K \subseteq^{\text{ess}} R_R$. We claim that $aR \oplus K = R$. If not, then there exists a maximal right ideal M such that $aR \oplus K \subseteq M$, and so M is WMGP-injective or a right annihilator. If M is WMGP-injective, then there exists a positive integer n such that every monomorphism from a^nR to M extends to a homomorphism of R to M. Now define $f\colon a^nR \to M$ by $f(a^nx) = ax$, where $x \in R$, then f is well defined as a is a nonzero divisor, and so $a = f(a^n) = ba^n$ for some $b \in M$. This follows that $1 - ba^{n-1} \in \mathbf{l}(a) = 0$, and then $1 \in M$, a contradiction. If M is a right annihilator, then since M is a maximal right ideal, there exists $0 \neq t \in R$ such that $M = \mathbf{r}(t)$. Hence, $t \in \mathbf{lr}(t) = \mathbf{l}(M) \subseteq \mathbf{l}(a) = 0$, i.e., $t = 0$, a contradiction too. Thus, $aR \oplus K = R$. Write $aR = eR$, where $e^2 = e$, then $a = ea$ and $e = ac$ for some $c \in R$, and so $a = aca$. Noting that a is a nonzero divisor, we have $ac = ca = 1$. This shows that R is a classical quotient ring.

Proposition 2.2 is proved.

At the end of this paper, we give an important property of semiprime right MGP-injective rings.

Proposition 2.3. *If R is a semiprime right MGP-injective ring, then R contains a unique largest reduced ideal I, and $I = \mathbf{rl}(I) = \mathbf{lr}(I)$, $Z({}_RI) = Z(I_R) = 0$.*

Proof. Let $I = \sum_{\alpha \in A} I_\alpha$ be the sum of all reduced ideals I_α of R. It may be assumed that $I \neq 0$. We prove that $\mathbf{rl}(I)$ is reduced. Otherwise, then there exists $0 \neq x \in \mathbf{rl}(I)$ such that $x^2 = 0$.

Case 1: $xR \cap I_\alpha = 0$ for all $\alpha \in A$. Then $xRI_\alpha \subseteq xR \cap I_\alpha = 0$ for all $\alpha \in A$, and so $xRI = 0$, $xR \subseteq \mathbf{l}(I)$. It follows that $xRx = 0$. But R is semiprime, $x = 0$, a contradiction.

Case 2: There is $i \in A$ such that $xR \cap I_i \neq 0$. Take $0 \neq a \in xR \cap I_i$, then aR is reduced. For any $y \in \mathbf{r}(a^2)$, since $(aya)^2 = ay(a^2y)a = 0$ and $aya \in aR$, we have $aya = 0$, and then $(ay)^2 = (aya)y = 0$, which implies that $ay = 0$. Hence, $\mathbf{r}(a^2) = \mathbf{r}(a)$. By the proof of [2] (Lemma 3.20), we have that $\mathbf{r}(a^k) = \mathbf{r}(a)$ for every positive integer k. If $a^2 = 0$, then $a = 0$, a contradiction. If $a^2 \neq 0$. Since R is right MGP-injective, by [2] (Theorem 3.2), there exists a positive integer n such that $a^{2n} \neq 0$ and $a = ba^{2n}$ for some $b \in R$. Write $c = ba^{2n-2}$, then $a = ca^2$. It is easy to see that $(a - aca)^2 = 0$, $a - aca \in aR$, so $a = aca$. Let $e = ac$, then $e^2 = e$, $a = ea$, $e \in aR \subseteq xR$. Thus, there exists $d \in R$ such that $e = xd, (ex)^2 = ex^2dx = 0$. But $ex \in aR$, so $ex = 0$, and whence $e = e^2 = exd = 0$, this follows that $a = 0$, a contradiction too.

Therefore, $\mathbf{rl}(I)$ is reduced. Noting that $\mathbf{rl}(I)$ is an ideal and $I \subseteq \mathbf{rl}(I)$, we have $I = \mathbf{rl}(I)$, and so I is the unique largest reduced ideal. Since R is semiprime, it is easy to see that $\mathbf{r}(K) = \mathbf{l}(K)$ for every ideal K of R. Noting that I and $\mathbf{l}(I)$ are ideals, we have $\mathbf{lr}(I) = \mathbf{ll}(I) = \mathbf{rl}(I) = I$.

It is obvious that $Z(I_R) = I \cap Z_r$. Assume that $I \cap Z_r \neq 0$, then there exists $0 \neq y \in I \cap Z_r$. Since $\mathbf{r}(y)$ is an essential right ideal, $\mathbf{r}(y) \cap yR \neq 0$, and so there is $0 \neq yz \in \mathbf{r}(y)$. Thus, $y^2z = 0, (yzy)^2 = yz(y^2z)y = 0$. But $yzy \in I$ and I is reduced, so $yzy = 0$, $(yz)^2 = (yzy)z = 0$, $yz \in I$, and hence $yz = 0$, which contradicts $yz \neq 0$. Consequently, $Z(I_R) = 0$. Similarly, $Z({}_RI) = 0$.

Proposition 2.3 is proved.

1. *Nicholson W. K., Yousif M. F.* Quasi-Frobenius rings. – Cambridge: Cambridge Univ. Press, 2003.
2. *Zhu Z. M.* MP-injective rings and MGP-injective rings // Indian J. Pure and Appl. Math. – 2010. – **41**. – P. 627 – 645.
3. *Nicholson W. K., Yousif M. F.* Principally injective rings // J. Algebra. – 1995. – **174**. – P. 77 – 93.
4. *Nam S. B., Kim N. K., Kim J. Y.* On simple GP-injective modules // Communs Algebra. – 1995. – **23**. – P. 5437 – 5444.
5. *Chen J. L., Ding N. Q.* On general principally injective rings // Communs Algebra. – 1999. – **27**. – P. 2097 – 2116.
6. *Chen J. L., Ding N. Q.* On regularity of rings // Algebra Colloq. – 2001. – **8**. – P. 267 – 274.
7. *Chen J. L., Zhou Y. Q., Zhu Z. M.* GP-injective rings need not be P-injective // Communs Algebra. – 2005. – **33**. – P. 2395 – 2402.
8. *Yue Chi Ming R.* On regular rings and self-injective rings. II // Glas. mat. – 1983. – **18**. – P. 221 – 229.
9. *Nicholson W. K., Yousif M. F.* Mininjective rings // J. Algebra. – 1997. – **187**. – P. 548 – 578.
10. *Zhu Z. M., Chen J. L.* 2-Simple injective rings // Int. J. Algebra. – 2010. – **4**. – P. 25 – 37.
11. *Chen J. L., Li W. X.* On artiness of right CF rings // Communs Algebra. – 2004. – **32**. – P. 4485 – 4494.
12. *Rutter E. A.* Rings with the principle extension property // Communs Algebra. – 1975. – **3**. – P. 203 – 212.
13. *Zhou Y. Q.* Rings in which certain right ideals are direct summands of annihilators // J. Austral. Math. Soc. – **73**. – 2002. – P. 335 – 346.
14. *Rege M. B.* On von Neumann regular rings and SF-rings // Math. Jap. – 1986. – **31**. – P. 927 – 936.
15. *Stenström B.* Rings of quotients. – Berlin etc.: Springer, 1975.

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