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BIG PICARD THEOREM FOR MEROMORPHIC MAPPINGS WITH MOVING HYPERPLANES IN $P^n(C)$ *

ВЕЛИКА ТЕОРЕМА ПІКАРА ДЛЯ МЕРОМОРФНИХ ВІДОБРАЖЕНЬ З РУХОМИМИ ГІПЕРПЛОЩИНАМИ В $P^n(C)$

We give some extension theorems in the style of Big Picard theorem for meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with a few moving hyperplanes.

Наведено деякі теореми про продовження в стилі великої теореми Пікара для мероморфних відображень $\mathbf{C}^m$ в $\mathbf{P}^n(\mathbf{C})$ з деякими рухомими гіперплощинами.

1. Introduction and main results. As well known, in complex one variable, Picard proved the following theorems for meromorphic functions.

Theorem A (Little Picard theorem)**.** *Let $f(z)$ be a meromorphic function on the complex plane. If there exist three mutually distinct points w_1, w_2 and w_3 on the Riemann sphere such that $f(z)-w_i$, $i = 1, 2, 3$, has no zero on the complex plane, then f is a constant.*

Theorem B (Big Picard theorem)**.** *Let $f(z)$ be a meromorphic function on $\Delta^* = \{z \in \mathbf{C}: 1 \leq |z| < +\infty\}$. If there exist three mutually distinct points w_1, w_2 and w_3 on the Riemann sphere such that $f(z) - w_i$, $i = 1, 2, 3$, has no zero on Δ^*, then f does not have an essential singularity at ∞.*

In the case of higher dimension, H. Fujimoto [3] improved Theorem B as follows.

Theorem C ([3], Theorem A)**.** *Let M be a complex manifold and let S be a regular thin analytic subset of M and let f be a holomorphic map of $M \setminus S$ into the n-dimensional complex projective space $\mathbf{P}^n(\mathbf{C})$. If f is of rank r somewhere and if $f(M \setminus S)$ omits $2n - r + 2$ hyperplanes in general position, then f can be extended to a holomorphic map of M into $\mathbf{P}^n(\mathbf{C})$, where the rank of f at a point $x \in M \setminus S$ means the rank of the Jacobian matrix of f at x.*

In 2006, by using a criterion on normality and applying little Picard theorems for holomorphic mappings, Z. H. Tu generalized Big Picard's theorem to the case of moving hyperplanes as follows.

Theorem D ([11], Theorem 2.2)**.** *Let S be an analytic subset of a domain D in $\mathbf{C}^n$ with codimension one, whose singularities are normal crossings. Let f be a holomorphic mapping from $D \setminus S$ into $\mathbf{P}^n(\mathbf{C})$. Let $a_1(z), \ldots, a_q(z)$, $z \in D$, be q, $q \geq 2n + 1$, moving hyperplanes in $\mathbf{P}^n(\mathbf{C})$ located in pointwise general position such that $f(z)$ intersects $a_j(z)$ on $D \setminus S$ with multiplicity at least m_j, $j = 1, \ldots, q$, where $m_1, \ldots, m_q$ are positive integers and may be $+\infty$, with*

$$\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-(n+1)}{n}.$$

Then the holomorphic mapping f from $D \setminus S$ into $\mathbf{P}^n(\mathbf{C})$ extends to a holomorphic mapping from D into $\mathbf{P}^n(\mathbf{C})$.

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We would like to note that in Theorem D, the number q of moving hyperplanes is assumed to be at least $2n+1$ and the technique of its proof does not work in the case where $q < 2n+1$. Then the natural question arise here:

Are there any extension theorem which is similar to Theorem D for the case where the number of moving hyperplanes is less than $2n+1$?

In this paper, we will give some positive answers for this question.

Firstly, we recall some notions due to [8, 10, 11].

Let D be a domain in $\mathbf{C}^m$. We mean a moving hyperplane of $\mathbf{P}^n(\mathbf{C})$ on D a holomorphic mapping a from D into $\mathbf{P}^n(\mathbf{C})$ with a reduced representation $a = (a_0 : \ldots : a_n)$, where $a_0, \ldots, a_n$ are holomorphic functions on D without common zeros. Sometime we regard $a(z)$ as a hyperplane $a(z) = \Big\{(\omega_0 : \ldots : \omega_n) \in \mathbf{P}^n(\mathbf{C}) \colon \sum_{j=0}^n a_j(z)\omega_j = 0\Big\}$.

Let f be a meromorphic mapping of D into $\mathbf{P}^n(\mathbf{C})$. Denote by D_f the smallest linear subspace of $\mathbf{P}^n(\mathbf{C})$ which contains $f(D)$ and denote by L_f the dimension of D_f. For $z \in D$, take a reduced representation $f = (f_0 : \ldots : f_n)$ of f on a neighborhood U_z of z and set $(f, a) := \sum_{j=0}^n a_j f_j$ on U_z. We define $\operatorname{div}(f,a)(z) := \operatorname{div}\Big(\sum_{j=0}^n a_j f_j\Big)(z)$ if $(f,a) \not\equiv 0$ and $\operatorname{div}(f,a)(z) := \infty$ if $(f,a) \equiv 0$. Thus, $\operatorname{div}(f,a)$ is well-defined on D independently of the choice of reduced representations of f. If $\operatorname{div}(f,a)(z) \geq m_j$ for all $z \in D$, we say that f intersects a on D with multiplicity at least m_j.

Let $\mathcal{A} = \{a_0, \ldots, a_{q-1}\}$ be a set of q moving hyperplanes of $\mathbf{P}^n(\mathbf{C})$ on D. Assume that each a_i has a reduced representation $a_i = (a_{i0} : \ldots : a_{in})$. Denote by $\mathcal{R}\{a_i\}$ the smallest field which contains $\mathbf{C}$ and all functions $\dfrac{a_{ik}}{a_{ij}}$ with $a_{ij} \not\equiv 0$. Sometime we write $\mathcal{R}$ for $\mathcal{R}\{a_i\}$ if there is no confusion and denote by $(\mathcal{A})_{\mathcal{R}}$ the linear span of $\mathcal{A}$ over $\mathcal{R}$. We say that:

$\mathcal{A}$ is located in general position on D if and only if for any arbitrary $n+1$ moving hyperlanes $\{a_{i_k}\}_{1\leq k\leq n+1} \subset \mathcal{A}$ there exists a point $z \in D$ such that $\cap_{1\leq k\leq n+1} a_{i_k}(z) = \varnothing$.

$\mathcal{A}$ is located in pointwise N-subgeneral position on D if and only if for any arbitrary $N+1$ moving hyperplanes $\{a_{i_k}\}_{1\leq k\leq N+1} \subset \mathcal{A}$ then $\cap_{1\leq k\leq N+1} a_{i_k}(z) = \varnothing$ for all $z \in D$.

$\mathcal{A}$ is located in pointwise N-subgeneral position on D with respect to f if and only if for any arbitrary $N+1$ moving hyperlanes $\{a_{i_k}\}_{k=1}^{N+1} \subset \mathcal{A}$ then $\cap_{k=1}^{N+1} a_{i_k}(z) \cap D_f = \varnothing$ for all $z \in D$.

Then we see that if $\mathcal{A}$ is located in pointwise N-subgeneral position on D then it will be located in pointwise N-subgeneral position on D with repect to f for every mapping f, but not vice versa.

Our main result of this work is stated as follows.

Theorem 1.1. *Let f be a holomorphic mapping of a domain $D \setminus S$ into $\mathbf{P}^n(\mathbf{C})$, where D is a domain in $\mathbf{C}^m$ and S is an analytic subset of codimension one of D. Let $a_1, \ldots, a_{n+2}$ be $n+2$ moving hyperplanes in $\mathbf{P}^n(\mathbf{C})$ on D located in general position so that f is linearly nondegenerate over $\mathcal{R}\{a_i\}$. Assume that f intersects each a_i on $D \setminus S$ with multiplicity at least m_i, where $m_1, \ldots, m_{n+2}$ are fixed positive integers and may be $+\infty$, with*

$$\sum_{i=1}^{n+2} \frac{1}{m_i} < \frac{1}{n}.$$

Then f extends to a meromorphic mapping $\tilde{f}$ from D into $\mathbf{P}^n(\mathbf{C})$.

In the last section of this paper, we will consider the case where the condition linearly nondegeneracy of mappings is omitted.

2. Basic notions and auxiliary results from Nevanlinna theory. ***2.1.*** We set punctured discs on $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ about ∞ by

$$\Delta^* = \{z \in \mathbf{C} \colon |z| \geq 1\},$$

$$\Delta^*(t) = \{z \in \mathbf{C} \colon |z| \geq t\}, \quad t \geq 1,$$

and we set

$$\Gamma(t) = \{z \in \mathbf{C} \colon |z| = t\}, \quad t \geq 1.$$

In this paper, we always assume that functions on Δ^* and mappings from Δ^* are defined on a neighborhood of Δ^* in $\mathbf{C}$. Let ξ be a function on Δ^* satisfying that

(i) ξ is differentiable outside a discrete set of points,

(ii) ξ is locally written as a difference of two subharmonic functions.

Then by [7] (§ 1), we have

$$\int\limits_1^t \frac{dt}{t} \int\limits_{\Delta^*(t)} dd^c\xi = \frac{1}{4\pi} \int\limits_{\Gamma(r)} \xi(re^{i\theta})d\theta - \frac{1}{4\pi} \int\limits_{\Gamma(1)} \xi(re^{i\theta})d\theta - (\log r) \int\limits_{\Gamma(1)} d^c\xi, \tag{2.1}$$

where $dd^c\xi$ is taken in the sense of current.

2.2. A divisor E on Δ^* is given by a formal sum $E = \sum \mu_\nu p_\nu$, with $\{p_\nu\}$ is a locally finite family of distinct points in Δ^* and $\mu_\nu \in \mathbf{Z}$. We define the support of the divisor E by $\operatorname{Supp}(E) = \bigcup_{\mu_\nu \neq 0} p_\nu$. Let k be a positive integer or $+\infty$. We define the divisor $E^{(k)}$ by

$$E^{(k)} := \sum \min\{\mu_\nu, k\} p_\nu$$

and the *truncated counting function to level* k of E by

$$N^{(k)}(r, E) := \int\limits_1^r \frac{n^{(k)}(t, E)}{t} dt, \quad 1 < r < +\infty,$$

where

$$n^{(k)}(t, E) = \sum_{|z| \leq t} E^{(k)}(z).$$

We simply write $N(r, E)$ for $N^{(+\infty)}(r, E)$.

2.3. Let $f \colon \Delta^* \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve. For an arbitrary fixed homogeneous coordinates $(w_0 : \ldots : w_n)$ of $\mathbf{P}^n(\mathbf{C})$, there exist a neighborhood U of Δ^* in $\mathbf{C}^m$ and a reduced representation $(f_0 : \ldots : f_n)$ on U of f, which means that $f_0, \ldots, f_n$ are holomorphic functions on U without common zeros. We set $\|f\| := \big(|f_0|^2 + \ldots + |f_n|^2\big)^{1/2}$.

Denote by Ω the Fubini – Study form of $\mathbf{P}^n(\mathbf{C})$. The *order function* or *characteristic function* of f with respect to Ω is defined by

$$T_f(r) := T_f(r;\Omega) = \int_1^r \frac{dt}{t} \int_{\Delta^*(t)} f^*\Omega, \quad r > 1. \tag{2.2}$$

Applying (2.1) to $\xi = \log \|f\|$, we obtain

$$T_f(r) = \frac{1}{2\pi} \int_{\Gamma(r)} \log \|f(re^{i\theta})\| d\theta - \frac{1}{2\pi} \int_{\Gamma(1)} \log \|f(e^{i\theta})\| d\theta - (\log r) \int_{\Gamma(1)} d^c \log \|f\|. \tag{2.3}$$

Let a be a moving hyperplane in $\mathbf{P}^n(\mathbf{C})$ with a reduced representation $a = (a_0 : \ldots : a_n)$. We set $(f,a) = \sum_{i=0}^n a_i f_i$. Assume that $(f,a) \not\equiv 0$, we define the *proximity function* of f with respect to a by

$$m_f(r,a) = \frac{1}{2\pi} \int_{\Gamma(r)} \log \frac{\|f\|\|a\|}{|(f,a)|} d\theta - \frac{1}{2\pi} \int_{\Gamma(1)} \log \frac{\|f\|\|a\|}{|(f,a)|} d\theta,$$

where $\|a\| = \left(\sum_{i=0}^n |a_i|^2\right)^{1/2}$.

Applying (2.1) to $\xi = \log |(f,a)|$, we get

$$N(r, \operatorname{div}(f,a)) = \frac{1}{2\pi} \int_{\Gamma(r)} \log |(f,a)| d\theta - \frac{1}{2\pi} \int_{\Gamma(1)} \log |(f,a)| d\theta - (\log r) \int_{\Gamma(1)} d^c \log |(f,a)|. \tag{2.4}$$

Combining (2.2) and (2.4), we have the **First Main Theorem** as follows:

$$T_f(r) + T_a(r) = N(r, \operatorname{div}(f,a)) + m_f(r,a) + (\log r) \int_{\Gamma(1)} d^c \log \left(\frac{\|f\| \, \|a\|}{|(f,a)|} \right). \tag{2.5}$$

2.4. For a meromorphic function φ on Δ^*, applying (2.1) to $\xi = \log |\varphi|$, we obtain

$$N(r, \operatorname{div}_0(\varphi)) + N(r, \operatorname{div}_\infty(\varphi)) =$$

$$= \frac{1}{2\pi} \int_{\Gamma(r)} \log |\varphi| d\theta - \frac{1}{2\pi} \int_{\Gamma(1)} \log |\varphi| d\theta - (\log r) \int_{\Gamma(1)} d^c \log |\varphi|.$$

The proximity function $m(r,\varphi)$ is defined by

$$m(r,\varphi) := \frac{1}{2\pi} \int_{\Gamma(r)} \log^+ |\varphi| d\theta,$$

where $\log^+ x = \max \{ \log x, 0 \}$ for $x \geqslant 0$. The Nevanlinna's characteristic function is defined by

$$T(r,\varphi) := N(r, \operatorname{div}_\infty(\varphi)) + m(r,\varphi).$$

We regard φ as a meromorphic mapping from Δ^* into $\mathbf{P}^1(\mathbf{C})$. There is a fact that

$$T_\varphi(r) = T(r,\varphi) + O(\log r).$$

Theorem 2.1 (Lemma on logarithmic derivative [7]). *Let φ be a nonzero meromorphic function on Δ^*. Then*

$$\Big\| \quad m\left(r, \frac{\varphi'}{\varphi}\right) = O(\log^+ T_\varphi(r)) + C \log r, \tag{2.6}$$

where C is a positive constant which does not depend on φ.

As usual, by the notation "$\| \ P$" we mean the assertion P holds for all $r \in (1, +\infty)$ excluding a finite Lebesgue measure subset E of $(1, +\infty)$.

3. Extension of meromorphic mappings with $(n+2)$ moving hyperplanes. In this section, we will give the proof of Theorem 1.1. We need the following lemmas.

Firstly, we know the following characterization of a removable singularity, a classical result of J. Noguchi (cf. [7]).

Lemma 3.1. *Let $f\colon \Delta^* \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve. Then f extends at ∞ to a holomorphic curve $\tilde{f}$ from $\Delta = \Delta^* \bigcup\{\infty\}$ into $\mathbf{P}^n(\mathbf{C})$ if and only if*

$$\liminf_{r\to\infty} T_f(r)/(\log r) < \infty.$$

The following lemma is due to Brownawell and Masser [2].

Lemma 3.2. *Assume $\sum_{i=0}^{n+1} f_i = 0$ and $\sum_{i\in I} f_i \neq 0$ for every $I \subsetneq \{0, \ldots, n+1\}$. Then we can find a partition*

$$\{f_0, \ldots, f_{n+1}\} = A_1 \bigcup A_2 \bigcup \ldots \bigcup A_k, \quad k \geq 1,$$

into nonempty disjoint sets $A_1, \ldots, A_k$, and nonempty sets $A_1' \subset A_1$, $A_2' \subset A_1 \bigcup A_2, \ldots, A_{k-1}' \subset \subset A_1 \bigcup \ldots \bigcup A_{k-1}$ such that $A_1, A_2 \bigcup A_1', \ldots, A_k \bigcup A_{k-1}'$ are minimal. Here, we say that a subset A of $\{f_0, \ldots, f_{n+1}\}$ is minimal if it is linearly dependent, and any its proper subset is linearly independent.

We now prove a Second Main Theorem for meromorphic mappings from punctured disks with moving hyperplanes as follows.

Lemma 3.3. *Let f be a holomorphic curve from the punctured disc Δ^* into $\mathbf{P}^n(\mathbf{C})$ with a reduced representation $f = (f_0 : \ldots : f_n)$, and let $f_{n+1} = -f_0 - \ldots - f_n$ so that*

$$\sum_{i\in I} f_i \neq 0 \quad \forall I \subsetneq \{0, \ldots, n+1\}.$$

Then the following holds:

$$\Big\| \quad T_f(r) \leq \sum_{i=0}^{n+1} N^{(n)}(r, \mathrm{div}_0(f_i)) + O(\log^+ T_f(r)) + O(\log r).$$

Proof. Set $A = \{f_0, \ldots, f_{n+1}\}$. By the assumption, then there exist a partition $A = A_1 \bigcup \ldots \ldots \bigcup A_k$ and nonempty subsets A_s', $1 \leq s \leq k-1$, as in Lemma 3.3. By changing indices if necessary, we may assume that

$$A_1 = \{0, 1, \ldots, t_1\}, \qquad A_s = \{t_{s-1}+1, t_{s-1}+2, \ldots, t_s\}, \qquad t_0 = 0, \quad t_k = n+1, 2 \leq s \leq k.$$

Since A_1 is minimal, there exist nonzero constants c_{1i}, $0 \leq i \leq t_1$, so that

$$\sum_{i=0}^{t_1} \alpha_{1i} f_i = 0.$$

Similarly, for $s > 1$, since $A_s \bigcup A'_{s-1}$ is minimal, there exist nonzero constants c_{si}, $t_{s-1} < i \le t_s$, and constants c_{si}, $0 \le i \le t_{s-1}$, so that

$$\sum_{i=0}^{t_s} \alpha_{si} f_i = 0.$$

We set $c_{si} = 0$ for all $i > t_s$, $s \ge 1$. Then we have

$$\sum_{s=1}^{k} \sum_{i=t_{s-1}+1}^{t_s} \alpha_{si} f_i = 0. \tag{3.1}$$

Since $A_1 \setminus \{0\}$ and A_s, $s \ge 2$, are linearly independent, then

$$D_s = \det\Big(\mathcal{D}^l(\alpha_{si} f_i);\ 0 \le l \le t_s - t_{s-1} - 1,\ t_{s-1} + 1 \le i \le t_s\Big) \ne 0,$$

where $\mathcal{D}^l$ denotes the derivatives of order l. Consider an minor $(n+1) \times (n+2)$-matrices T and $\widetilde{\mathcal{T}}$ given by

$$\mathcal{T} = \Big(\mathcal{D}^l(\alpha_{si} f_i)\Big)_{\substack{0 \le l \le t_s - t_{s-1} - 1,\, 1 \le s \le k \\ 0 \le i \le n+1}},$$

$$\widetilde{\mathcal{T}} = \left(\mathcal{D}^l\left(\frac{\alpha_{si} f_i}{f_0}\right)\right)_{\substack{0 \le l \le t_s - t_{s-1} - 1,\, 1 \le s \le k \\ 0 \le i \le n+1}}.$$

Denote by B_i (resp. $\widetilde{B}_i$) the determinant of the matrix obtained by deleting the $(i+1)$-th column of the minor matrix $\mathcal{T}$ (resp. $\widetilde{\mathcal{T}}$). Since the sum of each row of $\mathcal{T}$ (resp. $\widetilde{\mathcal{T}}$) is zero, we actually have

$$B_i = (-1)^i B_0 = (-1)^i \prod_{i=1}^{k} D_i = (-1)^i f_0^{n+1} \prod_{i=1}^{k} \widetilde{D}_i = (-1)^i f_0^{n+1} \widetilde{B}_0 = f_0^{n+1} \widetilde{B}_i.$$

We see that there exists a constant $C > 0$ so that $\|f(z)\| \le C\colon \max\{|f_0(z)|, \ldots, |f_{n+1}(z)|\}$ for all $z \in \Delta^*$. Therefore, we get

$$\frac{\|f(z)\|\, |B_0(z)|}{\prod_{i=0}^{n+1} |f_i(z)|} \le \frac{C \max\{|f_0(z)|, \ldots, |f_{n+1}(z)|\} |B_0(z)|}{\prod_{i=0}^{n+1} |f_i(z)|}.$$

This yields that

$$\log \|f(z)\| + \log \frac{|B_0(z)|}{\prod_{i=0}^{n+1} |f_i(z)|} \le \sum_{i=0}^{n+1} \log^+ \frac{|B_i(z)|}{\prod_{j=0, j \ne i}^{n+1} |f_j(z)|} + O(1) =$$

$$= \sum_{i=0}^{n+1} \log^+ \frac{|\widetilde{B}_i(z)|}{\prod_{j=0, j\neq i}^{n+1} \left|\dfrac{f_j}{f_0}(z)\right|} + O(1).$$

Integrating both sides of this inequality over $\Gamma(r)$ and applying the lemma on logarithmic derivatives, we obtain

$$\Bigg\| \quad T_f(r) + \frac{1}{2\pi} \int\limits_{\Gamma(r)} \log \frac{|B_0|}{\prod_{i=0}^{n+1} |f_i|} d\theta \leq O(\log^+ T_f(r)) + C_1 \log r.$$

Thus

$$\| \; T_f(r) \leq \sum_{s=1}^{k} \left(\sum_{f_i \in A_s} N(r, \operatorname{div}(f_i)) - N(r, \operatorname{div}(A_s)) \right) + O(\log^+ T_f(r)) + C_1 \log r \leq$$

$$\leq \sum_{i=1}^{k} \sum_{f_i \in A_s} N^{t_s - t_{s-1} - 1}(r, \operatorname{div}(f_i)) + O(\log^+ T_f(r)) + C_1 \log r \leq$$

$$\leq \sum_{i=0}^{n+1} N^{(n)}(r, \operatorname{div}(f_i)) + O(\log^+ T_f(r)) + C_1 \log r.$$

The lemma is proved.

Lemma 3.4. *Let f be a holomorphic curve from a punctured disc Δ^* into $\mathbf{P^n}(\mathbf{C})$, and let $a_1, \ldots, a_{n+2}$ be $n+2$ moving hyperplanes in $\mathbf{P^n}(\mathbf{C})$ on Δ located in general position so that there exist nonzero moromorphic functions α_i, $1 \leq i \leq n+2$, on Δ satisfying: $\sum_{i=1}^{n+2} \alpha_i(f, a_i) = 0$ and*

$$\sum_{i \in I} \alpha_i(f, a_i) \neq 0 \quad \forall I \subsetneq \{1, \ldots, n+2\}.$$

Assume that f intersects each a_i on Δ^* with multiplicity at least m_i, where $m_1, \ldots, m_{n+2}$ are fixed integers and may be $+\infty$, with

$$\sum_{i=1}^{n} \frac{1}{m_i} < \frac{1}{n}.$$

Then f extends at ∞ to a holomorphic curve $\widetilde{f}$ from $\Delta = \Delta^* \bigcup \{\infty\}$ to $\mathbf{P^n}(\mathbf{C})$.

Proof. Without loss of generality, we may assume that α_i, $1 \leq i \leq n+2$, have no neither common zero nor pole. We consider the following divisor on Δ^* as follows:

$$\nu(z) = \min\big\{ \operatorname{div}(\alpha_i(f, a_i))(z);\ 1 \leq i \leq n+2 \big\}.$$

Since f is holomorphic, it easy to see that $\operatorname{Supp}(\nu)$ is subset of

$$\bigcup_{1 \leq i_0 < \ldots < i_n \leq n+2} \{z \mid \operatorname{rank}_{\mathbf{C}}(a_{i_0}(z), \ldots, a_{i_n}(z)) \leq n\} \bigcup \bigcup_{i=1}^{n+1} \operatorname{Supp}(\operatorname{div}(\alpha_i)),$$

which is an analytic subset of Δ. Therefore we may consider ν as a divisor on Δ. Choose a holomorphic function h on $\mathbf{C}^m$ so that $\operatorname{div}(h) = \nu$ and set $F_i = \dfrac{1}{h}\alpha_i(f, a_i)$. Then we see that $\sum_{i=1}^{n+2} F_i = 0$, $(F_1 : \ldots : F_{n+1})$ is a reduced representation of a holomorphic curve F and

$$\sum_{i\in I} F_i \not\equiv 0 \quad \forall I \subsetneq \{1, \ldots, n+2\}.$$

Hence F satisfy the assumption of Lemma 3.3, then

$$\| \; T_F(r) \leq \sum_{i=1}^{n+2} N^{(n)}\left(r, \operatorname{div}_0\left(\frac{\alpha_i}{h}(f, a_i)\right)\right) + O(\log^+ T_F(r)) + O(\log r) =$$

$$= \sum_{i=1}^{n+2} N^{(n)}(r, \operatorname{div}_0(f, a_i))) + O\left(\sum_{i=1}^{n+2} T\left(r, \frac{\alpha_i}{h}\right)\right) + O(\log^+ T_F(r)) + O(\log r) \leq$$

$$\leq \sum_{i=1}^{n+2} \frac{n}{m_i} N(r, \operatorname{div}_0(f, a_i)) + O\left(\sum_{i=1}^{n+2} T\left(r, \frac{\alpha_i}{h}\right)\right) + O(\log^+ T_F(r)) + O(\log r) \leq$$

$$\leq \left(\sum_{i=1}^{n+2} \frac{n}{m_i}\right) T_f(r) + O\left(\sum_{i=1}^{n+2} T_{a_i}(r)\right) + O\left(\sum_{i=1}^{n+2} T\left(r, \frac{\alpha_i}{h}\right)\right) + O(\log^+ T_F(r)) + O(\log r).$$

Since a_i, $\dfrac{\alpha_i}{h}$, $1 \leq i \leq n+2$, are holomorphic on Δ, then by Lemma 3.2 we have

$$\sum_{i=1}^{n+2} \left(T_{a_i}(r) + T\left(r, \frac{\alpha_i}{h}\right)\right) = O(\log r).$$

Thus

$$\| \; T_F(r) \leq \left(\sum_{i=1}^{n+2} \frac{n}{m_i}\right) T_f(r) + O(\log^+ T_F(r)) + O(\log r).$$

On the other hand, we easily see that

$$\| \; T_F(r) = T_f(r) + O\left(\sum_{i=1}^{n+2} T_{a_i}(r)\right) = T_f(r) + O(\log r).$$

Hence, it follows that

$$\| \; T_f(r) \leq \left(\sum_{i=1}^{n+2} \frac{n}{m_i}\right) T_f(r) + O(\log^+ T_f(r)) + O(\log r).$$

This implies that

$$\| \; T_f(r) = O(\log^+ T_f(r)) + O(\log r).$$

Therefore

$$\liminf_{r\to+\infty} T_f(r)/(\log r) < +\infty.$$

By again Lemma 3.1 we have the required extension of f.

The lemma is proved.

***Proof of Theorem* 1.1.** Since $\{a_i\}_{i=1}^{n+2}$ are located in general position and f is linearly nondegenerate over $\mathcal{R}\{a_i\}$, there exist nonzero meromorphic functions α_i so that

$$\sum_{i=1}^{n+2} \alpha_i(f, a_i) = 0,$$

and

$$\sum_{i\in I} \alpha_i(f, a_i) \not\equiv 0 \quad \forall I \subsetneq \{1, \ldots, n+2\}.$$

We define the analytic subset S_0 of $D \setminus S$ by

$$S_0 = \bigcup_{I\subsetneq\{1,\ldots,n+2\}} \left\{ z \in D \setminus S \,\middle|\, \sum_{i\in I} \alpha_i(f(z), a_i(z)) = 0 \right\}.$$

Then S_0 is an analytic subset of codimension at least one of $D \setminus S$.

Put $\tilde{S} = \bigcup_{1\le i_0<\ldots<i_n\le n+2}\{z \in D \mid \bigcap_{j=0}^{n} a_{i_j}(z) \neq \varnothing\}$. It is easy to see that $\tilde{S}$ is an analytic subset of codimension at least one of D. Denote by S_1 the regular part of $S \bigcup \tilde{S}$ and S_2 the singular part of $S \bigcup \tilde{S}$. By [5], Corollary 3.3.44, it is enough to prove that f extends to a meromorphic mapping on $D \setminus S_2$.

Since the extendibility of the map f is a local property, it is suffices to prove that f is extendable on a neighborhood of each point in S_1.

For $z_0 \in S_1$, we take a small neighborhood U of z_0 in $D \setminus S_2$ so that U is biholomorphic with $\Delta \times \Delta^{m-1}$. Then for convenience, we may assume that $U = \Delta \times \Delta^{m-1}$ and $S_1 \cap U = \{\infty\} \times \Delta^{m-1}$.

Take a homogeneous coordinates $(\omega_0 : \ldots : \omega_n)$ of $\mathbf{P}^n(\mathbf{C})$ and set $f_i = \omega_i \circ f$. It suffices to show that for each $1 \le i \le n$, $\dfrac{f_i}{f_0}$ extends meromorphically over $\Delta \times \Delta^{m-1}$.

We easily see that there exists $(a, b) \in \mathbf{C} \times \mathbf{C}^{m-1}$, $a \neq 0$, so that the complex line $L = \{(ta, z_0 + tb)\}$ satisfying $L \cap (U \setminus S) \not\subset S_0$. Therefore, by changing the complex coordinates, we may assume that $\Delta^* \times \{z_0\} \not\subset S_0$. It follows that there exists a neighborhood U_1 of z_0 in Δ^{m-1} so that $\Delta^* \times \{z\} \not\subset S_0$ for all $z \in U_1$. By choosing a smaller neighborhood if necessary, we may assume that $U_1 = U = \Delta^* \times \Delta^{m-1}$. Then $\Delta^* \times \{z\} \not\subset S_3$ for all $z \in \Delta^{m-1}$.

We consider the holomorphic curve $f(\cdot, z_0)\colon z \in \Delta^* \longmapsto f(z, z_0)$, which intersect $a_i |_{\Delta^*\times\{z_0\}}$ with multiplicity at least m_i for all $0 \le i \le n+2$. Therefore, the curve $f(\cdot, z_0)$ and the family $\{a_i |_{\Delta^*\times\{z_0\}}\}_{a_i\in\mathcal{A}}$ satisfy the assumption of Lemma 3.3.

By Lemma 3.3, $f(\cdot, z_0)$ is extendable over Δ^*, hence $\dfrac{f_i}{f_0}(\cdot, z_0)$ extends to a meromorphic function on Δ denoted again by $\dfrac{f_i}{f_0}(\cdot, z_0)$. We put $\dfrac{f_i}{f_0}(z_1, z_0) = z_1^{\mu(z_0)} g(z_1, z_0)$, where $\mu(z_0) \in \mathbf{Z}$ and $g(\cdot, z_0)$ is a holomorphic function on Δ, $g(\infty, z_0) \neq 0, \infty$. Take a small neighborhood U_2 of z_0. Then $\mu(z')$ is bounded in $z' \in U_2$. Then there is a neighborhood of (∞, z_0), we may assume again that it is $\Delta \times \Delta^{m-1}$, so that $\dfrac{f_i}{f_0}$ is written as

$$\frac{f_i}{f_0}(z_1, z') = (z_1^a)g(z_1, z'),$$

where $a \in \mathbf{Z}$, g is a nowhere vanishing holomorphic function on $\Delta^* \times \Delta$. $g(\infty, z') \neq 0, \infty$ and $g(z^1, z')$ is holomorphic in $z^1 \in \Delta^*$ for each $z' \in \Delta^{m-1}$.

We consider the expansion of $g(z^1, z')$ in z^1 at the point ∞ as follows:

$$g(z^1, z') = \sum_{i=0}^{\infty} b_i(z') \left(\frac{1}{z^1}\right)^i.$$

Since $g = \dfrac{f_i}{z_1^a f_0}$, which is a holomorphic function on $\Delta^* \times \Delta^{m-1}$, it easy to see that each coefficient $b_i(z')$ is holomorphic on Δ^{m-1}. Hence g is holomorphic on $\Delta \times \Delta^{m-1}$. Therefore $\dfrac{f_i}{f_0}$ is meromorphic on $\Delta \times \Delta^{m-1}$.

The theorem is proved.

4. Big Picard theorem for the case of degenerate meromorphic mappings. In this section we consider holomorphic mappings without the condition on linearly nondegeneracy of mappings, we will prove the following theorem.

Theorem 4.1. *Let f be a holomorphic mapping of a domain $D \setminus S$ into $\mathbf{P}^n(\mathbf{C})$, where D is a domain in $\mathbf{C}^m$ and S is an analytic subset of codimension one of D with only normal crossings. Let N be a positive integer. Let $\mathcal{A} = \{a_0, \ldots, a_{q-1}\}$ be a set of q, $q \geq 2N+1$, moving hyperplanes on D of $\mathbf{P}^n(\mathbf{C})$ located in pointwise N-subgeneral position with respect to f. Assume that f intersects each a_i on $D \setminus S$ with multiplicity at least m_i, where $m_0, \ldots, m_{q-1}$ are fixed positive integers and may be $+\infty$, with*

$$\sum_{i=0}^{q-1} \frac{1}{m_i} < \frac{q - 2N - 1}{L_f} + 1.$$

Then f extends to a holomorphic mapping $\tilde{f}$ from D into $\mathbf{P}^n(\mathbf{C})$.

In order to prove Theorem 4.1, we need some following.

Definition 4.1 ([11], Definition 3.1)**.** *Let Ω be a hyperbolic domain and let M be a complete complex Hermitian manifold with metric ds_M^2. A holomorphic mapping $f(z)$ from Ω into M is said to be a normal holomorphic mapping from Ω into M if and only if there exists a positive constant C such that for all $z \in \Omega$ and all $\xi \in T_z(\Omega)$,*

$$ds_M^2\big(f(z), df(z)(\xi)\big) \leq CK_\Omega(z, \xi),$$

where $df(z)$ is the mapping from $T_z(\Omega)$ into $T_{f(z)}(M)$ induced by f and K_Ω denotes the infinitesimal Kobayashi metric on Ω.

Lemma 4.1 (see [11])**.** *Let f be a holomorphic mapping from a bounded domain Ω in $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ such that for every sequence of holomorphic mappings $\varphi_k(z)$ from the unit disc U in $\mathbf{C}$ into Ω, the sequence $\{f \circ \varphi_k(z)\}_{k=1}^{\infty}$ from U into $\mathbf{P}^n(\mathbf{C})$ is a normal family on U. Then f is a normal holomorphic mapping from Ω into $\mathbf{P}^n(\mathbf{C})$.*

Theorem 4.2 ([1], Theorem 3.1, [9], Theorem 2.5)**.** *Let Ω be a domain in $\mathbf{C}^m$. Let M be a compact complex Hermitian space. Let $\mathcal{F} \in \mathrm{Hol}(\Omega, M)$. Then the family $\mathcal{F}$ is not normal if and only if there exist sequences $\{p_j\} \in \Omega$ with $\{p_j\} \to p_0,\ (f_j) \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0$ such that*

$$g_j(\xi) := f_j(p_j + \rho_j \xi)$$

converges uniformly on compact subsets of $\mathbf{C}^m$ to a nonconstant holomorphic map $g \colon \mathbf{C}^m \to M$.

The following theorem is due to Noguchi [6].

Theorem 4.3 ([6], Theorem 3.1)**.** *Let f be a linearly nondegenerate holomorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^k(\mathbf{C})$ and let $\{H_0, \ldots H_{q-1}\}$ be q, $q \geq 2N - k + 1$, hyperplanes of $\mathbf{P}^k(\mathbf{C})$ located in N-subgeneral position with respect to f. Then the following holds*:

$$\| \ (q - 2N + k - 1) T_f(r) \leq \sum_{i=0}^{q-1} N^{(k)}(r, \mathrm{div}(f, H_i)) + o(T_f(r)).$$

For our purpose, we need a reformulation of Theorem 4.3 as follows.

Lemma 4.2. *Let f be a holomorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and let $\{H_0, \ldots H_{q-1}\}$ be q, $q \geq 2N - L_f + 1$, hyperplanes of $\mathbf{P}^n(\mathbf{C})$ located in $N-$subgeneral position with respect to f. Then the following holds*:

$$\| \ (q - 2N + L_f - 1) T_f(r) \leq \sum_{\substack{i=0 \\ (f, H_i) \not\equiv 0}}^{q-1} N^{(L_f)}(r, \mathrm{div}(f, H_i)) + o(T_f(r)).$$

Proof. We give a sketch of its proof as follows. Denote by D_f the linearly span of $f(\mathbf{C})$. Then we may consider D_f as a complex projective space of dimension L_f. Set $Q = \{j; (f, H_j) \equiv 0\}$. It is clear that an index $j \in Q$ if and only if $D_f \subset H_j$. For each $i \notin Q$, we set $H_i^* = H_i \cap D_f$, which is a hyperplane in D_f, and easily see that

$$\mathrm{div}(f, H_i) = \mathrm{div}(f, H_i^*),$$

here for the right-hand side of the equality we consider f as a map of $\mathbf{C}$ into D_f. We may verify that $\{H_i^*;\ 0 \leq i \leq q-1,\ i \notin Q\}$ is located in $(N - \sharp Q)$-subgeneral position in D_f. Indeed, for any subset $\{i_0, \ldots, i_{N-\sharp Q}\}$ of $\{0, \ldots, q-1\} \backslash Q$, we have

$$\bigcap_{j=0}^{N-\sharp Q} H_{i_j}^* = D_f \cap \bigcap_{j=0}^{N-\sharp Q} H_{i_j} = D_f \cap \bigcap_{i \in Q} H_i \cap \bigcap_{j=0}^{N-\sharp Q} H_{i_j} = \varnothing.$$

Applying Theorem 4.3, we obtain

$$\| \ ((q - \sharp Q) - 2(N - \sharp Q) + L_f - 1) T_f(r) \leq \sum_{\substack{i=0 \\ i \notin Q}}^{q-1} N^{(L_f)}(r, \mathrm{div}(f, H_i^*)) + o(T_f(r)).$$

This easily implies that

$$\| \ (q - 2N + L_f - 1) T_f(r) \leq \sum_{\substack{i=0 \\ i \notin Q}}^{q-1} N^{(L_f)}(r, \mathrm{div}(f, H_i)) + o(T_f(r)).$$

The lemma is proved.

***Proof of Theorem* 4.1.** For $z_0 \in S$, we take a relative compact subdomain Ω containing z_0 of D. It suffices to prove that f extends over $\Omega \setminus S$ to a holomorphic mapping.

Firstly, we shall prove that f is normal on $\Omega \setminus S$. Indeed, suppose that f is not normal on $\Omega \setminus S$, then there exists a sequence of holomorphic mappings $\{\varphi_i : U \to \Omega \setminus S\}_{j=1}^{\infty}$ such that $\{f \circ \varphi_j\}$ is not normal, where U denotes the unit disc in $\mathbf{C}$. By Lemma 4.2, we may assume that there exist sequences $\{p_j\} \in U$, $\{r_j\} \in \mathbf{R}$ with $r_j > 0$ and $r_j \searrow 0$, $p_j \to p_0 \in U$ such that $g_j(\xi) := f \circ \varphi_j(p_j + r_j\xi)$ converges uniformly on compact subsets of $\mathbf{C}$ to a nonconstant holomorphic mapping g of $\mathbf{C}$ into $\mathbf{P}^n(\mathbf{C})$. Because $\Omega \setminus S$ is bounded, then $\{\varphi_j\}$ is a normal family of holomorphic mappings. Hence, there exists a subsequence (again denoted by $\{\varphi_j\}$) of $\{\varphi_j\}$ which converges uniformly on compact subsets of U to a holomorphic $\varphi \colon U \to \overline{\Omega}$. Then $\lim_{j\to\infty} \varphi_j(p_j + r_j\xi) = \varphi(p_0) \in \overline{\Omega}$. Since $f(z)$ intersects $a_i(z)$ with multiplicity at least m_i, $g_j(\xi)$ intersects $a_i(\varphi_j(p_j+r_j\xi))$ with multiplicity at least m_j for all $0 \le i \le q-1$ and $1 \le j$. By Hurwitz's theorem g intersects $a_i(\varphi(p_0))$ with multiplicity at least m_j or $g(\mathbf{C})$ is included in $a_i(\varphi(p_0))$ for all $0 \le i \le q-1$.

Applying Lemma 4.2, we obtain

$$\| \ (q - 2N + L_g - 1)T_g(r) \le \sum_{(g,a_j(\varphi(p_0)))\not\equiv 0} N^{(L_g)}(r, \operatorname{div}(g, H_j)) + o(T_g(r)) \le$$

$$\le \sum_{(g,a_j(\varphi(p_0)))\not\equiv 0} \frac{L_g}{m_j} N(r, \operatorname{div}(g, H_j)) + o(T_g(r)) \le$$

$$\le \sum_{(g,a_j(\varphi(p_0)))\not\equiv 0} \frac{L_g}{m_j} T_g(r) + o(T_g(r)).$$

Letting $r \longrightarrow +\infty$, we get

$$q - 2N + L_g - 1 \le \sum_{j=0}^{q-1} \frac{L_g}{m_j} \Leftrightarrow \sum_{j=0}^{q-1} \frac{1}{m_j} \ge \frac{q-2N-1}{L_g} + 1.$$

It is clear that $L_g \le L_f$, then $\sum_{j=0}^{q-1} \frac{1}{m_j} \ge \frac{q-2N-1}{L_f} + 1$. This is a contradiction. Hence f is normal on $\Omega \setminus S$.

By the assumption of Theorem 4.1, $S \cap \Omega$ is an analytic subset of domain Ω with codimension 1, whose singularities are normal crossings. Then f extends to a holomorphic mapping from Ω into $\mathbf{P}^n(\mathbf{C})$ by Theorem 2.3 in Joseph and Kwack [4].

Theorem 4.1 is proved.

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