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WEYL'S THEOREM FOR ALGEBRAICALLY $wF(p, r, q)$ OPERATORS WITH $p, r > 0$ AND $q \geq 1$

ТЕОРЕМА ВЕЙЛЯ ДЛЯ ОПЕРАТОРІВ, ЩО АЛГЕБРАЇЧНО НАЛЕЖАТЬ КЛАСУ $wF(p, r, q)$ ПРИ $p, r > 0$ І $q \geq 1$

If T or T^* is an algebraically $wF(p, r, q)$ operator with $p, r > 0$ and $q \geq 1$ acting on an infinite-dimensional separable Hilbert space, then we prove that the Weyl theorem holds for $f(T)$, for every $f \in \mathrm{Hol}(\sigma(T))$, where $\mathrm{Hol}(\sigma(T))$ denotes the set of all analytic functions in an open neighborhood of $\sigma(T)$. Moreover, if T^* is a $wF(p, r, q)$ operator with $p, r > 0$ and $q \geq 1$, then the a-Weyl theorem holds for $f(T)$. Also, if T or T^* is an algebraically $wF(p, r, q)$ operators with $p, r > 0$ and $q \geq 1$, then we establish spectral mapping theorems for the Weyl spectrum and essential approximate point spectrum of T for every $f \in \mathrm{Hol}(\sigma(T))$, respectively. Finally, we examine the stability of the Weyl theorem and a-Weyl theorem under commutative perturbation by finite-rank operators.

У випадку, коли T або T^* — оператори, що алгебраїчно належать класу $wF(p, r, q)$, де $p, r > 0$, $q \geq 1$, і діють на нескінченновимірному сепарабельному гільбертовому просторі, доведено, що теорема Вейля виконується для $f(T)$ при кожному $f \in \mathrm{Hol}(\sigma(T))$, де $\mathrm{Hol}(\sigma(T))$ — множина всіх аналітичних функцій у відкритому околі $\sigma(T)$. Крім того, якщо T^* — оператор класу $wF(p, r, q)$, де $p, r > 0$ і $q \geq 1$, то a-теорема Вейля виконується для $f(T)$. У випадку, коли T або T^* — оператори, що алгебраїчно належать класу $wF(p, r, q)$ при $p, r > 0$ і $q \geq 1$, встановлено теореми про спектральне відображення, відповідно, для спектра Вейля та для істотного наближеного точкового спектра оператора T для кожного $f \in \mathrm{Hol}(\sigma(T))$. Досліджено стійкість теореми Вейля та a-теореми Вейля при комутативному збуренні операторами скінченного рангу.

1. Introduction. Throughout this paper let $\mathbf{B}(\mathcal{H})$, $\mathbf{F}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, denote, respectively, the algebra of bounded linear operators, the ideal of finite rank operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in \mathbf{B}(\mathcal{H})$ we shall write $\ker(T)$ and $\mathcal{R}(T)$ for the null space and range of T, respectively. Also, let $\alpha(T) := \dim \ker(T)$, $\beta(T) := \dim \mathcal{R}(T)$, and let $\sigma(T), \sigma_a(T), \sigma_p(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T \in \mathbf{B}(\mathcal{H})$ is called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite codimension. The index of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T).$$

T is called *Weyl* if it is Fredholm of index 0, and *Browder* if it is Fredholm "of finite ascent and descent".

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. The essential spectrum $\sigma_F(T)$, the Weyl spectrum $\sigma_W(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_F(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$$

respectively. Evidently

$$\sigma_F(T) \subseteq \sigma_W(T) \subseteq \sigma_b(T) \subseteq \sigma_F(T) \cup acc\sigma(T),$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$.

Following [1], we say that *Weyl's theorem* holds for T if $\sigma(T) \setminus \sigma_W(T) = E_0(T)$, where $E_0(T)$ is the set of all eigenvalues λ of finite multiplicity isolated in $\sigma(T)$. And *Browder's theorem* holds for T if $\sigma(T) \setminus \sigma_W(T) = \pi_0(T)$, where π_0 is the set of all poles of T of finite rank.

Let $SF(\mathcal{H})$ be the class of all semi-Fredholm operators on $\mathcal{H}$. Let $SF_+(\mathcal{H})$ be the class of all upper semi-Fredholm operators, $SF_+^-(\mathcal{H})$ be the class of all $T \in SF_+(\mathcal{H})$ with $i(T) \leq 0$, and for any $T \in \mathbf{B}(\mathcal{H})$, let

$$\sigma_{SF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF(\mathcal{H})\},$$

$$\sigma_{SF_+^-}(T) = \left\{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(\mathcal{H})\right\},$$

$\rho_{SF}(T) = \mathbb{C} \setminus \sigma_{SF}(T)$ and $\rho_{SF_+^-}(T) = \mathbb{C} \setminus \sigma_{SF_+^-}(T)$.

In [2] Berkani define the class of B-Fredholm operators as follows. For each integer n, define T_n to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_0 = T$). If for some n the range $\mathcal{R}(T^n)$ is closed and T_n is Fredholm (resp. semi-Fredholm) operator, then T is called a *B-Fredholm* (resp. *semi-B-Fredholm*) operator. In this case and from [2] T_m is a Fredholm operator and $i(T_m) = i(T_n)$ for each $m \geq n$. The index of a B-Fredholm operator T is defined as the index of the Fredholm operator T_n, where n is any integer such that the range $\mathcal{R}(T^n)$ is closed and T_n is Fredholm operator (see [2]). Let $SBF(\mathcal{H})$ be the class of all semi-B-Fredholm operators on $\mathcal{H}$. For $T \in \mathbf{B}(\mathcal{H})$, let

$$\sigma_{SBF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF(\mathcal{H})\},$$

$\rho_{SBF}(T) = \mathbb{C} \setminus \sigma_{SBF}(T)$.

Let E_0^a be the set of all eigenvalues of T of finite multiplicity which are isolated in $\sigma_a(T)$. According to [3], we say that T satisfies *a-Weyl's theorem* if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$ and a-Browder's theorem holds for T if $\sigma_{SF_+^-}(T) = \sigma_{ab}(T)$. It follows from [3] (Corollary 2.5) a-Weyl's theorem implies Weyl's theorem.

It follows from [3, 4] that

$$a\text{-Weyl's theorem} \Longrightarrow \text{Weyl's theorem} \Longrightarrow \text{Browder's theorem},$$

$$a\text{-Weyl's theorem} \Longrightarrow a\text{-Browder's theorem} \Longrightarrow \text{Browder's theorem}.$$

The investigation of operators obeying Weyl's theorem, a-Weyl's theorem, Browder's theorem or a-Browder's theorem was studied by many mathematicians [1, 3 – 9] and the references cited therein.

Following [10], we say that $T \in \mathbf{B}(\mathcal{H})$ has the *single-valued extension property* (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f \colon U_\lambda \longrightarrow \mathcal{H}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathbf{B}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic function it easily follows that $T \in \mathbf{B}(\mathcal{H})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [11] (Proposition 1.8), Laursen proved that if T is of finite ascent, then T has SVEP.

Proposition 1.1 [12]. *Let $T \in \mathbf{B}(\mathcal{H})$.*

(i) *If T has the SVEP, then $i(T-\lambda I) \leq 0$ for every $\lambda \in \rho_{SBF}(T)$.*

(ii) *If T^* has the SVEP, then $i(T-\lambda I) \geq 0$ for every $\lambda \in \rho_{SBF}(T)$.*

Definition 1.1 [13]. *Let $T \in \mathbf{B}(\mathcal{H})$ and $n, d \in \mathbb{N}$. Then T has a uniform descent for $n \geq d$ if $\mathcal{R}(T) + \ker(T^n) = \mathcal{R}(T) + \ker(T^d)$ for all $n \geq d$. If, in addition, $\mathcal{R}(T) + \ker(T^d)$ is closed, then T is said to have topological uniform descent for $n \geq d$.*

2. Properties of algebraically $wF(p,r,q)$ operators with $p,r>0$ and $q \geq 1$. A bounded linear operator $T \in \mathbf{B}(\mathcal{H})$ belongs to the class $wF(p,q,r)$ for each $p,r>0$ and $q \geq 1$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{1/q} \geq |T^*|^{2(p+r)/q}$$

and

$$|T|^{2(p+r)(1-1/q)} \geq \left(|T|^p|T^*|^{2r}|T|^p\right)^{(1-1/q)}.$$

This class has been introduced by Yang and Yuan, see [14]. An operator $T \in \mathbf{B}(\mathcal{H})$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator $T \in \mathbf{B}(\mathcal{H})$ is called *normaloid* if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of T. $T \in \mathbf{B}(\mathcal{H})$ is called *convexoid* if $\operatorname{conv}\sigma(T) = \overline{W(T)}$, where $W(T)$ is the numerical range of T. $X \in \mathbf{B}(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. $S \in \mathbf{B}(\mathcal{H})$ is said to be a *quasiaffine transform* of $T \in \mathbf{B}(\mathcal{H})$ (notation: $S \prec T$) if there is a quasiaffinity $X \in \mathbf{B}(\mathcal{H})$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$ then we say that S and T are *quasisimilar.*

In general, the following implications hold:

class $wF(p,r,q) \Longrightarrow$ algebraically class $wF(p,r,q)$ for each $p,r>0$ and $q \geq 1$.

The following facts follow from the above definition and some well known facts about class $wF(p,r,q)$ for each $p,r>0$ and $q \geq 1$.

(i) If $T \in \mathbf{B}(\mathcal{H})$ is algebraically class $wF(p,r,q)$ for each $p,r>0$ and $q \geq 1$ then so is $T-\lambda I$ for each $\lambda \in \mathbb{C}$.

(ii) If $T \in \mathbf{B}(\mathcal{H})$ is algebraically class $wF(p,r,q)$ for each $p,r>0$ and $q \geq 1$ and M is a closed T-invariant subspace of $\mathcal{H}$ then $T|_M$ is algebraically class $wF(p,r,q)$ for each $p,r>0$ and $q \geq 1$.

***Remark* 2.1.** In what follows, we use the notation wF to denote the class $wF(p,r,q)$ operators with $p,r>0$ and $q \geq 1$.

Lemma 2.1. *Let $T \in \mathbf{B}(\mathcal{H})$ belong to class $wF(p,r,q)$ with $p,r>0$ and $q \geq 1$. Let $\lambda \in \mathbb{C}$. Assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$.*

Proof. We consider two cases:

Case I ($\lambda = 0$): Since T belongs class wF for each $p,r>0$ and $q \geq 1$, T is normaloid. Therefore $T = 0$.

Case II ($\lambda \neq 0$): Here T is invertible, and since T belongs class wF for each $p,r>0$ and $q \geq 1$, we see that T^{-1} is also belongs to class wF for each $p,r>0$ and $q \geq 1$. Therefore T^{-1} is normaloid. On the other hand, $\sigma(T^{-1}) = \{1/\lambda\}$, so $\|T\|\|T^{-1}\| = |\lambda|\,|1/\lambda| = 1$. It follows that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda$.

Lemma 2.1 is proved.

Proposition 2.1. *Let T be a quasinilpotent algebraically wF operator. Then T is nilpotent.*

Proof. Assume that $p(T)$ is wF operator for some nonconstant polynomial p. Since $\sigma(p(T)) = p(\sigma(T))$, the operator $p(T) - p(0)$ is quasinilpotent. Thus Lemma 2.1 would imply that

$$cT^m(T - \lambda_1 I)\ldots(T - \lambda_n I) \equiv p(T) - p(0) = 0,$$

where $m \geq 1$. Since $T - \lambda_j I$ is invertible for every $\lambda_j \neq 0$, we must have $T^m = 0$.

Proposition 2.1 is proved.

An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be polaroid if $\operatorname{iso}\sigma(T) \subseteq \pi(T)$, where $\pi(T)$ is the set of all poles of T. In general, if T is polaroid then it is isoloid. However, the converse is not true. Consider the following example. Let $T \in \ell^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right).$$

Then T is a compact quasinilpotent operator with $\alpha(T) = 1$, and so T is isoloid. However, since T does not have finite ascent, T is not polaroid.

In [15] they showed that every wF operator is isoloid. We can prove more:

Proposition 2.2. *Let T be an algebraically wF operator. Then T is polaroid.*

Proof. Suppose T is an algebraically wF operator. Then $p(T)$ is wF for some nonconstant polynomial p. Let $\lambda \in \operatorname{iso}(\sigma(T))$. Using the spectral projection $P :=$ $:= \dfrac{1}{2i\pi}\displaystyle\int_{\partial D}(\mu - T)^{-1}\,d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \sigma(T_1) = \{\lambda\}, \quad \text{and} \quad \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is algebraically class wF and $\sigma(T_1) = \{\lambda\}$. But $\sigma(T_1 - \lambda I) = \{0\}$ it follows from Proposition 2.1 that $T_1 - \lambda I$ is nilpotent. Therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda I$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda I$ has finite ascent and descent. Therefore λ is a pole of the resolvent of T. Thus if $\lambda \in \operatorname{iso}(\sigma(T))$ implies $\lambda \in \pi(T)$, and so $\operatorname{iso}(\sigma(T)) \subset \pi(T)$. Hence T is polaroid.

Proposition 2.2 is proved.

***Corollary* 2.1.** *Let T be an algebraically wF operator. Then T is isoloid.*

For $T \in \mathbf{B}(\mathcal{H})$, $\lambda \in \sigma(T)$ is said to be a regular point if there exists $S \in \mathbf{B}(\mathcal{H})$ such that $T - \lambda I = (T - \lambda I)S(T - \lambda I)$. T is is called reguloid if every isolated point of $\sigma(T)$ is a regular point. It is well known [16] (Theorems 4.6.4 and 8.4.4) that $T - \lambda I = (T - \lambda I)S(T - \lambda I)$ for some $S \in \mathbf{B}(\mathcal{H}) \Longleftrightarrow T - \lambda I$ has a closed range.

Theorem 2.1. *Let T be an algebraically wF operator. Then T is reguloid.*

Proof. Suppose T is an algebraically wF operator. Then $p(T)$ is wF for some nonconstant polynomial p. Let $\lambda \in \operatorname{iso}(\sigma(T))$. Using the spectral projection $P :=$ $:= \dfrac{1}{2i\pi}\displaystyle\int_{\partial D}(\mu - T)^{-1}\,d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \sigma(T_1) = \{\lambda\}, \quad \text{and} \quad \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is algebraically class wF and $\sigma(T_1) = \{\lambda\}$, it follows from Lemma 2.1 that $T_1 = \lambda I$. Therefore by [15] (Theorem 2.10),

$$\mathcal{H} = E(\mathcal{H}) \oplus E(\mathcal{H})^{\perp} = \ker(T - \lambda I) \oplus \ker(T - \lambda I)^{\perp}. \tag{2.1}$$

Relative to decomposition (2.1), $T = \lambda I \oplus T_2$. Therefore $T - \lambda I = 0 \oplus T - \lambda I$ and hence $\operatorname{ran}(T - \lambda I) = (T - \lambda I)(\mathcal{H}) = 0 \oplus (T_2 - \lambda I)(\ker(T - \lambda I)^{\perp})$. Since $T_2 - \lambda I$ is invertible, $T - \lambda I$ has closed range.

Theorem 2.1 is proved.

3. Weyl's theorem for algebraically wF operators with $p, r > 0$ and $q \geq 1$.

Theorem 3.1. *Suppose T or T^* is an algebraically class $wF(p, r, q)$ operator for $p, r > 0$ and $q \geq 1$. Then Weyl's theorem holds for $f(T)$ for every $f \in \operatorname{Hol}(\sigma(T))$.*

Proof. Suppose that T is algebraically class wF. We first show that Weyl's theorem holds for T. Let $\lambda \in \sigma(T) \setminus \sigma_W(T)$. Then $T - \lambda I$ is Weyl but not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume to the contrary that λ is an interior point of $\sigma(T)$. Then there exists a neighborhood U of λ such that $\dim \ker(T - \mu) > 0$ for all $\mu \in U$. Then it follows from [10] (Theorem 10) that T does not have SVEP. On the other hand, since $p(T)$ is of class $wF(p, r, q)$ for some nonconstant polynomial p, it follows from [15] and [10] (Proposition 1.8) that $p(T)$ has SVEP. Hence by [17], T has SVEP. This is a contradiction. Therefore $\lambda \in \partial\sigma(T) \setminus \sigma_W(T)$, and it follows from the punctured neighborhood theorem that $\lambda \in E_0(T)$. Conversely, suppose that $\lambda \in E_0(T)$. Using the spectral projection $P := \dfrac{1}{2i\pi}\displaystyle\int_{\partial D} (\mu - T)^{-1}\, d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \sigma(T_1) = \{\lambda\}, \quad \text{and} \quad \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is of class wF and $\sigma(T_1) = \{\lambda\}$, it follows from Lemma 2.1 that $T_1 - \lambda I$ is nilpotent. Since $\lambda \in E_0(T)$, $T - \lambda I$ is a finite dimensional operator, so $T - \lambda I$ is Weyl. Since $T_2 - \lambda I$ is invertible, $T_2 - \lambda I$ is Weyl. Thus Weyl's theorem holds for T. Now we claim that $\sigma_W(f(T)) = f(\sigma_W(T))$ for all $f \in \operatorname{Hol}(\sigma(T))$. Let $f \in \operatorname{Hol}(\sigma(T))$. Since $\sigma_W(f(T)) \subseteq f(\sigma_W(T))$ with no other restriction on T, it suffices to show that $f(\sigma_W(T)) \subseteq \sigma_W(f(T))$. Suppose that $\lambda \notin \sigma_W(f(T))$. Then $f(T) - \lambda$ is Weyl and

$$f(T) - \lambda I = c(T - \alpha_1 I)(T - \alpha_2 I) \ldots (T - \alpha_n) g(T), \tag{3.1}$$

where $c, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $g(T)$ is invertible. Since the operators in the right-hand side of equation (3.1) commute, every $T - \alpha_i I$ is Fredholm. Since T is algebraically class $wF(p, r, q)$ for each $p, r > 0$ and $n \geq 1$, T has SVEP. It follows from Proposition 1.1 that $i(T - \alpha_k I) \leq 0$ for each $k = 1, \ldots, n$. Therefore $\lambda \notin f(\sigma_W(T))$. Now recall [17] that if T is isoloid then

$$f(\sigma(T) \setminus E_0(T)) = f(\sigma(T)) \setminus f(E_0(T)) \quad \text{for every} \quad f \in \operatorname{Hol}(\sigma(T)).$$

Since T is isoloid by Corollary 2.1 and Weyl's theorem holds for T,

$$f(\sigma(T)) \setminus f(E_0(T)) = f(\sigma(T) \setminus E_0(T)) = f(\sigma_W(T)) = \sigma_W(f(T))$$

$$\text{for every} \quad f \in \mathrm{Hol}(\sigma(T)),$$

which implies that Weyl's theorem holds for $f(T)$.

Now suppose that T^* is algebraically class $wF(p,r,q)$ for each $p,r>0$ and $q \geq 1$. We first show that Weyls theorem holds for T. Suppose that $\lambda \in \sigma(T) \setminus \sigma_W(T)$. Observe that $\sigma(T^*) = \overline{\sigma(T)}$ and $\sigma_W(T^*) = \overline{\sigma_W(T)}$. So $\bar{\lambda} \in \sigma(T) \setminus \sigma_W(T)$, and hence $\bar{\lambda} \in E_0(T^*)$. Therefore λ is an isolated point of $\sigma(T)$, and so $\lambda \in E_0(T)$. Conversely, suppose that $\lambda \in E_0(T)$. Then λ is an isolated point of $\sigma(T)$ and $0 < \alpha(T-\lambda I) < \infty$. Since $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$ and T^* is algebraically class $wF(p,r,q)$ for each $p,r>0$ and $q \geq 1$, it follows from that $\lambda \in \pi(T^*)$. Therefore there exists a natural number n_0 such that $n_0 = a(T^* - \bar{\lambda}I) = d(T^* - \bar{\lambda}I)$. Hence we have $\mathcal{H} = \ker((T^* - \bar{\lambda}I)^{n_0}) \oplus \mathrm{ran}((T^* - \bar{\lambda}I)^{n_0})$ and $\mathrm{ran}((T^* - \bar{\lambda}I)^{n_0})$ is closed. Therefore $\mathrm{ran}((T-\lambda I)^{n_0})$ is closed and $\mathcal{H} = \ker((T^* - \bar{\lambda}I)^{n_0})^{\perp} \oplus \mathrm{ran}((T^* - \bar{\lambda}I)^{n_0})^{\perp} = \ker((T-\lambda I)^{n_0}) \oplus \mathrm{ran}((T-\lambda I)^{n_0})$. So $\lambda \in \pi(T)$, and hence $T - \lambda I$ is Weyl. Consequently, $\lambda \in \sigma(T) \setminus \sigma_W(T)$. Thus Weyl's theorem holds for T. Now we show that $\sigma_W(f(T)) = f(\sigma_W(T))$ for each $f \in \mathrm{Hol}(\sigma(T))$. Let $f \in \mathrm{Hol}(\sigma(T))$. To show that $\sigma_W(f(T)) = f(\sigma_W(T))$ it is sufficient to show that $\sigma_W(f(T)) \supseteq f(\sigma_W(T))$. Suppose that $\lambda \notin \sigma_W(f(T))$. Then $f(T) - \lambda I$ is Weyl. Since T^* is algebraically class wF, it has SVEP. It follows from Proposition 1.1 that $i(T-\alpha_j) \geq 0$ for each $j = 1,2,\ldots,n$. Since

$$0 \leq \sum_{j=1}^{n} i(T-\alpha_j) = i(f(T) - \lambda I) = 0,$$

$T - \alpha_j$ is Weyl for each $j = 1,\ldots,n$. Hence $\lambda \notin f(\sigma_W(T))$, and so $f(\sigma_W(T)) \subseteq \sigma_W(f(T))$. Thus $f(\sigma_W(T)) = \sigma_W(f(T))$ for each $f \in \mathrm{Hol}(\sigma(T))$. Since Weyl's theorem holds for T and T is isoloid, Weyl's theorem holds for $f(T)$ for every $f \in \mathrm{Hol}(\sigma(T))$.

Theorem 3.1 is proved.

From the proof of the Theorem 3.1, we obtain the following useful consequence.

***Corollary* 3.1.** *Suppose T or T^* is an algebraically class $wF(p,r,q)$ operator for each $p,r>0$ and $q \geq 1$. Then $\sigma_W(f(T)) = f(\sigma_W(T))$ for every $f \in \mathrm{Hol}(\sigma(T))$.*

4. a-Weyl's theorem for algebraically wF operators with $p,r>0$ and $q \geq 1$. Let $T \in \mathbf{B}(\mathcal{H})$. It is well known that the inclusion $\sigma_{SF_+^-}(f(T)) \subseteq f(\sigma_{SF_+^-}(T))$ holds for every $f \in \mathrm{Hol}(\sigma(T))$ with no restriction on T [18]. The next theorem shows that the spectral mapping theorem holds for the essential approximate point spectrum for algebraically class wF.

Theorem 4.1. *Suppose T^* or T is an algebraically class wF operator. Then*

$$\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T)).$$

Proof. Assume first that T is algebraically wF and let $f \in \mathrm{Hol}(\sigma(T))$. It suffices to show that $\sigma_{SF_+^-}(f(T)) \supseteq f(\sigma_{SF_+^-}(T))$. Suppose that $\lambda \notin \sigma_{SF_+^-}(f(T))$. Then $f(T) - \lambda I \in SF_+^-(\mathcal{H})$ and

$$f(T) - \lambda I = c(T-\mu_1 I)(T-\mu_2 I)\ldots(T-\mu_n I)g(T),$$

where $c, \mu_1, \mu_2, \ldots, \mu_n \in \mathbb{C}$, and $g(T)$ is invertible. Since T is algebraically class wF, it has SVEP. It follows from [19] (Theorem 2.6) that $i(T - \mu_j) \le 0$ for each $j = 1, 2, \ldots, n$. Therefore $\lambda \notin f(\sigma_{SF_+^-}(T))$, and hence $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$. Suppose now that T^* is algebraically class wF. Then T^* has SVEP, and so by [19] (Theorem 2.6) $i(T - \mu_j I) \ge 0$ for each $j = 1, 2, \ldots, n$. Since

$$0 \le \sum_{j=1}^{n} i(T - \mu_j I) = i(f(T) - \lambda I) \le 0,$$

$T - \mu_j I$ is Weyl for each $j = 1, 2, \ldots, n$. Hence $\lambda \notin f(\sigma_{SF_+^-}(T))$, and so $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$.

Theorem 4.1 is proved.

An operator $T \in \mathbf{B}(\mathcal{H})$ is called a-isoloid if $iso\sigma_a(T) \subseteq \sigma_p(T)$. Clearly, if T is a-isoloid then it is isoloid. However, the converse is not true. Consider the following example: Let $U \oplus Q$, where U is the unilateral forward shift on ℓ^2 and Q is an injective quasinilpotent on ℓ^2, respectively. Then $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ and $\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$. Therefore T is isoloid but not a-isoloid.

It is easily seen that quasinilpotent operators do not satisfy a-Weyl's theorem, in general. for instance, if

$$T(x_1, x_2, \ldots) = \left(0, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right), \quad (x_n) \in \ell^2(\mathbb{N}),$$

then T is quasinilpotent but a-Weyl's theorem fails for T, since $\sigma(T) = \sigma_a(T) = \sigma_{SF_+^-}(T) = \{0\} = E_0^a(T)$.

Theorem 4.2. *Suppose T^* is an algebraically class $wF(p, r, q)$. Then a-Weyl's theorem holds for $f(T)$ for every $f \in \mathrm{Hol}(\sigma(T))$.*

Proof. Suppose T^* is an algebraically class wF operator. We first show that a-Weyl's theorem holds for T. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Then $T - \lambda I$ is upper semi-Fredholm and $i(T - \lambda I) \le 0$. Since T^* is algebraically class wF, T^* has SVEP. Therefore by [19] (Theorem 2.6) that $i(T - \lambda I) \ge 0$, and hence $T - \lambda I$ is Weyl. Since T^* has SVEP, it follows from [10] (Corollary 7) that $\sigma_a(T) = \sigma(T)$. Also, since Weyl's theorem holds for T by Theorem 3.1, $\lambda \in \pi_0^a(T)$.

Conversely, suppose that $\lambda \in \pi_0^a(T)$. Since T^* has SVEP, it follows from [10] (Corollary 7) that $\sigma_a(T) = \sigma(T)$. Therefore λ is an isolated point of $\sigma(T)$, and hence $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$. But T^* is algebraically class wF operator, hence by Proposition 2.2 that $\bar{\lambda} \in \pi(T^*)$. Therefore there exists a natural number n_0 such that $n_0 = a(T^* - \bar{\lambda} I) = d(T^* - \bar{\lambda} I)$. Hence we have $\mathcal{H} = \ker((T^* - \bar{\lambda} I)^{n_0}) \oplus \mathrm{ran}((T^* - \bar{\lambda} I)^{n_0})$ and $\mathrm{ran}((T^* - \bar{\lambda} I)^{n_0})$ is closed. Therefore $\mathrm{ran}((T - \lambda I)^{n_0})$ is closed and $\mathcal{H} = \ker((T^* - \bar{\lambda} I)^{n_0})^{\perp} \oplus \mathrm{ran}((T^* - \bar{\lambda} I)^{n_0})^{\perp} = \ker((T - \lambda I)^{n_0}) \oplus \mathrm{ran}((T - \lambda I)^{n_0})$. So $\lambda \in \sigma_p(T)$, and hence $T - \lambda I$ is Weyl. Consequently, $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Thus a-Weyl's theorem holds for T.

Now we show that T is a-isoloid. Let λ be an isolated point of $\sigma_a(T)$. Since T^* has SVEP, λ is an isolated point of $\sigma(T)$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus T is a-isoloid.

Finally, we shall show that a-Weyl's theorem holds for $f(T)$ for every $f \in \mathrm{Hol}(\sigma(T))$. Let $f \in \mathrm{Hol}(\sigma(T))$. Since a-Weyl's theorem holds for T, it satisfies a-Browder's theorem. Therefore $\sigma_{ab}(T) = \sigma_{SF_+^-}(T)$. It follows from Theorem 4.1 that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{SF_+^-}(T)) = \sigma_{SF_+^-}(f(T)),$$

and hence a-Browder's theorem holds for $f(T)$. So $\sigma_a()f(T) \setminus \sigma_{SF_+^-}(f(T)) \subset \pi_0^a(T)$. Conversely, suppose that $\lambda \in \pi_0^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T) - \lambda I) < 1$. Since λ is an isolated point of $f(\sigma_a(T))$, if $\mu_j \in \sigma_a(T)$ then μ_j is an isolated point of $\sigma_a(T)$. Since T is a-isoloid, $0 < \alpha(T - \mu_j) < 1$ for each $j = 1, 2, \ldots, n$. Since a-Weyl's theorem holds for T, $T - \mu_j$ is upper semi-Fredholm and $i(T - \mu_j) \le 0$ for each $j = 1, 2, \ldots, n$. Therefore $f(T) - \lambda I$ is upper semi-Fredholm and $f(T) - \lambda I = \sum_{j=1}^{n} i(T - \mu_j I) \le 0$. Hence $\lambda \in \sigma_a()f(T) \setminus \sigma_{SF_+^-}(f(T))$, and so a-Weyl's theorem holds for $f(T)$ for each $f \in \mathrm{Hol}(\sigma(T))$.

Theorem 4.2 is proved.

From the proof of the Theorem 4.2, we obtain the following useful consequence.

***Corollary* 4.1.** *Suppose T^* is an algebraically class $wF(p,r,q)$. Then T is a-isoloid.*

5. Finite rank perturbations for Hilbert space operators. For each nonnegative integer n define T_n to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_0 = T$). If for some n, $\mathcal{R}(T^n)$ is closed and T_n is an upper (resp. lower) semi-Fredholm operator then T is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or lower semi-B-Fredholm operators. If moreover, T_n is a Fredholm operator then T is called a *B-Fredholm* operator. The *index* of a semi-B-Fredholm is defined as the index of the semi-Fredholm operator T_n (see [13]). In [13] it is proved that an operator T is a B-Fredholm operator if and only if $T = F \oplus N$, where F is a Fredholm operator and N is a nilpotent operator. An operator $T \in \mathcal{B}(H)$ is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} \colon\ T - \lambda I \text{ is not a B-Weyl operator}\}.$$

Following [13] *generalized Browder's theorem* holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T);$$

where $\pi(T)$ is the set of all poles of T. Recently, in [20] it is proved that

$$\text{generalized Browder's theorem} \Leftrightarrow \text{Browder's theorem}.$$

Recall that an operator $T \in \mathbf{B}(\mathcal{H})$ is a *Drazin invertible* if and only if it has a finite ascent and descent, which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is nilpotent operator and T_1 is invertible operator (see [21] (Proposition A). The Drazin spectrum is given by

$$\sigma_D(T) := \{\lambda \in \mathbb{C} \colon T - \lambda I \ \text{ is not Drazin invertible}\}.$$

We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$, where $\pi(T)$ is the set of all poles.

Theorem 5.1. *Suppose $T \in \mathbf{B}(\mathcal{H})$ be an algebraically wF. If F is finite rank on $\mathcal{H}$ such that $TF = FT$, then $T + F$ satisfies generalized Browder's theorem.*

Proof. From the characterization of $\sigma_{BW}(T)$ [2] (Theorem 4.3), it follows that if F is a finite rank operator, then $\sigma_{BW}(T + F) = \sigma_{BW}(T)$. Moreover, if F commutes with

T, it follows from [22] (Theorem 2.7) that $\sigma_D(T+F) = \sigma_D(T)$. Since T has SVEP, then it satisfies generalized Browder's theorem by [12] (Theorem 1.5), then $\sigma_{BW}(T) = \sigma_D(T)$. Hence $\sigma_{BW}(T+F) = \sigma_D(T+F)$, and so $T+F$ satisfies generalized Browder's theorem.

Theorem 5.1 is proved.

***Corollary* 5.1.** *Suppose $T \in \mathbf{B}(\mathcal{H})$ be an algebraically wF. If F is finite rank on $\mathcal{H}$ such that $TF = FT$, then $T+F$ satisfies Browder's theorem.*

Theorem 5.2. *If $T \in \mathbf{B}(\mathcal{H})$ is an algebraically wF, if $FT = TF$, $F \in \mathbf{F}(\mathcal{H})$. Then $T+F$ satisfies Weyl's theorem.*

Proof. Since by Corollary 5.1 Browder's theorem holds for $T+F$ it suffices to prove that $E_0(T+F) = \pi_0(T+F)$. Let $\lambda \in E_0(T+F)$ be given, then $\lambda \in \operatorname{iso}\sigma(T+F)$ and $\lambda \in \sigma_p(T+F)$, hence $\lambda \notin \operatorname{acc}(\sigma(T+F))$ and $\lambda \notin \operatorname{acc}(\sigma(T))$. We distinguish two cases:

Case I. If $\lambda \notin \sigma(T)$, then $T - \lambda I$ is invertible and $T - \lambda I$ is Fredholm of index zero, since F is a finite rank operator on $\mathcal{H}$, it follows that $T + F - \lambda I$ is Fredholm operator of index zero. Then $\lambda \notin \sigma_W(T+F)$ and $\lambda \in \pi_0(T+F)$.

Case II. If $\lambda \in \sigma(T)$, then $\lambda \in \operatorname{iso}(\sigma(T))$ and since T is isoloid $\lambda \in \sigma_p(T)$. Thus $\lambda \in \operatorname{iso}(\sigma(T)) \cap \sigma_p(T) = E_0(T)$. From the fact that T obeys Weyl's theorem, it follows that $\lambda \notin \sigma_W(T) = \sigma_W(T+F)$ and since $\lambda \in \operatorname{iso}(\sigma(T+F))$, it follows that $\lambda \in \pi_0(T+F)$. Finally $E_0(T+F) \subset \pi_0(T+F)$, and since the reverse inclusion is always true, $T+F$ obeys Weyl's theorem.

Theorem 5.2 is proved.

***Example* 5.1.** This example shows that the commutativity hypothesis in Theorem 5.2 is essential. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and T and F be defined by

$$T(x_1, x_2, \ldots) := \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots\right), \quad \{x_n\} \in \ell^2(\mathbb{N}),$$

and

$$F(x_1, x_2, \ldots) := \left(0, \frac{-x_1}{2}, 0, \ldots\right), \quad \{x_n\} \in \ell^2(\mathbb{N}).$$

Clearly, F is a nilpotent operator and hence of finite rank operator, and T is a quasinilpotent satisfying Weyl's theorem since $\sigma(T) = \sigma_W(T) = \{0\}$ and $E_0(T) = \varnothing$. Now T and F do not commute, $\sigma(T+F) = \sigma_W(T+F) = E_0(T+F) = \{0\}$, and $T+F$ does not satisfy Weyl's theorem.

Theorem 5.3. *Let T be an algebraically wF. If F is an operator that commutes with T and for which there exits a positive integer n such that F^n is finite rank, then $T+F$ satisfies Weyl's theorem.*

Proof. Form Corollary 2.1 and Theorem 3.1, T is isoloid and satisfies Weyl's theorem. Now the result follows at once from [9] (Theorem 2.4).

Theorem 5.3 is proved.

Theorem 5.4. *If T^* is an algebraically wF, then for any finite rank operator $F \in \mathbf{B}(\mathcal{H})$ commuting with T, the a-Weyl's theorem holds for $T+F$.*

Proof. (a) Firstly, We will prove that $\sigma_a(T+F) \setminus \sigma_{SF_+^-}(T+F) \subseteq E_0^a(T+F)$.

Let $\lambda_0 \in \sigma_a(T+F) \setminus \sigma_{SF_+^-}(T+F)$. Using the perturbation theorem of semi-Fredholm operator, $T - \lambda_0 I$ is upper semi-Fredholm with $i(T - \lambda_0 I) \leq 0$ because F is compact. Since a-Weyl's theorem holds for T by Theorem 4.2 if T is an algebraically class wF operator, it follows that $\lambda_0 \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, or $\lambda_0 \in \rho(T)$. Then $T - \lambda_0 I$

has finite ascent and hence $T + F - \lambda_0 I$ has finite ascent [23] (Theorem 1), [24] (Lemma 2.4), $\lambda_0 \in E_0^a(T + F)$.

(b) Secondly, We will prove that $\sigma_a(T + F) \setminus \sigma_{SF_+^-}(T + F) \supseteq E_0^a(T + F)$.

Let $\lambda_0 \in E_0^a(T+F)$, that is $\lambda_0 \in \mathrm{iso}(\sigma_a(T+F))$ and $0 < \dim\ker(T+F-\lambda_0 I) < < \infty$. Then $\dim\ker(T - \lambda_0 I) < \infty$ [25] (Lemma 2.1) and there exists $\epsilon > 0$ such that $T+F-\lambda_0 I$ is bounded below if $0 < |\lambda-\lambda_0| < \epsilon$. Then $T-\lambda I$ is upper semi-Fredholm if $0 < |\lambda - \lambda_0| < \epsilon$.

If there exists $\{\lambda_n\}_{n=1}^{\infty} \subseteq \sigma_a(T)$ such that $\lambda_i \neq \lambda_j$ and $\lambda_n \to \lambda_0$ as $n \to \infty$, without loss of generality, we suppose $0 < |\lambda - \lambda_0| < \epsilon$. Let $M_n = \ker(T - \lambda_n I)$ and let $F_n = F|_{M_n}$. Then F_n is linear and injective. In fact, if there exists $x \in M_n$ such that $F_n x = 0$, then $(T + F - \lambda_n I)x = F_n x = 0$. Since $T + F - \lambda_n I$ is bounded below, we have $x = 0$. We know that in finite dimensional linear space M_n, F_n is injective if and only if F_n is surjective. Then $\ker(T - \lambda_n I) = F_n \ker(T - \lambda_n I) \subseteq \mathrm{ran}(F)$, thus $\sum_{n=1}^{\infty} \bigoplus \ker(T - \lambda_n I) \subseteq \mathrm{ran}(F)$. We have that $\sum_{n=1}^{\infty} \dim\ker(T - \lambda_n I) \leq \leq \dim\mathrm{ran}(F)$. Since $\lambda_n \in \sigma_a(T) \setminus \sigma_{aw}(T)$, we have $\dim\ker(T - \lambda_n I) > 0$ for any $n \in \mathbb{N}$. Then $\dim\mathrm{ran}(F) = \infty$; it is impossible because $\dim\mathrm{ran}(F) < \infty$.

From the proof above, we get there exists $\epsilon' > 0$ (ϵ' should be less than ϵ) such that $T - \lambda I$ is bounded below if $0 < |\lambda - \lambda_0| < \epsilon'$. Then $\lambda_0 \in \mathrm{iso}(\sigma_a(T))$.

Since T is a-isoloid by Corollary 4.1, it follows that $0 < \ker(T - \lambda_0 I)\infty$, which means that $\lambda_0 \in E_0^a(T)$. The a-Weyl's theorem holds for T, then $\lambda_0 \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, and hence $T+F-\lambda_0 I$ is upper semi-Fredholm operator with $i(T+F-\lambda_0 I) \leq \leq 0$. Now we have that $\lambda_0 \in \sigma_a(T + F) \setminus \sigma_{SF_+^-}(T + F)$.

From (a) and (b), we get $\sigma_a(T + F) \setminus \sigma_{SF_+^-}(T + F) = E_0^a(T + F)$, which means that the a-Weyl's theorem holds for $T + F$.

Theorem 5.4 is proved.

In general, a-Weyl's theorem is not transmitted under commuting finite rank perturbation.

***Example* 5.2.** Let $S = \ell^2 \longrightarrow \ell^2$ be an injective quasinilpotent operator which is not nilpotent and let $U\colon \ell^2 \longrightarrow \ell^2$ be defined by $U\langle x_1, x_2, \ldots\rangle := \langle -x_1, 0, \ldots\rangle$, $x_n \in \ell^2(\mathbb{N})$. Define on $\mathcal{H} := \ell^2 \oplus \ell^2$ the operators T and K by $T := I \oplus S$ where I is the identity on ℓ^2 and $K := U \oplus 0$.

It is easily that $\sigma_a(T) = \{0, 1\}$, $E_0^a(T) = \{1\}$ and $\sigma_{SF_+^-}(T) = \{0\}$. Hence T satisfies a-Weyl's theorem. Now K is finite rank operator and $TK = KT$. Moreover, $\sigma_a(T + K) = \{0, 1\}$ and $E_0^a(T + K) = \{0, 1\}$. As $\sigma_{SF_+^-}(T + K) = \sigma_{SF_+^-}(T) = \{0\}$, Then $T + K$ does not satisfy a-Weyl's theorem.

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