

**ON A CONTINUED FRACTION OF ORDER TWELVE****ПРО ЛАНЦЮГОВИЙ ДРІБ ДВАНДЦЯТОГО ПОРЯДКУ**

We present some new relations between a continued fraction  $U(q)$  of order twelve (which is established by M. S. M. Naika et al.) and  $U(q^n)$  for  $n = 7, 9, 11$  and  $13$ .

Наведено деякі нові співвідношення між ланцюговим дробом  $U(q)$  дванадцятого порядку (який описано М. С. М. Найка та іншими авторами) і  $U(q^n)$  для  $n = 7, 9, 11$  та  $13$ .

**1. Introduction.** Throughout the paper, we assume  $|q| < 1$  and for a positive integer  $n$ , we use the standard notation

$$(a)_0 := (a; q)_0 = 1,$$

$$(a)_n := (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$$

and

$$(a)_\infty := (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

In Chapter 16 of his Second Notebook [12, p. 197; 3, p. 34] S. Ramanujan develops the theory of theta functions and his theta function is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \quad |ab| < 1.$$

Following Ramanujan [12, p. 197], we define

$$\phi(q) := f(q, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} = \frac{(-q; -q)_\infty}{(q; -q)_\infty},$$

$$\psi(q) := f(q, q^3) = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}} = (q; q)_\infty.$$

For convenience, we denote

$$f(-q^n) = f_n.$$

The celebrated Rogers–Ramanujan continued fraction is defined as

$$R(q) := \frac{q^{1/5} f(-q, -q^4)}{f(-q^2, -q^3)} = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots \quad (1.1)$$

On p. 365 of his Lost Notebook [13], Ramanujan recorded five identities, which shows the relations between  $R(q)$  and five continued fractions  $R(-q)$ ,  $R(q^2)$ ,  $R(q^3)$ ,  $R(q^4)$  and  $R(q^5)$ . He also recorded these identities at the scattered places of his Notebooks [12]. L. J. Rogers [14] found the modular equations relating  $R(q)$  and  $R(q^n)$  for  $n = 2, 3, 5$  and 11, the latter equations is not found in Ramanujan's works. For a proof of these, one may refer [4]. Recently K. R. Vasuki and S. R. Swamy [19] found the modular equation relating  $R(q)$  with  $R(q^7)$ .

The Ramanujan's cubic continued fraction  $G(q)$  be defined by

$$G(q) := \frac{q^{1/3} f(-q, -q^5)}{f(-q^3, -q^3)} = \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \dots \quad (1.2)$$

The continued fraction (1.2) was first introduced by Ramanujan in his second letter to G. H. Hardy [10]. He also recorded the continued fraction (1.2) on p. 365 of his Lost Notebook [13] and claims that there are many results of  $G(q)$  which are analogous to the famous Rogers–Ramanujan continued fraction (1.1).

Motivated by Ramanujan's claim H. H. Chan [5], N. D. Baruah [1], K. R. Vasuki and B. R. Srivatsakumar [18] have established the modular relations between  $G(q)$  and  $G(q^n)$  for  $n = 2, 3, 5, 7, 11$  and 13.

The Ramanujan Göllnitz–Gordon continued fraction [13, p. 44; 8; 9] is defined by

$$H(q) := \frac{q^{1/2} f(-q^3, -q^5)}{f(-q, -q^7)} = \frac{q^{1/2}}{1} + \frac{q^2}{1 + q^3} + \frac{q^4}{1 + q^5} + \dots \quad (1.3)$$

Chan and S. S. Hang [6], K. R. Vasuki and B. R. Srivatsakumar [17] have established the identities which gives the relations between  $H(q)$  and  $H(q^n)$  with  $n = 3, 4, 5, 11$  by employing the modular equations stated by Ramanujan. Recently B. Cho, J. K. Koo and Y. K. Park [7] have extended the above results on continued fraction (1.3) to all odd primes  $p$  by computing the affine models of modular curves  $X(\Gamma)$  with  $\Gamma = \Gamma_1(8) \cap \Gamma_0(16p)$ .

Motivated by the above works on the continued fractions (1.1)–(1.3), in this paper, we establish the relation between the following continued fraction  $U(q)$  with  $U(q^n)$  for  $n = 2, 7, 9, 11$  and 13

$$U(q) := \frac{q f(-q, -q^{11})}{f(-q^5, -q^7)} = \frac{q(1-q)}{(1-q^3)} + \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)} + \frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})} + \dots \quad (1.4)$$

The continued fraction (1.4) was established by M. S. M. Naika et al. [11] as a particular case of fascinating continued fraction identity recorded by Ramanujan in his Second Notebook [12, p. 74]. Furthermore they have also established the modular relation between the continued fraction  $U(q)$  with  $U(q^n)$  with  $n = 3$  and 5.

**2. Some preliminary results.****Theorem 2.1** [11]. *We have*

$$\frac{\phi(q)}{\phi(q^3)} = \frac{1+U(q)}{1-U(q)}. \quad (2.1)$$

**Theorem 2.2.** *We have*

$$\frac{\phi^2(-q)}{\phi^2(-q^3)} = 1 - \frac{4U(q)}{1+U^2(q)}. \quad (2.2)$$

**Proof.** From [2], we have

$$(\phi^2(q) + \phi^2(q^3))^2 = \frac{4\phi(q^3)\phi^3(-q^3)\phi(q)}{\phi(-q)}, \quad (2.3)$$

and also from [16], we have

$$(3\phi^2(q^3) - \phi^2(q))^2 = \frac{4\phi(q^3)\phi(q)\phi^3(-q)}{\phi(-q^3)}. \quad (2.4)$$

From (2.3) and (2.4), we deduce that

$$\frac{\phi^2(-q)}{\phi^2(-q^3)} = \frac{3 - \frac{\phi^2(q)}{\phi(q^3)}}{1 + \frac{\phi^2(q)}{\phi^2(q^3)}}.$$

Employing (2.1) in the right-hand side of the above identity, we obtain (2.2).

**Theorem 2.3.** *We have*

$$\frac{\psi^2(q^2)}{q\psi^2(q^6)} = U(q) + \frac{1}{U(q)} - 1. \quad (2.5)$$

**Proof.** From [2], we obtain

$$1 - \frac{\phi^2(-q)}{\phi^2(-q^3)} = \frac{4qf_1f_{12}^3}{f_4f_3^3}, \quad (2.6)$$

and also from [2], we have

$$1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)} = \frac{f_3^3f_4}{qf_1f_{12}^3}.$$

Using (2.6) and the above identity, we find that

$$1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)} = \frac{4}{1 - \frac{\phi^2(-q)}{\phi^2(-q^3)}},$$

which upon using (2.2) gives the required result.

**Theorem 2.4.** *We have*

$$\frac{\psi^2(q)}{q^{1/2}\psi^2(q^3)} = \frac{1+U(q)}{1-U(q)} \sqrt{U(q) + \frac{1}{U(q)} - 1}. \quad (2.7)$$

**Proof.** Consider,

$$\frac{\psi^4(q)}{q\psi^4(q^3)} = \frac{\phi^2(q)\psi^2(q^2)}{q\phi^2(q^3)\psi^2(q^6)} \quad (2.8)$$

where, we have used the identity  $\phi(q)\psi(q^2) = \psi^2(q)$  from Entry 25 [3, p. 40]. Now using (2.1) and (2.7) in the right-hand side of the above, we obtain the required result.

**Theorem 2.5.** *We have*

$$\frac{\psi^4(-q)}{q\psi^4(-q^3)} = \left[ \frac{U^2(q) + 1}{U(q)} + \frac{4U(q)}{1 + U^2(q)} - 5 \right]. \quad (2.9)$$

**Proof.** Changing  $q$  to  $-q$  in (2.8) and then using (2.2) and (2.5) on the right-hand side of (2.8), we obtain (2.9).

To prove our main results, we require following easily deducible identities:

$$\begin{aligned} \phi(q) &= \frac{f_2^5}{f_1^2 f_4^2}, & \psi(q) &= \frac{f_2^2}{f_1}, \\ \phi(-q) &= \frac{f_1^2}{f_2}, & \psi(-q) &= \frac{f_1 f_4}{f_2}, \\ f(q) &= \frac{f_2^3}{f_1 f_4}, & \chi(q) &= \frac{f_2^2}{f_1 f_4} \end{aligned}$$

and

$$\chi(-q) = \frac{f_1}{f_2}. \quad (2.10)$$

And also the following identities analogous to Rogers–Ramanujan forty identities:

**Theorem 2.6** [15]. *Let*

$$M(q) = \frac{f(-q^5, -q^7)}{f_4}$$

and

$$N(q) = \frac{f(-q, -q^{11})}{f_4}.$$

Then

$$M(q^2)N(q) + qM(q)N(q^2) = \frac{f_3 f_{24}}{f_4 f_8}, \quad (2.11)$$

$$M(q^2)N(q) - qM(q)N(q^2) = \frac{f_1^2 f_6 f_{24}}{f_2 f_3 f_4 f_8}, \quad (2.12)$$

$$M(q^3)N(q) + q^2 M(q)N(q^3) = \frac{f_1 f_6^5 f_9 f_{36}}{f_2^2 f_3^2 f_{12}^3 f_{18}}, \quad (2.13)$$

$$M(q^3)N(q) - q^2 M(q)N(q^3) = \frac{f_1 f_{18}^2}{f_4 f_9 f_{12}}, \quad (2.14)$$

$$M(q^5)N(q) - q^4M(q)N(q^5) = \frac{f_1f_6^2f_{30}}{f_2f_3f_{12}f_{20}} \quad (2.15)$$

and

$$M(q)M(q^5) - q^6N(q)N(q^5) = \frac{f_5f_6f_{30}^2}{f_4f_{10}f_{15}f_{60}}. \quad (2.16)$$

A proof of (2.11)–(2.16) can be found in [15].

**3. Main results.** In this section, we derive the relation between the continued fraction  $U(q)$  with  $U(q^n)$  for  $n = 2, 3, 5, 7, 9, 11$  and  $13$ .

**Theorem 3.1.** Let  $u := U(q)$  and  $v := U(q^2)$ . Then

$$u^2 - v + 2uv - u^2v + v^2 = 0.$$

**Proof.** By (1.4), we have

$$U(q) = \frac{qN(q)}{M(q)}. \quad (3.1)$$

Dividing (2.11) by (2.12) and on using (2.10), we obtain

$$\frac{M(q^2)N(q) + qM(q)N(q^2)}{M(q^2)N(q) - qM(q)N(q^2)} = \frac{\varphi(-q^3)}{\varphi(-q)}.$$

Squaring on both sides of the above identity and using (3.1) on the left-hand side and (2.2) on the right-hand side of the above identity and then dividing throughout by  $4u$ , we deduce the required result.

**Theorem 3.2** [11]. Let  $u = U(q)$  and  $v = U(q^3)$ . Then

$$u^3 - v^3 + v^2 - v + 3uv - 3u^2v^2 + u^3v^2 - u^3v = 0.$$

**Proof.** Dividing (2.14) by (2.13) and on using (2.10), we find that

$$\frac{M(q^3)N(q) - q^2M(q)N(q^3)}{M(q^3)N(q) + q^2M(q)N(q^3)} = \frac{\phi(-q^2)\phi(-q^3)\phi(-q^{18})}{\phi(-q^6)\phi(-q^9)\phi(-q^6)}.$$

Squaring on both sides of the above identity, and using (3.1) on the left-hand side and employing the identity

$$\phi(q)\phi(-q) = \phi^2(-q^2)$$

on the right-hand side of the above, we obtain

$$\left(\frac{U(q) - U(q^3)}{U(q) + U(q^3)}\right)^2 = \frac{\phi(q)}{\phi(q^3)} \frac{\phi(-q)}{\phi(-q^3)} \frac{\phi(-q^3)}{\phi(-q^9)} \frac{\phi(q^9)}{\phi(q^3)}.$$

Squaring again on both sides of the above, employing (2.1) and (2.2) on the right-hand side of the above identity and then factorizing using maple, we deduce that

$$\begin{aligned} & 8(-v^3 + v^2 - v + 3uv - 3u^2v^2 + u^3v^2 - u^3v + u^3) \times \\ & \times (u^5v - u^4v^3 + u^4v^2 - 3u^4v + 5u^3v^3 - u^3v^2 - u^3v + u^3 - u^2v^5 + u^4v^2 + \end{aligned}$$

$$+u^2v^3 - 5u^2v^2 + 3v^4u - uv^3 + uv^2 - v^4) = 0.$$

Now, from the definition of  $u$  and  $v$ , we have

$$u = U(q) = 1 - q + q^5 - q^6 + q^7 + \dots \quad (3.2)$$

and

$$v = U(q^3) = 1 - q^3 + q^{15} - q^{18} + q^{21} + \dots$$

Using (3.2) and the above in the above factors, we see that the first factor becomes

$$q^3(3 - 6q + 3q^2 - 3q^3 + 6q^4 + 3q^5 - 9q^6 + \dots),$$

and the second factor becomes

$$-q^3(10 - 22q + 15q^2 - 22q^3 + 33q^4 - 9q^5 + \dots).$$

Thus the second factor does not vanish. Hence by the identity Theorem, we must have

$$v^3 - v^2 + v - 3uv + 3u^2v^2 - u^3v^2 + u^3v - u^3 = 0.$$

Theorem 3.2 is proved.

**Theorem 3.3** [11]. *Let  $u := U(q)$  and  $v := U(q^5)$ . Then*

$$uv^6 - u^6v^5 + 5u^5v^5 - 5u^4v^5 - 5uv^5 + 10v^4u^3 + 5uv^4 - 10u^4v^3 - 10u^2v^3 + \\ + 5u^5v^2 + 10u^3v^2 - 5u^5v - 5u^2v + 5uv - v + u^5 = 0.$$

**Proof.** Dividing (2.15) by (2.16) and using (2.10), we find that

$$\frac{M(q^5)N(q) - q^4M(q)N(q^5)}{M(q)M(q^5) - q^6N(q)N(q^5)} = \frac{q\psi(-q)\psi(-q^{15})}{\psi(-q^3)\psi(-q^5)}.$$

Taking power 4 on both sides of the above and employing (2.9) on the right-hand side of the above and then factorizing using Maple, we deduce that

$$(u^2 + u^2v^2 + 1 - 4uv + v^2)(uv^6 - u^6v^5 + 5u^5v^5 - 5u^4v^5 - 5uv^5 + 10v^4u^3 + 5uv^4 - \\ - 10u^4v^3 - 10u^2v^3 + 5u^5v^2 + 10u^3v^2 - 5u^5v - 5u^2v + 5uv - v + u^5) = 0.$$

Changing  $q$  to  $q^5$  in (3.2), we find that

$$v = U(q^5) = 1 - q^5 + q^{25} - q^{30} + q^{35} + \dots$$

Using (3.2) and the above in the above factors, we see that the first factor becomes

$$q^2(2 - 4q^4 + 2q^5 - 4q^6 + 4q^8 - 2q^9 + 3q^{10} + \dots),$$

and the second factor becomes

$$-q^7(40 - 40q + 20q^2 + 10q^3 - 55q^4 + 110q^5 - 180q^6 + \dots).$$

Thus the first factor does not vanish. Hence by the identity theorem, we must have

$$uv^6 - u^6v^5 + 5u^5v^5 - 5u^4v^5 - 5uv^5 + 10v^4u^3 + 5uv^4 - 10u^4v^3 - 10u^2v^3 + \\ + 5u^5v^2 + 10u^3v^2 - 5u^5v - 5u^2v + 5uv - v + u^5 = 0.$$

Theorem 3.3 is proved.

**Theorem 3.4.** Let  $u = U(q)$  and  $v = U(q^7)$ . Then

$$\begin{aligned} &7u^3v^2 - 35u^3v^4 - 7u^3v - 7u^3v^3 + 35u^4v^3 - u^7 + \\ &+ 7u^5v - 7u^6v + 7u^7v - 28u^5v^2 + 28u^6v^2 - 14v^2u^7 + 7v^3u^5 - 7v^3u^6 + \\ &+ 7v^3u^7 - 35v^4u^5 + 7v^5u - 7v^5u^2 + 7v^5u^3 + 35v^5u^4 - 7v^5u^5 + 28v^5u^6 - \\ &- 7v^5u^7 - 14v^6u + 28v^6u^2 - 28v^6u^3 + 7v^6u^5 - 28v^6u^6 + 7v^6u^7 + 7v^7u - \\ &- 7v^7u^2 + 7v^7u^3 - 7v^7u^5 + 14v^7u^6 - 7v^7u^7 + v^7u^8 - v^8u + v - 7uv + \\ &+ 7uv^2 + 14u^2v - 28u^2v^2 - 7uv^3 + 28u^2v^3 = 0. \end{aligned}$$

**Proof.** If

$$P = \frac{\phi(q)}{\phi(q^3)}$$

and

$$Q = \frac{\phi(q^7)}{\phi(q^{21})},$$

then

$$\begin{aligned} &\left(\frac{Q}{P}\right)^4 - \left(\frac{P}{Q}\right)^4 + 14 \left[ \left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2 \right] = \\ &(PQ)^3 + \frac{27}{(PQ)^3} + 7 \left( PQ + \frac{3}{PQ} \right) \left[ 1 - \left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2 \right]. \end{aligned} \quad (3.3)$$

We have deduced the identity (3.3) on the lines remarked by N. D. Baruah [1]. Now employing (2.1) in (3.3), we deduce the required result.

**Theorem 3.5.** Let  $u = U(q)$  and  $v = U(q^9)$ , then

$$\begin{aligned} &9u^4v + 126v^4u - 90uv^3 + 27v^8u^3 + v - 9vu + 45v^2u + 378u^7v^5 + \\ &+ 135u^8v^4 + 54u^8v^2 - 9u^8v + 18u^7v + 558v^4u^6 - 558v^5u^3 + 561v^4u^3 - 10u^9v^2 - \\ &- 333u^6v^3 + 4u^9v - 561u^6v^5 + 16u^9v^3 - 19u^9v^4 + 369v^5u^2 - \\ &- 99u^5v^2 - 378v^4u^2 + 135v^7u^2 + 99uv^6 + 16u^9v^5 + 333u^3v^6 + \\ &+ 243u^7v^3 - 243u^2v^6 - 126u^8v^5 - 27u^6v - 54v^7u + 9v^8u - 135u^7v^2 + \\ &+ 10v^3 - 198v^6u^4 + 369u^4v^5 - 16v^6 + 19v^5 - 16v^4 - u^9 - 4v^8 + \\ &+ 10v^7 - 243v^4u^4 - 153v^2u^2 + 99v^7u^4 + 297v^3u^2 + 9u^8v^8 + 243u^5v^5 - \end{aligned}$$

$$\begin{aligned}
& -18v^8u^2 - 297v^6u^7 + 30u^6v^8 + 153v^7u^7 - 30vu^3 - 165v^7u^3 + 198u^5v^3 - \\
& \quad -99u^8v^3 - 423v^3u^3 - 9u^4v^8 - 369u^7v^4 + 165u^6v^2 + \\
& \quad +423u^6v^6 - 153u^5v^6 - 171u^6v^7 - 369u^5v^4 - 9u^5v^8 + 54u^5v^7 + \\
& \quad +27vu^2 + 90u^8v^6 + v^9 + 9u^5v - 10u^9v^6 - 45u^8v^7 + 171v^2u^3 + 153v^3u^4 + \\
& \quad +4u^9v^7 - 4v^2 - 27u^7v^8 - 54u^4v^2 - u^9v^8 - 135v^5u = 0.
\end{aligned}$$

**Proof.** Let  $w = U(q^3)$  then, by Theorem 3.2, we have

$$u^3 - w^3 + w^2 - w + 3uw - 3u^2w^2 + u^3w^2 - u^3w = 0. \quad (3.4)$$

Changing  $q$  to  $q^3$  in (3.4), we obtain

$$w^3 - v^3 + v^2 - v + 3wv - 3w^2v^2 + w^3v^2 - w^3v = 0. \quad (3.5)$$

Now eliminating  $w$  between (3.4) and (3.5), we deduce the required result.

**Theorem 3.6.** Let  $u = U(q)$  and  $v = U(q^{11})$ , then

$$\begin{aligned}
& -363v^7u^{10} - 528v^{10}u^8 + 11v^{10}u^{11} - 1089v^9u^8 + 1463v^4u^8 - 1012v^7u^8 + \\
& \quad +11v^{11}u^2 + vu - 968v^4u^4 + 44v^3u - 374v^3u^2 + 77v^4u - \\
& \quad -528v^4u^2 - 11vu^2 - 363v^5u^2 + 55v^7u^{11} + v^{11}u^{11} + \\
& \quad +759v^7u^9 + 55v^5u + 55v^11u^7 - 759v^3u^7 - 55v^7u + \\
& \quad +363v^7u^2 - 77v^8u + 528v^8u^2 + 176v^{10}u^2 + \\
& \quad +44v^{11}u^9 + 363v^2u^5 - 44vu^9 - 11v^2u^{11} - 11v^{10}u + \\
& \quad +528v^{10}u^4 + 528v^4u^{10} + 374v^9u^2 - 44v^9u - \\
& \quad -110v^2u^2 + 11v^2u + 77vu^8 + 374v^3u^{10} - 363v^2u^7 - \\
& \quad -77vu^4 + 11vu^{10} + 1089v^4u^3 - 374v^9u^{10} + \\
& \quad +44vu^3 - 759v^9u^5 + 55vu^5 + 759v^5u^3 + \\
& \quad +363v^5u^{10} + 1012v^5u^8 - 1012v^5u^4 - 55vu^7 + 1496v^5u^5 - \\
& \quad -759v^5u^9 - 528v^2u^4 - 704v^5u^7 - 55v^5u^{11} - 968v^8u^8 + \\
& \quad +1463v^8u^4 + 1089v^3u^8 - 528v^8u^{10} - 1089v^3u^4 - \\
& \quad -1089v^8u^3 - 1012v^8u^5 + 1496v^7u^7 + 77v^8u^{11} +
\end{aligned}$$



$$\begin{aligned}
& +1089v^8u^9 - 363v^{10}u^5 + 1012v^8u^7 + u^{12} - 374v^2u^9 + \\
& +1023v^3u^3 - 110v^{10}u^{10}1012v^7u^4 - 704v^7u^5 - 77v^{11}u^8 + \\
& +1012v^4u^5 + 11vu^{11} - 44v^{11}u^3 - 1012v^4u^7 + 924v^6u^6 + \\
& +1089v^9u^4 + 1023v^9u^9 - 759v^7u^3 + 759v^3u^5 + 44v^9u^{11} + \\
& +528v^2u^8 + 759v^9u^7 - 803v^3u^9 - 44v^3u^{11} + 363v^{10}u^7 + \\
& +1089v^4u^9 + 11v^{11}u + 374v^2u^3 - 77v^4u^{11} + 77v^{11}u^4 + 374v^{10}u^9 - \\
& -55v^{11}u^5 + v^{12} - 374v^{10}u^3 - 11v^{11}u^{10} - 803v^9u^3 + 176v^2u^{10} = 0.
\end{aligned}$$

**Proof.** Let

$$P = \frac{\varphi(q)}{\varphi(q^3)}$$

and

$$Q = \frac{\varphi(q^{11})}{\varphi(q^{33})},$$

then from [18], we have

$$\begin{aligned}
& (PQ)^5 + \frac{3^5}{(PQ)^5} - 11 \left[ (PQ)^3 + \frac{3^3}{(PQ)^3} \right] + 308 \left[ PQ + \frac{3}{PQ} \right] = \\
& = \left( \frac{P}{Q} \right)^6 + \left( \frac{Q}{P} \right)^6 + 22 \left[ \left( \frac{P}{Q} \right)^4 + \left( \frac{Q}{P} \right)^4 \right] \left[ 3 - \left( PQ + \frac{3}{PQ} \right) \right] + \\
& + 11 \left[ \left( \frac{P}{Q} \right)^2 + \left( \frac{Q}{P} \right)^2 \right] \left[ (PQ)^3 + \frac{3^3}{(PQ)^3} - 15 \left( PQ + \frac{3}{PQ} \right) + 45 \right] + 924.
\end{aligned}$$

Using (2.1) in the above, we deduce the required result.

**Theorem 3.7.** Let  $u = U(q)$  and  $v = U(q^{13})$ , then

$$\begin{aligned}
& -8346u^8v^5 + 611u^{11}v^{12} + 6318u^4v^8 + 8723u^8v^6 - 8346u^5v^8 - 78u^7v^5 + \\
& + 910u^{10}v^7 + 1508u^8v^2 - 780u^8v^7 + 8476u^8v^9 - 10322u^4v^9 - 5148u^8v^8 - \\
& - 481u^3v^{12} - 7228u^8v^{10} + 10894u^5v^9 + 611u^3v^2 - 208u^6v^{13} + 910u^7v^{10} - \\
& - 10322u^5v^{10} - 143u^{12}v^{12} + 6409u^3v^9 - 1911u^5v^{12} + 1313u^{10}v^2 - \\
& - 832u^7v^{11} - 5317u^3v^{10} - 130u^7v^{12} - 2184u^{11}v^{11} - 182u^{13}v^9 + 2652u^3v^{11} - \\
& - 5148u^6v^6 - 5317u^{10}v^3 - 91u^{13}v^4 + 1508u^6v^{12} + 78uv^7 + 8723u^6v^8 +
\end{aligned}$$

$$\begin{aligned}
& +4706u^6v^3 + 1846u^5v^2 + 1508u^2v^8 + v^{14} + 9607u^{10}v^4 - 5317u^4v^{11} - \\
& -182uv^5 + 78u^6v - 1170u^6v^2 - 2184u^3v^3 - 7228u^4v^6 - 6331u^{11}v^9 + \\
& +195u^9v + 10894u^9v^5 + 6409u^9v^3 - 7228u^{10}v^8 + 6318u^{10}v^6 + 78u^7v^{13} + \\
& +4706u^3v^6 - 780u^6v^7 + 13u^{13}v^{12} - 130u^2v^7 + 52u^2v^{13} - 1911u^9v^2 + \\
& +6318u^8v^4 + 78uv^6 + 4407u^{11}v^{10} + 1846u^2v^5 - 8346u^6v^9 - 1768u^7v^7 - \\
& -8346u^9v^6 - 832u^7v^3 - 10322u^{10}v^5 - 7631u^4v^4 - 3926u^8v^3 - 780u^7v^8 + \\
& +910u^7v^4 - 130u^7v^2 + 78u^7v + 10257u^4v^5 + u^{14} - 832u^3v^7 - 78u^7v^9 - \\
& -78u^9v^7 + 10257u^{10}v^9 + 8476u^9v^8 - 780u^7v^6 - 12909u^9v^9 - 7631u^{10}v^{10} + \\
& +846u^{12}v^9 - u^{13}v^{13} + 156u^{13}v^{10} - 10322u^9v^4 + 4407u^4v^3 + 13uv^2 + \\
& +10257u^9v^{10} + 6409u^{11}v^5 - 6331u^5v^3 - 1378u^4v^2 + 13u^2v + 4407u^{10}v^{11} - \\
& -3926u^{11}v^6 - 3926u^3v^8 - 143u^2v^2 - 1911u^{12}v^5 + 6409u^5v^{11} + 8476u^6v^5 + \\
& +156uv^4 - 65u^3v + 1508u^{12}v^6 - 208uv^8 + 13u^{12}v^{13} - 65u^{11}v^{13} + 156u^{10}v^{13} + \\
& +195u^{13}v^5 - 1378u^{10}v^{12} + 1313u^4v^{12} - 91u^4v^{13} - 130u^{12}v^7 - 3926u^6v^{11} - \\
& -832u^{11}v^7 - 65uv^3 - 208u^{13}v^6 - 1170u^{12}v^8 + 9607u^4v^{10} - 13u^3v^{13} + \\
& +6318u^6v^{10} + 78u^{13}v^8 + 611u^{12}v^{11} + 156u^4v + 611u^2v^3 + 78u^{13}v^7 + \\
& +4706u^{11}v^8 - 1378u^2v^4 + 10257u^5v^4 - 208u^8v - 26u^{13}v - 7228u^6v^4 - \\
& -91u^{10}v + 52u^{12}v + 4407u^3v^4 - 13u^{11}v + 8476u^5v^6 - 481u^{11}v^2 - 1378u^{12}v^{10} - \\
& -1170u^8v^{12} + 78u^8v^{13} + 4706u^8v^{11} - 182u^5v - 13uv^{11} + 195uv^9 - 91uv^{10} - \\
& -65u^{13}v^{11} - uv - 39u^2v^{12} - 481u^2v^{11} - 26uv^{13} - 1170u^2v^6 + 52uv^{12} - \\
& -1911u^2v^9 + 1846u^9v^{12} + 910u^4v^7 - 6331u^9v^{11} - 13u^{13}v^3 + 2652u^{11}v^3 - \\
& -481u^{12}v^3 - 182u^9v^{13} + 52u^{13}v^2 + 195u^5v^{13} + 1313u^2v^{10} - 39u^{12}v^2 - \\
& -6331u^3v^5 - 5317u^{11}v^4 + 1313u^{12}v^4 - 12909u^5v^5 - 78u^5v^7 = 0.
\end{aligned}$$

**Proof.** Let  $P = \frac{\varphi(q)}{\varphi(q^3)}$  and  $Q = \frac{\varphi(q^{13})}{\varphi(q^{39})}$ , then from [18], we have

$$\begin{aligned}
& \left(\frac{Q}{P}\right)^7 + \left(\frac{P}{Q}\right)^7 + 13 \left[ \left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6 \right] - 26 \left[ \left(\frac{P}{Q}\right)^5 + \left(\frac{Q}{P}\right)^5 \right] - \\
& -13 \left[ 3(PQ)^2 + \frac{27}{(PQ)^2} + 10 \right] \left[ \left(\frac{Q}{P}\right)^4 + \left(\frac{P}{Q}\right)^4 \right] + 13 \left[ 10(PQ)^2 + \frac{90}{(PQ)^2} + 68 \right] \times \\
& \times \left[ \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 \right] + 13 \left[ 10(PQ)^2 + \frac{90}{(PQ)^2} + 68 \right] \left[ \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 \right] + \\
& +13 \left[ (PQ)^4 + \frac{81}{(PQ)^4} - 20 \left( (PQ)^2 + \frac{9}{(PQ)^2} \right) - 115 \right] \left[ \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 \right] - \\
& -13 \left[ (PQ)^4 + \frac{81}{(PQ)^4} - 10 \left( (PQ)^2 + \frac{9}{(PQ)^2} \right) - 131 \right] \left[ \left(\frac{Q}{P}\right) + \left(\frac{P}{Q}\right) \right] = \\
& = (PQ)^6 + \frac{729}{(PQ)^6} - 26 \left[ (PQ)^4 + \frac{81}{(PQ)^4} \right] + 169 \left[ (PQ)^2 + \frac{9}{(PQ)^2} \right] + 832.
\end{aligned}$$

Now using (2.1) in the above identity, we deduce the required result.

**Acknowledgement.** The authors are thankful to DST, New Delhi for awarding research project [No. SR/S4/MS:517/08] under which this work has been done. Further, authors thank the referee for comments to improve the manuscript.

1. Baruah N. D. Modular equations for Ramanujan's cubic continued fraction // J. Math. Anal. and Appl. – 2002. – **268**. – P. 244–255.
2. Baruah N. D., Barman R. Certain Theta-function identities and Ramanujan's modular equations of degree 3 // Indian J. Math. – 2006. – **48**, № 1. – P. 113–133.
3. Berndt B. C. Ramanujan notebooks. Pt III. – New York: Springer, 1991.
4. Berndt B. C. Ramanujan Notebooks. Pt V. – New York: Springer, 1998.
5. Chan H. H. On Ramanujan's cubic continued fraction // Acta arithm. – 1995. – **73**, № 4. – P. 343–355.
6. Chan H. H., Huang S. S. On the Ramanujan–Göllnitz–Gordan continued fraction // Ramanujan J. – 1997. – **1**. – P. 75–90.
7. Cho B., Koo J. K., Park Y. K. Arithmetic of the Ramanujan Göllnitz Gordan continued fraction // J. Number Theory. – 2009. – **4**, № 129. – P. 922–947.
8. Gollnitz H. Partition mit Differenzbedingungen // J. reine und angew. Math. – 1967. – **25**. – S. 154–190.
9. Gordon B. Some continued fractions of the Rogers Ramanujan type // Duke Math. J. – 1965. – **32**. – P. 741–748.
10. Hardy G. H. Ramanujan. – 3<sup>rd</sup> ed. – New York: Chelsea, 1978.
11. Naika M. S. M., Dharmendra B. N., Shivashankra K. A continued fraction of order twelve // Cent. Eur. J. Math. – DOI: 10.2478/s 11533-008-0031-y.
12. Ramanujan S. Notebooks (2 volumes). – Tata Inst. Fundam. Research. – Bombay, 1957.
13. Ramanujan S. The "Lost" Notebook and other unpublished papers. – New Delhi: Narosa, 1988.
14. Rogers L. J. On a type of modular relation // Proc. London Math. Soc. – 1920. – **19**. – P. 387–397.
15. Vasuki K. R., Guruprasad P. S. On certain new modular relations for the Rogers–Ramanujan type functions of order twelve // Proc. Jangjeon Math. Soc. (to appear).
16. Vasuki K. R., Sharath G., Rajanna K. R. Two modular equations for squares of the cubic functions with Applications // Note Math. (to appear).
17. Vasuki K. R., Srivastakumar B. R. Certain identities for Ramanujan–Göllnitz–Gordan continued fraction // J. Comput. Appl. Math. – 2006. – **187**. – P. 87–95.
18. Vasuki K. R., Srivastakumar B. R. Two identities for Ramanujan's cubic continued fraction. – Preprint.
19. Vasuki K. R., Swamy S. R. A new identity for the Rogers–Ramanujan continued fraction // J. Appl. Math. Anal. and Appl. – 2006. – **2**, № 1. – P. 71–83.

Received 27.10.09,  
after revision – 08.07.10