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ON SINGULARITIES OF THE GALILEAN SPHERICAL DARBOUX RULED SURFACE OF A SPACE CURVE IN G_3

ПРО ОСОБЛИВОСТІ СФЕРИЧНО-ГАЛІЛЕЄВОЇ ЛІНІЙЧАТОЇ ПОВЕРХНІ ДАРБУ ПРОСТОРОВОЇ КРИВОЇ В G_3

We study the singularities of Galilean height functions intrinsically related to Frenet frame along a curve embedded into Galilean space. We establish the relationships between singularities of discriminant and bifurcation sets of the function and geometric invariants of curves in Galilean space.

Досліджено особливості галілеївських функцій висоти, що внутрішньо пов'язані із рамкою Френе вздовж кривої, вкладеної у галілеївський простір. Встановлено співвідношення між особливостями множини дискримінантів та множини біфуркацій функції і геометричними інваріантами кривих у галілеївському просторі.

1. Introduction. Singularity theory, being a direct descendant of differential calculus, is certain to have a great deal of interest to say about geometry and therefore about all the branches of mathematics, physics and other disciplines where the geometrical spirit is a guiding light.

The crucial idea of a versal unfolding is contributed by R. Thom in 1975 which was also emerging in algebraic geometry at the same time. Most of the deeper and more interesting results in [1] hinged on Thom's versal unfolding idea, and it became a central tool in almost all applications of singularity theory inside and outside mathematics.

Several geometers were interested in studying the singularities and generic differential geometry in Euclidean space [1 – 6]. The main point of studying singularity is defining real-valued functions such as squared-distance function and height function defined on a curve or on a surface. The classical invariants of extrinsic differential geometry can be treated as singularities of these two functions. Also, some good approximations to singularity theory in affine geometry can be found in [7 – 10]. Related to the theory, some geometrical applications can be found in [11, 12].

Besides Euclidean geometry, a range of new types of geometries have been invented and developed in the last two centuries. They can be introduced in a variety of manners. One possible way is through projective manner, where one can express metric properties through projective relations. For this purpose a fixed conic (called absolute) in infinity is taken and all metric relations may be considered as projective relations with respect to the absolute. This approach is due to A. Cayley and F. Klein. F. Klein noticed that due to the nature of the absolute, various geometries are possible [13]. Among these geometries, there is also Galilean geometry which is our matter in this paper.

In this paper we will introduce the notion of *Galilean height function* on space curves in G_3, Galilean space. This function is quite useful for the study of singularities of Galilean spherical Darboux ruled surface of space curves in G_3. We also introduce the notion of the line of striction of the Galilean spherical Darboux ruled surface and Galilean spherical Darboux images of space curves in G_3.

As a consequence, we apply ordinary techniques of singularity theory for the function and describe the relationships between the singularities of the above three subjects and differential geometric invariants of space curves in G_3. We also explain by an example that Galilean spherical Darboux ruled surface of space curves in G_3 is a planed surface while Euclidean spherical Darboux ruled surface of space curves in E^3 is a non-planed surface (see Fig. 2).

The techniques used in this paper depend heavily on those in the book of Bruce and Giblin [1].

2. Preliminaries on Galilean geometry. "All geometry is projective geometry" (A. Cayley). From A. Cayley point of view, G_3 is a real 3-dimensional projective space $P^3(\mathbb{R})$, is the set of equivalence classes of $\sim$ on $\mathbb{R}^4 - \{0\}$ by equivalence relation $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Thus, $P^3(\mathbb{R})$ obtained as a factor space on $\mathbb{R}^4 \setminus \{0\}$ by $\sim$, i.e., $P^3(\mathbb{R}) \cong \left(\mathbb{R}^4 - \{0\}\right) / \sim$ [14]. We can think of $P^3(\mathbb{R})$ more geometrically as set of lines through the origin in $\mathbb{R}^4$. G_3 is a real Cayley – Klein space equipped with the projective metric of signature $(0, 0, +, +)$, as showed in [15]. The absolute of the Galilean geometry is an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed elliptic involution of points of f. The points, the lines and the planes of $P^3(\mathbb{R})$ are the one-dimensional, two-dimensional and three-dimensional subspaces of $\mathbb{R}^4$, respectively [16]. Therefore, G_3 contains $\mathbb{R}^3$ as a proper subset and the complement in G_3 to w is diffeomorphic to $\mathbb{R}^3$.

Let P be any point of $\mathbb{R}^3$ with affine coordinates (x, y, z). Write (x, y, z) as $\left(\dfrac{X_1}{X_0}, \dfrac{X_2}{X_0}, \dfrac{X_3}{X_0}\right)$, where X_0 is some common deminator. Call (X_0, X_1, X_2, X_3) the *homogeneous coordinates of* P. Thus, the homogeneous coordinates $(X_0 : X_1 : X_2 : X_3)$ and $\rho\,(X_0 : X_1 : X_2 : X_3)$ refer to the same point, for all $\rho \in \mathbb{R} - \{0\}$ [16]. We now can introduce homogeneous coordinates in G_3 in such a way that the absolute plane w is given by $X_0 = 0$, the absolute line f by $X_0 = X_1 = 0$ and the elliptic involution I by

$$(0 : 0 : X_2 : X_3) \to (0 : 0 : X_3 : -X_2).$$

In affine coordinates, the distance between the points $P_i = (x_i, y_i, z_i)$ for $i = 1, 2$, is defined by

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1| \;, & \text{if } \; x_1 \neq x_2, \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2} & \text{if } \; x_1 = x_2. \end{cases} \tag{1}$$

In the nonhomogeneous coordinates the isometries group B_6 has the form

$$\begin{aligned} \overline{x} &= a + x, \\ \overline{y} &= b + cx + y\cos\varphi + z\sin\varphi, \\ \overline{z} &= d + ex - y\sin\varphi + z\cos\varphi, \end{aligned} \tag{2}$$

where a, b, c, d, e and φ are real numbers. The group of motions of G_3 is a six-parameter group [17].

A vector $A(x, y, z)$ is said to be non-isotropic if $x \neq 0$. All unit non-isotropic vectors are of the form $(1, y, z)$. For isotropic vectors $x = 0$ holds.

For a curve $\gamma : I \to G_3$, $I \subset \mathbb{R}$ parametrized by the invariant parameter $s = x$, given in the coordinate form

$$\gamma(x) = (x, y(x), z(x)),$$

the curvature $\kappa(x)$ and the torsion $\tau(x)$ are defined by

$$\kappa(x) = \sqrt{y''(x)^2 + z''(x)^2},$$

$$\tau(x) = \frac{\det(\gamma'(x), \gamma''(x), \gamma'''(x))}{\kappa^2(x)} \tag{3}$$

and the associated moving trihedron is given by

$$\begin{aligned} t(x) &= \gamma'(x) = (1, y'(x), z'(x)), \\ n(x) &= \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \\ b(x) &= \frac{1}{\kappa(x)}(0, -z''(x), y''(x)). \end{aligned} \tag{4}$$

The vectors $t(x), n(x)$ and $b(x)$ are called the vectors of the tangent, principal normal and the binormal line, respectively [17]. Therefore, the Frenet–Serret formulas can be written in matrix notation as

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{5}$$

From the equations in (4) and (5) one gets an important relation

$$\gamma'''(x) = \kappa'(x)\, n(x) + \kappa(x)\, \tau(x)\, b(x).$$

For any unit special curve $\gamma \colon I \to G_3$, we call $D(x) = \tau(x)\, t(x) + \kappa(x)\, b(x)$ a *Darboux vector of* γ [18]. By using the Darboux vector, Frenet–Serret formulas can be rewritten as follows:

$$\begin{aligned} t(x) &= D(x) \times_G t(x), \\ n(x) &= D(x) \times_G n(x), \\ b(x) &= D(x) \times_G b(x), \end{aligned} \tag{6}$$

where the Galilean cross product $\times_G$is defined by

$$\mathbf{a} \times_G \mathbf{b} = \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{7}$$

for $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ [19, 20].

According to the absolute figure, there are two types of (ideal) lines in the Galilean space-isotropic lines which intersect the absolute line f and non-isotropic lines which do not. A plane is called Euclidean if it contains f, otherwise it is called isotropic. In the given affine coordinates, isotropic vectors are of the form $(0, y, z)$, whereas Euclidean planes are of the form $x = k, k \in \mathbb{R}$.

A ruled surface in the Galilean G_3 is a surface that admits a parametrization

$$\varphi(u, v) = \beta(u) + v a(u),$$

where β is an admissible curve (the directrix), $\mathbf{a}$ is a nowhere vanishing vector field (field of generators) along the curve β and u, v are parameters, $u \in I \subset \mathbb{R}$, $v \in \mathbb{R}$. According to the absolute figure of G_3, we distinguish the following three types of ruled surfaces in G_3 :

Type A. Nonconodial or conodial ruled surfaces whose striction line does not lie in a Euclidean plane.

Type B. Ruled surfaces with the striction line in a Euclidean plane.

Type C. Conodial ruled surfaces with the absolute line as the directional line in infinity [18].

The Galilean sphere S_G^2 is defined by $S_G^2 = \{(x, y, z) \in G_3 \ \mid \ |x - x_0| = r\}$.

For more on Galilean geometry, one can refer to [18, 20] and references there in.

3. Singularities of some functions in Galilean geometry. We define a spherical curve $d\colon I \to S_G^2$ by $d(x) = \dfrac{D(x)}{\|D(x)\|_G}$ and surface

$$dR(\gamma) = \{d(x) + un(x) \ \mid u \in \mathbb{R}, x \in I\}, \tag{8}$$

$$\beta(x) = \left\{ d(x) - \frac{1}{\tau(x)} \left(\frac{\kappa}{\tau}\right)'(x)\, n(x) \ \mid x \in I \right\}. \tag{9}$$

We call the image of d the Galilean spherical Darboux image, the surface $dR(\gamma)$ the Galilean spherical Darboux ruled surface of γ and the curve $\beta(x)$ the line of striction of the Darboux ruled surface.

Theorem 1. *Let $\gamma\colon I \to G_3$ be a unit speed curve. Then we have the following*:

(1) *The line of stiriction of the Galilean spherical Darboux ruled surface image is locally diffeomorphic to the ordinary cusp C at $\beta(x_0)$ if and if only*

$$\left(\frac{\kappa}{\tau}\right)''(x) = \frac{\tau'(x)}{\tau(x)} \left(\frac{\kappa}{\tau}\right)'(x) \quad \textit{and} \quad \left(\frac{\kappa}{\tau}\right)'''(x) \neq \frac{\tau''(x)}{\tau(x)} \left(\frac{\kappa}{\tau}\right)'(x).$$

(2) (a) *The Galilean spherical Darboux ruled surface is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at $d(x_0) + u_0 n(x_0)$ if and if only*

$$u_0 = -\frac{1}{\tau(x_0)} \left(\frac{\kappa}{\tau}\right)'(x_0) \quad \textit{and} \quad \left(\frac{\kappa}{\tau}\right)''(x) \neq \frac{\tau'(x)}{\tau(x)} \left(\frac{\kappa}{\tau}\right)'(x).$$

(b) *The Galilean spherical Darboux ruled surface is locally diffeomorphic to the swallowtail SW at $d(x_0) + u_0 n(x_0)$ if and if only*

$$u_0 = -\frac{1}{\tau(x_0)} \left(\frac{\kappa}{\tau}\right)'(x_0), \qquad \left(\frac{\kappa}{\tau}\right)''(x) = \frac{\tau'(x)}{\tau(x)} \left(\frac{\kappa}{\tau}\right)'(x),$$

and

$$\left(\frac{\kappa}{\tau}\right)'''(x) \neq \frac{\tau''(x)}{\tau(x)} \left(\frac{\kappa}{\tau}\right)'(x).$$

Here, $C = \{(x_1, x_2) : x_1^2 = x_2^3\}$ is ordinary cusp and

$$SW = \{(x_1, x_2, x_3) : x_1 = 3u^4 + u^2 v,\ x_2 = 4u^3 + 2uv,\ x_3 = v\}$$

is the swallowtail (*see Fig.* 1).

The main aim of this paper is proving the preceding theorem, Theorem 1. For this issue, we will study the singularities of height function in Galilean space in Section 3.1. Also, since we need the unfoldings of functions in G_3, we describe the content of them in Section 3.2.

3.1. Families of smooth functions on a space curve in Galilean geometry. In this section families of function on a space curve and surface will be defined which are useful

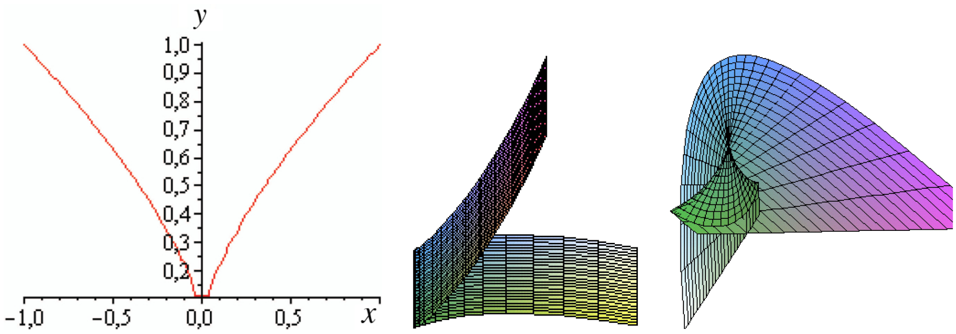

Fig. 1. The cusp curve, the cuspidal edge, the swallowtail surface.

for the study of singularities. Let $\gamma\colon I \to G_3$ be a unit speed curve with $\kappa(x) \neq 0$. We will assume that $\tau(x) \neq 0$ throughout this paper.

3.1.1. Height function in Galilean space. We now define a two-parameter family of smooth functions on I:

$$F_h\colon I \times S_G^2 \to \mathbb{R}$$

by $F_h(x, \mathbf{v}) = |t(x) \quad b(x) \quad \mathbf{v}|$. Here, $|\mathbf{a} \ \ \mathbf{b} \ \ \mathbf{c}|$ denotes the determinant of the matrix $(\mathbf{a} \ \ \mathbf{b} \ \ \mathbf{c})$. We call F_h *a Galilean height function* (*or a normal directed height function*) on γ. We denote that $f_{hv}(x) = F_h(x, \mathbf{v})$ for any $v \in S_G^2$. Then, we have the following proposition.

Proposition 1. *Let $\gamma\colon I \to G_3$ be a unit speed curve with $\kappa(x) \neq 0$ and $\tau(x) \neq$ $\neq 0$. Then,*

(1) $f'_{hv}(x) = 0$ *if and only if there exist real numbers $\mu \in \mathbb{R}$, such that*

$$\mathbf{v} = \pm t(x) + \mu n(x) \pm \left(\frac{\kappa}{\tau}\right)(x)\, b(x),$$

(2) $f'_{hv}(x) = f''_{hv}(x) = 0$ *if and only if*

$$\mathbf{v} = \pm\left(t(x) - \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)\, n(x) + \left(\frac{\kappa}{\tau}\right)(x)\, b(x)\right),$$

(3) $f'_{hv}(x) = f''_{hv}(x) = f'''_{hv}(x) = 0$ *if and only if*

$$\mathbf{v} = \pm\left(t(x) - \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)\, n(x) + \left(\frac{\kappa}{\tau}\right)(x)\, b(x)\right),$$

$$\left(\frac{\kappa}{\tau}\right)''(x) = \frac{\tau'(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x),$$

(4) $f'_{hv}(x) = f''_{hv}(x) = f'''_{hv}(x) = f^{(4)}_{hv}(x) = 0$ *if and only if*

$$\mathbf{v} = \pm\left(t(x) - \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)\, n(x) + \left(\frac{\kappa}{\tau}\right)(x)\, b(x)\right),$$

$$\left(\frac{\kappa}{\tau}\right)'''(x) = \frac{\tau''(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x).$$

Proof. By the Frenet–Serret formula, we have the following calculations:

(i) $f'_{hv}(x) = \kappa(x)\,|n(x)\, b(x)\, \mathbf{v}| - \tau(x)\,|t(x)\, n(x)\, \mathbf{v}|,$

(ii) $f''_{hv}(x) = \kappa'(x)\,|n(x)\, b(x)\, \mathbf{v}| - \tau'(x)\,|t(x)\, n(x)\, \mathbf{v}| - \tau^2(x)\,|t(x)\, b(x)\, \mathbf{v}|,$

(iii) $f_{hv}^{\prime\prime\prime}(x) = \left(\kappa''(x) - \kappa(x)\tau^2(x)\right)|n(x)b(x)\mathbf{v}| + \left(\tau^3(x) - \tau''(x)\right)|t(x)n(x)\mathbf{v}| -$
$-3\tau(x)\tau'(x)|t(x)b(x)\mathbf{v}|$,

(iv) $f_{hv}^{(4)}(x) = \left(\kappa'''(x) - \kappa'(x)\tau^2(x) - 5\kappa(x)\tau(x)\tau'(x)\right)|n(x)b(x)\mathbf{v}| + \left(6\tau^2(x)\tau'(x) -\right.$
$\left.-\tau'''(x)\right)|t(x)n(x)\mathbf{v}| + \left(\tau^4(x) - 3\tau'^2(x) - 4\tau(x)\tau''(x)\right)|t(x)b(x)\mathbf{v}|$.

(1) The assertion is trivial by the formula (i) from the above calculations. By the assumption that $\mathbf{v} \in S_G^2$, we have $\mathbf{v} = \pm t(x) + \mu n(x) + \lambda b(x)$. It follows (i) that $f'_{hv}(x) = \pm\kappa(x) - \lambda\tau(x)$. Since $\tau(x) \neq 0$, $f'_{hv}(x) = 0$ if and only if $\lambda = \pm\left(\frac{\kappa}{\tau}\right)(x)$. Therefore we have

$$\mathbf{v} = \pm t(x) + \mu n(x) \pm \left(\frac{\kappa}{\tau}\right)(x)\, b(x).$$

(2) By (1), we have $\mathbf{v} = \pm t(x) + \mu n(x) \pm \left(\frac{\kappa}{\tau}\right)(x)\, b(x)$. It follows from (ii) that $f''_{hv}(x) = \pm\kappa'(x) \pm \tau'(x)\left(\frac{\kappa}{\tau}\right)(x) + \mu\tau^2(x)$. Since $\tau(x) \neq 0$, $f''_{hv}(x) = 0$ if and only if $\mu = \mp\frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)(x)$. Therefore we have

$$\mathbf{v} = \pm\left(t(x) - \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)\, n(x) + \left(\frac{\kappa}{\tau}\right)(x)\, b(x)\right).$$

(3) If we substitute the formula

$$\mathbf{v} = \pm\left(t(x) - \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)\, n(x) + \left(\frac{\kappa}{\tau}\right)(x)\, b(x)\right)$$

into (iii), then we have

$$\kappa''(x)\tau^2(x) - \kappa(x)\tau(x)\tau''(x) - 3\kappa'(x)\tau(x)\tau'(x) + 3\kappa(x)\tau'^2(x) = 0.$$

Therefore, we have $\left(\frac{\kappa}{\tau}\right)''(x) = \frac{\tau'(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)$ the assertion (3) follows.

(4) We also substitute the formula (3) into (iv), then we have

$$\kappa'''(x)\tau^3(x) - \kappa(x)\tau^2(x)\tau'''(x) + 3\kappa(x)\tau'^3(x) + 4\kappa(x)\tau(x)\tau'(x)\tau''(x) =$$
$$= +3\kappa'(x)\tau(x)\tau'^2(x) + 4\kappa'(x)\tau^2(x)\tau''(x). \tag{10}$$

If $\left(\frac{\kappa}{\tau}\right)''(x) = \frac{\tau'(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)$ then we can show that $\left(\frac{\kappa}{\tau}\right)'''(x) = \frac{\tau''(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)$. We have assertion (4)

$$3\kappa(x)\tau'^3(x) + 4\kappa(x)\tau(x)\tau'(x)\tau''(x) = 0.$$

Proposition 1 is proved.

We now study the geometric properties of the spherical Darboux ruled surface of space curves in G_3. By the propositions in the last section, we can recognize that the function $\left(\frac{\kappa}{\tau}\right)'(x)$ and the modified Darboux vector $\left(\frac{\tau}{\kappa}\right)(x)\, t(x) + b(x)$ are important subjects. If $\left(\frac{\kappa}{\tau}\right)(x) \equiv c$ (constant) then the curve $\gamma(x)$ in G_3 has been classically known as a *helix* in Galilean space [18]. Galilean cycle is the only curves of constant curvature in plane [20]. For a unit speed regular curve $\gamma(x)$ has tangent curve $\sigma\colon I \to S_G^2$, $\sigma(x) = t(x)$ is called the Galilean spherical tangential image of $\gamma(x)$.

Proposition 2. *Let $\gamma\colon I \to G_3$ be a unit speed regular curve. Then $\gamma(x)$ is a helix if and only if the modified Darboux vector $d\,(x)$ is a constant vector. In this case we have the following assertions*:

(1) *The Galilean spherical tangential image $\sigma\,(x)$ of $\gamma\,(x)$ is a cycle on the unit Galilean sphere S^2_G.*

(2) *The Galilean spherical Darboux ruled surface of $\gamma\,(x)$ is a plane given by $e + un\,(x)$. Where $e = d\,(x)$.*

Proof. By the Frenet–Serret formulas, we can show that $\widetilde{D}'\,(x) = \left(\dfrac{\tau}{\kappa}\right)'(x)\,t\,(x)$. Therefore, $\gamma(x)$ is a helix if and only if $\widetilde{D}'\,(x) \equiv 0$. This condition is equivalent to the condition that $\widetilde{D}\,(x)$ is a constant vector. In this case we have

$$\sigma\,(x) = t\,(x)\,,$$

$$\sigma'\,(x) = \kappa\,(x)\,n\,(x)\,,$$

$$\sigma''\,(x) = \kappa'\,(x)\,n\,(x) + \kappa\,(x)\,\tau\,(x)\,b\,(x)\,.$$

The curvature of $\sigma\,(x)$ is $\kappa_\sigma\,(x) = \left(\dfrac{\kappa}{\tau}\right)(x) =$ constant. This means that the Galilean spherical tangential image $\sigma\,(x)$ is a cycle on the unit Galilean sphere S^2_G [20]. The assertion (2) is clear by definition.

Proposition 2 is proved.

The singularities of the Galilean spherical Darboux image describe how the shape of the curve γ is similar to a helix.

3.2. Unfoldings of functions by one-variable. In this section, we will use some general results on singularity theory for families of function germs. Let

$$F\colon\ (I \times \mathbb{R}^r, (x_0, w_0)) \to \mathbb{R}$$

be a function germ. We call F an *r-parameter unfolding of* f, where $f\,(x) = F_{w_0}\,(x, w_0)$. We say that f has *A_k-singularity at x_0* if $f^{(p)}\,(x_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}\,(x_0) \neq 0$. We also say that f has *$A_{\geq k}$-singularity at x_0* if $f^{(p)}\,(x_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f\,(x)$ has *A_k-singularity* $(k \geq 1)$ *at* x_0. We denote the $(k-1)$-jet of the partial derivative $\dfrac{\partial F}{\partial w_i}$ *at* x_0 by $J^{k-1}\left(\dfrac{\partial F}{\partial w_i}\,(x, w_0)\right)(x_0) = \sum_{j=1}^{k-1} \alpha_{ij} x^j$ for $i = 1, \ldots, r$. Then F is called *a (p)-versal unfolding* if the $\big((k-1)\times r\big)$-matrix of coefficients (α_{ij}) has rank $k-1$, $k-1 \leq r$. Under the same conditions as the above, then F is called *a versal unfolding* if the $(k \times r)$-matrix of coefficients $(\alpha_{0i}, \alpha_{ij})$ has rank k, $k \leq r$, where $\alpha_{0i} = \dfrac{\partial F}{\partial w_i}\,(x_0, w_0)$.

We now introduce important sets concerning the unfoldings relative to the above notions. The *bifurcation set* B_F of F is the set

$$B_F = \left\{ w \in \mathbb{R}^r \mid \frac{\partial F}{\partial w} = \frac{\partial^2 F}{\partial w^2} = 0 \ at\ \ (x, w) \right\}.$$

The *discriminant set* of F is the set

$$D_F = \left\{ w \in \mathbb{R}^r \mid \frac{\partial F}{\partial w} = 0 \ at\ \ (x, w) \right\}.$$

Then we have the following well-known result [1].

Theorem 2. *Let* $F:(I\times\mathbb{R}^r,(x_0,w_0))\to\mathbb{R}$ *be an r-parameter unfolding of* $f(x)$ *which has* A_k*-singularity at* x_0.

(1) *Suppose that* F *is a (p)-versal unfolding*:

(a) *if* $k=2$, *then* B_F *is locally diffeomorphic to* $\{0\}\times\mathbb{R}^{r-1}$;

(b) *if* $k=3$, *then* B_F *is locally diffeomorphic to* $C\times\mathbb{R}^{r-2}$;

(c) *if* $k=4$, *then* B_F *is locally diffeomorphic to* $SW\times\mathbb{R}^{r-3}$.

(2) *Suppose that* F *is a versal unfolding*:

(a) *if* $k=1$, *then* D_F *is locally diffeomorphic to* $\{0\}\times\mathbb{R}^{r-1}$;

(b) *if* $k=2$, *then* D_F *is locally diffeomorphic to* $C\times\mathbb{R}^{r-2}$;

(c) *if* $k=3$, *then* D_F *is locally diffeomorphic to* $SW\times\mathbb{R}^{r-3}$.

Here, $C=\left\{(x_1,x_2):x_1^2=x_2^3\right\}$ *is ordinary cusp and*

$$SW=\left\{(x_1,x_2,x_3):x_1=3u^4+u^2v,\ x_2=4u^3+2uv,\ x_3=v\right\}$$

is the swallowtail (*see Fig.* 1).

For the proof of Theorem 1, we have the following key propositions.

Proposition 3. *Let* $F_h\colon I\times S_G^2\to\mathbb{R}$ *be the Galilean height function on a unit speed curve* $\gamma(x)$. *If* f_{hv_0} *has* A_k*-singularity* $(k=2,3)$ *at* x_0, *then* F_h *is a* (p)*-versal unfolding of* f_{hv_0}.

Proof. We denote by $\gamma(x)=(x,y(x),z(x))$ and $v=(1,v_2,v_3)$. By definition, we have

$$F_h(x,\mathbf{v})=|t(x)\ \ b(x)\ \ \mathbf{v}|=$$

$$=\frac{1}{\kappa(x)}\left[-z''(x)\,v_3-y''(x)\,v_2+y'(x)\,y''(x)+z'(x)\,z''(x)\right].$$

Let $J^{k-1}\left(\dfrac{\partial F_h}{\partial v_i}(x,v_0)\right)(x_0)$ be the $(k-1)$-jet of $\dfrac{\partial F_h}{\partial v_i}$, $i=2,3$, at x_0; then we have

$$J^3\left(\frac{\partial F_h}{\partial v_i}(x,v_0)\right)(x_0)=-n_i'(x_0)\,x-\frac{1}{2}n_i''(x_0)\,x^2-\frac{1}{6}n_i'''(x_0)\,x^3, i=2,3.$$

Here, $n(x)=(0,n_2,n_3)=\dfrac{1}{\kappa(x)}(0,y''(x),z''(x))$ by the equation (5). We distinguish two cases.

Case (1). When f_{hv_0} has the A_2*-singularity at* x_0, we can define (1×2)-matrix A as follows:

$$A=\left[\left(-\frac{y''(x_0)}{\kappa(x_0)}\right)'\quad\left(-\frac{z''(x_0)}{\kappa(x_0)}\right)'\right].$$

We also have $A(x)=-n'(x)=-\tau(x)\,b(x)\neq 0$ by the equation (5). Therefore we have $\operatorname{Rank}A=1$.

Case (2). When f_{hv_0} has the A_3*-singularity at* x_0, we define (2×2)-matrix A as follows:

$$B=\begin{bmatrix}\left(-\dfrac{y''(x_0)}{\kappa(x_0)}\right)' & \left(-\dfrac{z''(x_0)}{\kappa(x_0)}\right)'\\ \left(-\dfrac{y''(x_0)}{\kappa(x_0)}\right)'' & \left(-\dfrac{z''(x_0)}{\kappa(x_0)}\right)''\end{bmatrix}$$

to be nonsingular. That is to say $\det B=|B|\neq 0$. Here, we can show by direct calculations but rather long calculation. We will use a method simpler than that. By the

Frenet – Serret formulas (5), we have the following calculation:

$$|B| = |t(x_0)\ \ n'(x_0)\ \ n''(x_0)|\,.$$

If the necessary derivatives of the Frenet – Serret formulas (5) is written, then we have $|B| = -\tau^3(x_0)$. Since $\tau(x) \neq 0$, the rank of B is 2.

Proposition 3 is proved.

Let's define a function $\widetilde{F}_h\colon I \times S^2_G \times \mathbb{R} \to \mathbb{R}$ by $\widetilde{F}_h(x, v, w) = F(x, v) - w$ and $f_{hv,w}(x) = \widetilde{F}_h(x, v, w)$.

Proposition 4. *If f_{hv_0,w_0} has A_k-singularity $(k = 1, 2, 3)$ at x_0, then F_h is a versal unfolding of f_{hv_0,w_0}.*

Proof. Using the same notations of Proposition 3, we have

$$\widetilde{F}_h(x, v, v_1) = \frac{1}{\kappa(x)}\left[-z''(x)\,v_3 - y''(x)\,v_2 + y'(x)\,y''(x) + z'(x)\,z''(x)\right] - v_1.$$

Let $J^{k-1}\left(\dfrac{\partial \widetilde{F}_h}{\partial v_i}(x, v_0)\right)(x_0)$ be the $(k-1)$-jet of $\dfrac{\partial \widetilde{F}_h}{\partial v_i}$, $i = 1, 2, 3$, at x_0; then we have

$$\frac{\partial \widetilde{F}_h}{\partial v_1}(x_0, v_0) + J^2\left(\frac{\partial \widetilde{F}_h}{\partial v_1}(x, v_0)\right)(x_0) = -1,$$

$$\frac{\partial \widetilde{F}_h}{\partial v_i}(x_0, v_0) + J^2\left(\frac{\partial \widetilde{F}_h}{\partial v_i}(x, v_0)\right)(x_0) = -n_i(x_0) - n_i'(x_0)\,x - n_i''(x_0)\frac{x^2}{2}, \quad i = 2, 3.$$

Now, we will distinguish three cases.

Case (1). When f_{hv_0,w_0} has the A_1*-singularity at* x_0, we define (1×2)-matrix C as follows:

$$C = \begin{bmatrix} -1 & \left(-\dfrac{y''(x_0)}{\kappa(x_0)}\right) & \left(-\dfrac{z''(x_0)}{\kappa(x_0)}\right) \end{bmatrix}.$$

The rank of C is clearly 1.

Case (2). When f_{hv_0,w_0} has the A_2*-singularity at* x_0, we require (2×3)-matrix:

$$D = \begin{bmatrix} -1 & \left(-\dfrac{y''(x_0)}{\kappa(x_0)}\right) & \left(-\dfrac{z''(x_0)}{\kappa(x_0)}\right) \\[2ex] 0 & \left(-\dfrac{y''(x_0)}{\kappa(x_0)}\right)' & \left(-\dfrac{z''(x_0)}{\kappa(x_0)}\right)' \end{bmatrix}$$

to have the maximal rank. By the case 1 in Proposition 3, the second line of D does not vanish. Thus the rank of D is 2.

Case (3). When f_{hv_0,w_0} has the A_3*-singularity at* x_0, we define (3×3)-matrix:

$$E = \begin{bmatrix} -1 & \left(-\dfrac{y''(x_0)}{\kappa(x_0)}\right) & \left(-\dfrac{z''(x_0)}{\kappa(x_0)}\right) \\[2ex] 0 & \left(-\dfrac{y''(x_0)}{\kappa(x_0)}\right)' & \left(-\dfrac{z''(x_0)}{\kappa(x_0)}\right)' \\[2ex] 0 & \left(-\dfrac{y''(x_0)}{\kappa(x_0)}\right)'' & \left(-\dfrac{z''(x_0)}{\kappa(x_0)}\right)'' \end{bmatrix}$$

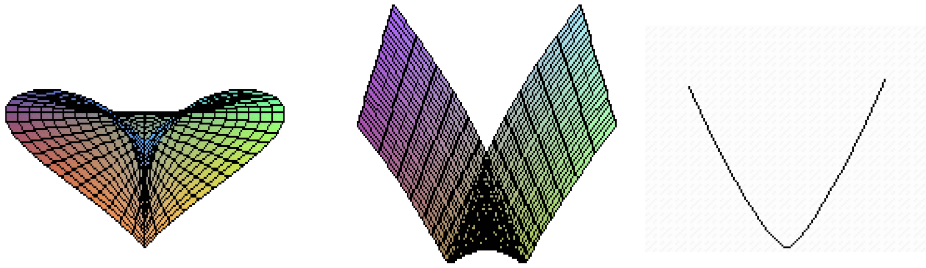

Fig. 2. The Euclidean spherical Darboux ruled surface, the Galilean spherical Darboux ruled surface, the line of striction of $dR(\gamma)$.

to be nonsingular. By the case 2 in Proposition 4, determinant of E does not vanish. It means that the rank of E is 3.

Proposition 4 is proved.

Proof of Theorem 1 follows from, Propositions 1, 3, 4 and Theorem 2.

***Example* 1.** Consider the curve $\gamma\colon I \subset \mathbb{R} \to E^3$, $\gamma(x) = \left(x, \dfrac{x^2}{\sqrt{2}}, \dfrac{x^3}{3}\right)$. For an arbitrary speed curve $\gamma\colon I \to E^3$, $I \subset \mathbb{R}$ the associated moving trihedron is given by

$$T(x) = \frac{\gamma'(x)}{\|\gamma'(x)\|}, \quad B(x) = \frac{\gamma'(x) \times \gamma''(x)}{\|\gamma'(x) \times \gamma''(x)\|}, \quad N(x) = B(x) \times T(x)$$

and the curvature $\kappa(x)$ and the torsion $\tau(x)$ are defined by

$$\kappa(x) = \frac{\|\gamma'(x) \times \gamma''(x)\|}{\|\gamma'(x)\|^3}, \quad \tau(x) = \frac{\det(\gamma'(x), \gamma''(x), \gamma'''(x))}{\kappa^2(x)}.$$

The vectors $t(x), n(x)$ and $b(x)$ are called the vectors of the tangent, principal normal and the binormal line, respectively [21].

We compute the Frenet apparatus of the curve $\gamma(x) = \left(x, \dfrac{x^2}{\sqrt{2}}, \dfrac{x^3}{3}\right)$. If the necessary derivatives of the Frenet–Serret formulas is written, then we have

$$T(x) = \frac{1}{1+x^2}\left(1, \sqrt{2}x, x^2\right), \quad B(x) = \frac{1}{1+x^2}\left(x^2, -\sqrt{2}x, 1\right),$$

$$N(x) = \frac{1}{1+x^2}\left(-\sqrt{2}x, 1-x^2, \sqrt{2}x\right), \quad \kappa(x) = \tau(x) = \frac{\sqrt{2}}{(1+x^2)^2}.$$

Therefore we compute $dR(\gamma) = \{d(x) + uN(x) \mid u \in \mathbb{R}, x \in I\}$ surface, here a spherical curve $d\colon I \to S^2$ by $d(x) = \dfrac{D(x)}{\|D(x)\|}$ (see Fig. 2). Hence, we have

$$d(x) + uN(x) = \frac{\kappa(x)}{\sqrt{\kappa^2(x) + \tau^2(x)}}\left(\frac{\tau(x)}{\kappa(x)}T(x) + B(x)\right) + uN(x) =$$

$$= \left(\frac{1+x^2}{\sqrt{2}} - u\frac{\sqrt{2}x}{1+x^2}, u\frac{1-x^2}{1+x^2}, \frac{1+x^2}{\sqrt{2}} + u\frac{\sqrt{2}x}{1+x^2}\right). \tag{11}$$

We also consider the curve $\gamma\colon I \subset \mathbb{R} \to G_3$, $\gamma(x) = \left(x, \dfrac{x^2}{\sqrt{2}}, \dfrac{x^3}{3}\right)$. If the necessary derivatives of the Frenet–Serret formulas (5) is written, then we have

$$t(x) = \left(1, \sqrt{2}x, x^2\right), \quad n(x) = \frac{1}{\sqrt{2+4x^2}}\left(0, \sqrt{2}, 2x\right),$$

$$b(x) = \frac{1}{\sqrt{2+4x^2}}\left(0, -2x, \sqrt{2}\right), \quad \kappa(x) = \sqrt{2+4x^2}, \quad \tau(x) = \frac{\sqrt{2}}{2+4x^2}.$$

Therefore we compute $dR(\gamma) = \{d(x) + un(x) \mid u \in \mathbb{R}, x \in I\}$ surface, here a spherical curve $d\colon I \to S_G^2$ by $d(x) = \dfrac{D(x)}{\|D(x)\|_G}$. Hence, we have

$$d(x) + un(x) = \left(t(x) + \frac{\kappa(x)}{\tau(x)}b(x)\right) + un(x) =$$

$$= \left(1, -\sqrt{2}x - 4\sqrt{2}x^3 + \frac{u}{\sqrt{1+2x^2}}, 2 + 5x^2 + \frac{2ux}{\sqrt{2+4x^2}}\right).$$

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