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ON IMPULSIVE STURM – LIOUVILLE OPERATORS WITH COULOMB POTENTIAL AND SPECTRAL PARAMETER LINEARLY CONTAINED IN BOUNDARY CONDITIONS

ON IMPULSIVE STURM – LIOUVILLE OPERATORS WITH COULOMB POTENTIAL AND SPECTRAL PARAMETER LINEARLY CONTAINED IN BOUNDARY CONDITIONS

The Sturm – Liouville problem with linear discontinuities is investigated in the case where an eigenparameter appears not only in the differential equation but also in the boundary conditions. Properties and asymptotic behaviours of spectral characteristic are studied for the Sturm – Liouville operators with the Coulomb potential which have discontinuity conditions inside a finite interval. Moreover, the Weyl function for this problem under consideration is defined and uniqueness theorems for solution of inverse problem according to this function are proved.

In this study, Sturm – Liouville problem with discontinuities linearly is investigated when an eigenparameter appears not only in the differential equation but it also appears in the boundary conditions. Properties and asymptotic behaviours of spectral characteristic are studied for Sturm – Liouville operators with Coulomb potential which have discontinuity conditions inside a finite interval. Also Weyl function for this problem under consideration has been defined and uniqueness theorems for solution of inverse problem according to this function have been proved.

1. Introduction. In spectral theory, the inverse problem is the usual name for any problem in which it is required to ascertain the spectral data that will determine a differential operator uniquely and a method of construction of this operator from the data. This kind of problem was first formulated and investigated by Ambartsumyan in 1929 [7]. Since 1946, various forms of the inverse problem have been considered by numerous authors — G. Borg [15], N. Levinson [8], B. M. Levitan [9], etc. and now there exists an extensive literature on the [10 – 14]. Later, the inverse problems having specified singularities were considered by a number of authors [18 – 20].

We consider the boundary-value problem L for the equation:

$$\ell(y) := -y'' + \frac{C}{x}y + q(x)y = k^2 y \tag{1.1}$$

on the interval $0 < x < \pi$ with the boundary conditions

$$U(y) := y(0) = 0, V(y) := \left(\alpha_1 k^2 + \alpha_2\right) y(\pi) + \left(\beta_1 k^2 + \beta_2\right) y'(\pi) = 0 \tag{1.2}$$

and with the jump conditions

$$\begin{aligned} y(d+0) &= \alpha y(d-0), \\ y'(d+0) &= \alpha^{-1} y'(d-0), \end{aligned} \tag{1.3}$$

where k is spectral parameter; C, α, α_1, α_2, β_1, $\beta_2 \in R$, $\alpha_2\beta_1 - \beta_2\alpha_1 > 0$, $\alpha \neq 1$, $\alpha > 0$, $d \in \left(\frac{\pi}{2}, \pi\right)$, $q(x)$ is a real valued bounded function and $q(x) \in L_2(0,\pi)$.

In case of $q(x) \equiv 0$, since this operator is the singular Sturm – Liouville operator with Coulomb potential, linearly independent solutions of this kind of differential equation could be given with hypergeometric functions and this integral representation is also a representation for hypergeometric functions.

The boundary-value problems that contain the spectral parameter in boundary conditions linearly were investigated in [30 – 32]. In [30, 41], an operator-theoretic formulation of the problems of the form (1.1) – (1.3) has been given. Oscillation and comparison results have been obtained in [33 – 35]. In case of $\alpha_1 \neq 0$, problem (1.1) – (1.3) is associated with the physical problem of cooling a thin solid bar one end of which is placed in contact with a finite amount of liquid at time zero (see [30] and also [37] in it). Assuming that heat flows only into the liquid which has un-uniform density $\rho(x)$ and is convected only form the liquid into the surrounding medium, the initial boundary-value problem for a bar of length one takes the form

$$u_t = \rho(x) u_{xx}, \tag{1.4}$$

$$u_x\,(0,t) = 0, \tag{1.5}$$

$$-kAu_x\left(\pi^-, t\right) = qM\,(dv/dt) + k_1 Bv\,(t) \quad \text{for all} \quad t, \tag{1.6}$$

$$u\,(x,0) = u_0(x) \quad \text{for} \quad x \in [0,\pi], \tag{1.7}$$

$$v(0) = v_0$$

after factoring out the steady-state solution, where

$$\rho(x) = \begin{cases} 1, & 0 < x < d, \\ \alpha^2, & d < x < \pi. \end{cases}$$

Assuming that the rate of heat transfer across the liquid-solid interface is proportional to the difference in temperature between the end of the bar and the liquid with which it is in contract (Newton's law of cooling), and applying Fourier's law of heat conduction at $x = \pi$, we get

$$v\,(t) = u\,(\pi, t) + kc^{-1} u_x\left(\pi^{-1}, t\right) \quad \text{for} \quad t > 0,$$

where $c > 0$ is the coefficient of heat transfer for the liquid. If we put $u(x,t) = y(x)\exp(-\lambda t)$ then the problem (1.1) – (1.3) will appear to be consequence of the above problem. Indeed, the condition (1.2) is obtained from (1.5) and the condition (1.3) is obtained from (1.6) easily. Here $\alpha_1 = \dfrac{c}{k}$, $\beta_2 = -\dfrac{cA + k_1 B}{qM}$ and $\alpha_2 = -\dfrac{k_1 Bc}{qMk}$. Finally, if we put

$$t = \begin{cases} x, & 0 < x < d, \\ \alpha x, & d < x < \pi, \end{cases}$$

then the discontinuity conditions (1.3) and a particular case of (1.1) will appear. This corresponds to the case of nonperfect thermal contact. Since, the density is changed at one point in interval, both of the intensity and the instant velocity of heat change at this point. Hence, (1.1) – (1.3) will appear to be consequence of the above problem.

Boundary-value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disiplines ranging from engineering to the geo-sciences.

For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [21, 22]. After reducing corresponding mathematical model we come to boundary-value problem L where $q(x)$ must be constructed from the given spectral information which describes desirable amplitude and phase characteristics. Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimensional discontinuous medium [23, 24]. Boundary-value problems with discontinuties in an interiorpoint also appear in geophysical models for oscillations of the Earth [25, 26]. Here, the main disconutinuity is cased by reflection of the shear waves at the base of the crust. Further, it is known that inverse spectral problems play an important role for investigating some nonlinear evolution equations of mathematical physics. Discontinuous inverse problems help to study the blow-up behaviour of solutions for such nonlinear equations. We also note that inverse problem considered here appears in mathematics for investigating spectral properties of some classes of differential, integrodifferential and integral operators.

It must be noted that some special cases of the considered problem (1.1) – (1.3) arise after an application of the method of seperation of variables to the varied assortment of physical problems. For example, some boundary-value problems with transmission condition arise in heat and mass transfer problems (see, for example, [40]), in vibrating string problems when the string loaded additionally with point masses (see, for example, [37]) and in diffraction problems (see, for example, [39]). Moreover, some of the problems with boundary conditions depend on the spectral parameter occur in the theory of small vibrations of a damped string and freezing of the liquid (see, for example, [36 – 38]).

In the study of [29], there isn't existed spectral parameter in boundary conditions in the point $x = \pi$. However some certain physical problems are reduced to boundary-value problems which contain spectral parameter in boundary conditions, so they are reduced to investigate the type of problems (1.1) – (1.3).

In this study, representation with transformation operator has been obtained as in [28, 29].

Moreover, properties of characteristic function of L_0 and asymptotic behaviours of spectral characteristics of considering operator have been given such that the remaining parts are in the space ℓ_2 as in [29].

2. Representation for the solution. We define $y_1(x) = y(x)$, $y_2(x) = (\Gamma y)(x) = y'(x) - u(x)y(x)$, $u(x) = C \ln x$ and let's write the expression of left-hand side of equation (1.1) as follows

$$\ell\,(y) = -\left[(\Gamma y)(x)\right]' - u(x)(\Gamma y)(x) - u^2(x)y + q(x)y = k^2 y \tag{2.1}$$

then equation (1.1) reduces to the system;

$$\begin{aligned} &y_1' - y_2 = u(x)y_1, \\ &y_2' + k^2 y_1 = -u(x)y_2 - u^2(x)y_1 + q(x)y_1 \end{aligned} \tag{2.2}$$

with the boundary conditions

$$y_1(0)=0, \qquad \left(\alpha_1 k^2+\alpha_2\right) y_1(\pi)+\left(\beta_1 k^2+\beta_2\right) y_2(\pi)=0 \tag{2.3}$$

and with the jump conditions

$$\begin{aligned} &y_1(d+0)=\alpha y_1(d-0), \\ &y_2(d+0)=\alpha^{-1} y_2(d-0). \end{aligned} \tag{2.4}$$

Matrix form of system (2.2)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^{\prime}=\begin{pmatrix} u & 1 \\ -k^2-u^2+q & -u \end{pmatrix}\begin{pmatrix} [c]cy_1 \\ y_2 \end{pmatrix} \tag{2.5}$$

or $y'=Ay$ such that $A=\begin{pmatrix} u(x) & 1 \\ -k^2-u^2(x)+q(x) & -u(x) \end{pmatrix}$, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

$x=0$ is a regular-singular end point for equation (2.5) and Theorem 2 in [1, p. 56] (see Remark 1-2) extends to interval $[0,\pi]$. For this reason, by [1], there exists only one solution of the system (2.2) which satisfies the initial conditions $y_1(\xi)=\upsilon_1$, $y_2(\xi)=\upsilon_2$ for each $\xi\in[0,\pi]$, $\upsilon=(\upsilon_1,\upsilon_2)^T\in C^2$, especially the initial conditions $y_1(0)=1$, $y_2(0)=ik$.

Definition 2.1. *The first component of the solution of system* (2.2) *which satisfies the initial condititons* $y_1(\xi)=\upsilon_1$, $y_2(\xi)=(\Gamma y)(\xi)=\upsilon_2$ *is called the solution of equation* (1.1) *which satisfies these same initial conditions.*

It was showed in [29] by the successive approximations method that (see [16]) the following theorem is true.

Theorem 2.1. *For each solution of system* (2.2) *which satisfying the initial conditions* $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0)=\begin{pmatrix} 1 \\ ik \end{pmatrix}$ *and the jump conditions* (2.4), *the following expression is true*:

$$\begin{aligned} &y_1=e^{ikx}+\text{int}_{-x}^{x}K_{11}(x,t)e^{ikt}dt, \\ &y_2=ike^{ikx}+b(x)e^{ikx}+\text{int}_{-x}^{x}K_{21}(x,t)e^{ikt}dt+ik\text{int}_{-x}^{x}K_{22}(x,t)e^{ikt}dt, \end{aligned} \qquad x<d,$$

$$\begin{aligned} &y_1=\alpha^+e^{ikx}+\alpha^-e^{ik(2d-x)}+\text{int}_{-x}^{x}K_{11}(x,t)e^{ikt}dt, \\ &y_2=ik\left(\alpha^+e^{ikx}-\alpha^-e^{ik(2d-x)}\right)+b(x)\left[\alpha^+e^{ikx}+\alpha^-e^{ik(2d-x)}\right], \quad x>d, \\ &\qquad +\text{int}_{-x}^{x}K_{21}(x,t)e^{ikt}dt+ik\text{int}_{-x}^{x}K_{22}(x,t)e^{ikt}dt, \end{aligned}$$

where

$$b(x)=-\frac{1}{2}\text{int}_0^x\left[u^2(s)-q(s)\right]e^{-\frac{1}{2}\text{int}_s^x u(t)dt}ds,$$

$$K_{11}(x,x)=\frac{\alpha^+}{2}u(x),$$

$$K_{21}(x,x)=b'(x)-\frac{1}{2}\text{int}_0^x\left[u^2(s)-q(s)\right]K_{11}(s,s)\,ds-\frac{1}{2}\text{int}_0^x u(s)\,K_{21}(s,s)\,ds,$$

$$K_{22}(x,x)=-\frac{\alpha^+}{2}\left[u(x)+2b(x)\right],$$

$$K_{11}\left(x,2d-x+0\right)-K_{11}\left(x,2d-x-0\right)=\frac{\alpha^{-}}{2}u(x),$$

$$\frac{\partial K_{ij}\left(x,.\right)}{\partial x},\ \frac{\partial K_{ij}\left(x,.\right)}{\partial t}\in L_2(0,\pi),\quad i,\,j=1,2.$$

3. Properties of the spectrum. In this section, properties of the spectrum of problem L will be learned. Let us denote problem L as L_0 in the case of $C=0$ and $q(x)\equiv 0$.

When $C=0$ and $q(x)\equiv 0$, it is easily shown that solution $\varphi_0(x,k)$ satisfying the initial conditions $\varphi_0(0,k)=0$, $(\Gamma\varphi_0)(0,k)=k$ and the jump conditions (2.4) is shown as

$$\varphi_0(x,k)=\begin{cases}\sin kx, & \text{for }\ x<d,\\ \alpha^{+}\sin kx+\alpha^{-}\sin k(2d-x), & \text{for }\ x>d,\end{cases}$$

$$(\Gamma\varphi_0)(x,k)=\begin{cases}k\cos kx, & \text{for }\ x<d,\\ k\alpha^{+}\cos kx-k\alpha^{-}\cos k(2d-x), & \text{for }\ x>d.\end{cases} \tag{3.1}$$

We denote characteristic function, eigenvalues sequence and normalizing constant sequence by $\Delta(k)$, $\{k_n\}$ and $\{a_n\}$ respectively. Denote

$$\Delta(k)=\left\langle\psi(x,k),\varphi(x,k)\right\rangle, \tag{3.2}$$

where

$$\langle y(x),z(x)\rangle:=y(x)(\Gamma z)(x)-(\Gamma y)(x)z(x).$$

Also we defined normalizing constants by

$$a_n:=\text{int}_0^{\pi}\varphi^2\left(x,k_n\right)dx+\frac{1}{\rho}\big[\alpha_1\varphi(\pi,k_n)+\beta_1(\Gamma\varphi)(\pi,k_n)\big]^2, \tag{3.3}$$

where $\rho=\alpha_2\beta_1-\beta_2\alpha_1$. According to the Liouville formula, $\langle\psi(x,k),\varphi(x,k)\rangle$ is not depend on x.

We shall assume that $\varphi(x,k)$ and $\psi(x,k)$ are solutions of equation (1.1) under the following initial conditions:

$\varphi(0,k)=0$, $(\Gamma\varphi)(0,k)=k$, $\psi(\pi,k)=\left(\beta_1k^2+\beta_2\right)$, $\left(\Gamma\psi\right)(\pi,k)=-(\alpha_1k^2+$ $+\,\alpha_2)$.

Clearly, for each x, functions $\langle\psi(x,k),\varphi(x,k)\rangle$ are entire in k and

$$\Delta(k)=V\left(\varphi\right)=U\left(\psi\right)=$$
$$=\left(\alpha_1k^2+\alpha_2\right)\varphi(\pi,k)+\left(\beta_1k^2+\beta_2\right)\varphi(\pi,k)=\psi(0,k). \tag{3.4}$$

By using the representation of the function $y(x,k)$ for the solution $\varphi(x,k)$:

$$\varphi(x,k)=\varphi_0(x,k)+\text{int}_0^{\pi}\widetilde{K}_{11}\left(\pi,t\right)\sin ktdt \tag{3.5}$$

is obtained.

Lemma 3.1 (Lagrange's fomula)**.** *Let $y, z \in D\left(L_0^*\right)$. Then*

$$\left(L_0^* y, z\right) = \mathrm{int}_0^\pi \ell\left(y\right)\overline{z}dx = \left(y, L_0^* z\right) + \left[y, \overline{z}\right]\left(|_0^{d-0} + |_{d+0}^\pi\right),$$

where $[y, z]\left(|_0^{d-0} + |_{d+0}^\pi\right) = \left[\left(\Gamma\overline{z}\right)(x)y(x) - (\Gamma y)(x)\overline{z(x)}\right]\left(|_0^{d-0} + |_{d+0}^\pi\right)$.

Proof. We have

$$\left(L_0^* y, z\right) = -\mathrm{int}_0^\pi \left(y' - u\ y\right)'\overline{z}dx - \mathrm{int}_0^\pi u\left(y' - u\ y\right)\overline{z}dx - \mathrm{int}_0^\pi \left(u^2 - q(x)\right)y\overline{z}dx =$$

$$= \mathrm{int}_0^\pi \left(y' - u\ y\right)\left(\overline{z}' - u\overline{z}\right)dx - \mathrm{int}_0^\pi \left(u^2 - q(x)\right)y\overline{z}dx - (\Gamma y)(x)\overline{z(x)}\left(|_0^{d-0} + |_{d+0}^\pi\right) =$$

$$= \mathrm{int}_0^\pi y\ell\left(\overline{z}\right)dx + \left[y, \overline{z}\right]\left(|_0^{d-0} + |_{d+0}^\pi\right) = \left(y, L_0^* z\right) + \left[y, \overline{z}\right]\left(|_0^{d-0} + |_{d+0}^\pi\right).$$

Lemma 3.1 is proved.

Lemma 3.2. *The zeros $\{k_n\}$ of the characteristic function coincide with the eigenvalues of the boundary-value problem L. The functions $\varphi\left(x, k_n\right)$ and $\psi\left(x, k_n\right)$ are eigenfunctions and there exists a sequence $\{\gamma_n\}$ such that*

$$\psi\left(x, k_n\right) = \gamma_n\varphi\left(x, k_n\right), \quad \gamma_n \neq 0. \tag{3.6}$$

Proof. 1) Let k_0 be a zero of the function $\Delta(k)$. Then by virtue of equation (3.2) and (3.4), $\psi\left(x, k_0\right) = \gamma_0\varphi\left(x, k_0\right)$ and the functions $\psi\left(x, k_0\right), \varphi\left(x, k_0\right)$ satisfy the boundary conditions (1.2). Hence k_0 is an eigenvalue and $\psi\left(x, k_0\right), \varphi\left(x, k_0\right)$ are eigenfunctions related to k_0.

2) Let k_0 be an eigenvalue of L, y_0 be a corresponding eigenfunctions. Then $U(y_0) = V(y_0) = 0$. Clearly $y_0(0) = 0$. Without loss of generality we put $(\Gamma y_0)(0) = ik$. Hence $y_0(x) \equiv \varphi\left(x, k_0\right)$. Thus, from equation (3.4), $\Delta\left(k_0\right) = V\left(\varphi\left(x, k_0\right)\right) = V\left(y_0(x)\right) = 0$ is obtained.

Lemma 3.2 is proved.

Lemma 3.3. *Eigenvalues of the problem L are simple and separated.*

Proof. Since $\varphi(x, k)$ and $\psi(x, k)$ are solutions of equation (1.1),

$$-\psi''(x, k) + \left[u'(x) + q(x)\right]\psi(x, k) = k\psi(x, k) -$$

$$-\varphi''\left(x, k_n\right) + \left[u'(x) + q(x)\right]\varphi\left(x, k_n\right) = k_n\varphi\left(x, k_n\right).$$

If first equation is multiplied by $\varphi\left(x, k_n\right)$, second equation is multiplied by $\psi(x, k)$ and substracting them side by side and finally integrating over the interval $[0, \pi]$, the equality

$$\left\langle\psi(x, k),\ \varphi\left(x, k_n\right)\right\rangle\left[|_0^{d-0} + |_{d+0}^\pi\right] = \left(k - k_n\right)\mathrm{int}_0^\pi\psi(x, k)\varphi\left(x, k_n\right)dx \tag{3.7}$$

is obtained.

If jump conditions (1.3) and equation (3.3) are considered, then

$$\mathrm{int}_0^\pi\ \psi\left(x, k_n\right)\varphi\left(x, k_n\right)dx +$$

$$+\frac{1}{\rho}\left[\alpha_1\psi(\pi, k_n) + \beta_1\left(\Gamma\psi\right)\left(\pi, k_n\right)\right]\left[\alpha_1\varphi(\pi, k_n) + \beta_1(\Gamma\varphi)(\pi, k_n)\right] = -\dot{\Delta}\left(k_n\right)$$

as $k \to k_n$ is obtained. From Lemma 3.2, we get that

$$a_n\gamma_n = -\dot{\Delta}\left(k_n\right). \tag{3.8}$$

It is obvious that $\dot{\Delta}(k_n) \neq 0$.

Since the function $\Delta(k)$ is an entire function of k, the zeros of $\Delta(k)$ are separated.

Lemma 3.3 is proved.

Now, let problems be

$$L\colon \begin{cases} -y'' + [u'(x) + q(x)]\, y = \lambda y, \\ (\Gamma y)(0) - hy(0) = 0, \\ (\beta_1\lambda + \beta_2)\,(\Gamma y)\,(\pi) + (\alpha_1\lambda + \alpha_2)\, y\,(\pi) = 0, \\ y\,(d+0) = \alpha y\,(d-0)\,, \\ (\Gamma y)\,(d+0) = \alpha^{-1}\,(\Gamma y)\,(d-0) \end{cases}$$

and

$$\widetilde{L}\colon \begin{cases} -y'' + [u'(x) + q(x)]\, y = \mu y, \\ (\Gamma y)(0) - hy(0) = 0, \\ \left(\widetilde{\beta}_1\lambda + \widetilde{\beta}_2\right)(\Gamma y)\,(\pi) + (\widetilde{\alpha}_1\lambda + \widetilde{\alpha}_2)\, y\,(\pi) = 0, \\ y\,(d+0) = \alpha y\,(d-0)\,, \\ (\Gamma y)\,(d+0) = \alpha^{-1}\,(\Gamma y)\,(d-0)\,, \end{cases}$$

where $\alpha_1\widetilde{\beta}_1 = \widetilde{\alpha}_2\beta_2$, $\alpha_1\widetilde{\beta}_2 = \widetilde{\alpha}_2\beta_1$, $\alpha_2\widetilde{\beta}_1 = \widetilde{\alpha}_1\beta_2$. Let $\{\lambda_n\}_{n\geq 0}$ and $\{\mu_n\}_{n\geq 0}$ be the eigenvalues of the problems L and $\widetilde{L}$ respectively.

Lemma 3.4. *The eigenvalues of the problems L and $\widetilde{L}$ are interlace, i.e.,*

$$\begin{aligned} \lambda_n < \mu_n < \lambda_{n+1}, \quad &\textit{if} \quad \alpha_2\widetilde{\beta}_2 < \widetilde{\alpha}_2\beta_2, \\ \mu_n < \lambda_n < \mu_{n+1}, \quad &\textit{if} \quad \alpha_2\widetilde{\beta}_2 > \widetilde{\alpha}_2\beta_2, \quad n \geq 0. \end{aligned} \tag{3.9}$$

where $\alpha_1\widetilde{\alpha}_2 > \widetilde{\alpha}_1\alpha_2$ and $\beta_1\widetilde{\beta}_2 > \widetilde{\beta}_1\beta_2$.

Proof. As in the proof of Lemma 3, we get that

$$\frac{d}{dx}\,\langle\varphi(x,\lambda), \varphi\,(x,\mu)\rangle = (\lambda - \mu)\,\varphi\,(x,\lambda)\,\varphi(x,\mu)$$

and from here

$$\begin{aligned} (\lambda-\mu)\,\mathrm{int}_0^\pi \varphi(x,\lambda)\varphi(x,\mu)dx = \langle\varphi(x,\lambda), \varphi(x,\mu)\rangle\left[|_0^{d-0} + |_{d+0}^\pi\right] = \\ = \varphi(\pi,\lambda)(\Gamma\varphi)\,(\pi,\mu) - (\Gamma\varphi)(\pi,\lambda)\varphi(\pi,\mu) = \\ = \frac{\widetilde{\alpha}_1\alpha_2 - \alpha_1\widetilde{\alpha}_2}{\alpha_2\widetilde{\beta}_2 - \widetilde{\alpha}_2\beta_2}\,(\lambda-\mu)\,\varphi(\pi,\lambda)\,(\varphi)\,(\pi,\mu) + \\ + \frac{\widetilde{\beta}_1\beta_2 - \beta_1\widetilde{\beta}_2}{\alpha_2\widetilde{\beta}_2 - \widetilde{\alpha}_2\beta_2}\,(\lambda-\mu)\,(\Gamma\varphi)(\pi,\lambda)(\Gamma\varphi)\,(\pi,\mu) + \\ + \frac{1}{\alpha_2\widetilde{\beta}_2 - \widetilde{\alpha}_2\beta_2}\left[\widetilde{\Delta}(\lambda)\Delta\,(\mu) - \widetilde{\Delta}\,(\mu)\,\Delta(\lambda)\right]. \end{aligned}$$

Hence

$$(\lambda-\mu)\,\mathrm{int}_0^{\pi}\varphi(x,\lambda)\varphi(x,\mu)dx=$$
$$=\frac{\widetilde{\alpha}_1\alpha_2-\alpha_1\widetilde{\alpha}_2}{\alpha_2\widetilde{\beta}_2-\widetilde{\alpha}_2\beta_2}\,(\lambda-\mu)\,\varphi(\pi,\lambda)\,(\varphi)\,(\pi,\mu)+$$
$$+\frac{\widetilde{\beta}_1\beta_2-\beta_1\widetilde{\beta}_2}{\alpha_2\widetilde{\beta}_2-\widetilde{\alpha}_2\beta_2}\,(\lambda-\mu)\,(\Gamma\varphi)(\pi,\lambda)(\Gamma\varphi)(\pi,\mu)+$$
$$+\frac{1}{\alpha_2\widetilde{\beta}_2-\widetilde{\alpha}_2\beta_2}\left[\frac{\widetilde{\Delta}(\lambda)-\widetilde{\Delta}\,(\mu)}{\lambda-\mu}\Delta\,(\mu)-\frac{\Delta(\lambda)-\Delta\,(\mu)}{\lambda-\mu}\widetilde{\Delta}\,(\mu)\right].$$

As $\mu\to\lambda$

$$\mathrm{int}_0^{\pi}\varphi^2(x,\lambda)dx=\frac{1}{\alpha_2\widetilde{\beta}_2-\widetilde{\alpha}_2\beta_2}\times$$
$$\times\Bigg[\left(\widetilde{\alpha}_1\alpha_2-\alpha_1\widetilde{\alpha}_2\right)\varphi^2(\pi,\lambda)+\left(\widetilde{\beta}_1\beta_2-\beta_1\widetilde{\beta}_2\right)\left(\Gamma\varphi\right)^2(\pi,\lambda)+$$
$$+\dot{\widetilde{\Delta}}(\lambda)\Delta(\lambda)-\dot{\Delta}\,(\lambda)\,\widetilde{\Delta}(\lambda)\Bigg], \tag{3.10}$$

where $\dot{\Delta}(\lambda)=\dfrac{d}{d\lambda}\Delta(\lambda)$, $\dot{\widetilde{\Delta}}(\lambda)=\dfrac{d}{d\lambda}\widetilde{\Delta}(\lambda)$. From equation (3.10), if $\widetilde{\Delta}(\lambda)\neq 0$

$$\frac{1}{\widetilde{\Delta}^2(\lambda)}\left[\mathrm{int}_0^{\pi}\varphi^2(x,\lambda)dx-\frac{\left(\widetilde{\alpha}_1\alpha_2-\alpha_1\widetilde{\alpha}_2\right)\varphi^2\,(\pi,\lambda)+\left(\widetilde{\beta}_1\beta_2-\beta_1\widetilde{\beta}_2\right)(\Gamma\varphi)^2\,(\pi,\lambda)}{\widetilde{\alpha}_2\beta_2-\alpha_2\widetilde{\beta}_2}\right]=$$
$$=-\frac{1}{\left(\widetilde{\alpha}_2\beta_2-\alpha_2\widetilde{\beta}_2\right)}\,\frac{d}{d\lambda}\left(\frac{\Delta\,(\lambda)}{\widetilde{\Delta}(\lambda)}\right),\quad -\infty<\lambda<\infty,$$

is obtained.

If $\alpha_2\widetilde{\beta}_2<\widetilde{\alpha}_2\beta_2$, then $\dfrac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)}$ is monotonically decreasing in the set of $R\backslash\{\mu_n,n\geq 0\}$. Thus it is obvious that $\lim\limits_{\lambda\to\mu_n^{\pm 0}}\dfrac{\Delta\,(\lambda)}{\widetilde{\Delta}(\lambda)}=\pm\infty$.

When $\alpha_2\widetilde{\beta}_2>\widetilde{\alpha}_2\beta_2$, if we write the equality (3.10) as

$$\frac{1}{\Delta^2(\lambda)}\left[\mathrm{int}_0^{\pi}\varphi^2(x,\lambda)dx-\frac{\left(\widetilde{\alpha}_1\alpha_2-\alpha_1\widetilde{\alpha}_2\right)\varphi^2\,(\pi,\lambda)+\left(\widetilde{\beta}_1\beta_2-\beta_1\widetilde{\beta}_2\right)(\Gamma\varphi)^2\,(\pi,\lambda)}{\alpha_2\widetilde{\beta}_2-\widetilde{\alpha}_2\beta_2}\right]=$$
$$=-\frac{1}{\left(\alpha_2\widetilde{\beta}_2-\widetilde{\alpha}_2\beta_2\right)}\frac{d}{d\lambda}\left(\frac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)}\right),\qquad -\infty<\lambda<\infty,\quad \Delta(\lambda)\neq 0,$$

we get that the function $\dfrac{\widetilde{\Delta}\,(\lambda)}{\Delta(\lambda)}$ is monotonically decreasing in $R\setminus\{\lambda_n,\,n\geq 0\}$ and it is clear that $\lim_{\lambda\to\lambda_n^{\pm 0}}\dfrac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)}=\pm\infty$. From here (3.9) is obtained.

Theorem 3.1. *The eigenvalues k_n, eigenfunctions $\varphi(x, k_n)$ and the normalizing numbers α_n of problem L have the following asymptotic behaviour*

$$k_n = k_n^0 + \frac{d_n}{k_n^0} + \frac{\delta_n}{k_n^0}, \tag{3.11}$$

$$\varphi(x, k_n) = \alpha^+ \sin\left(k_n^0 + \varepsilon\right) x + \alpha^- \sin\left(k_n^0 + \varepsilon\right)(2d - x) + \frac{s_n}{k_n^0} + \frac{b_n}{k_n^0}, \tag{3.12}$$

$$a_n = \left[\left(\alpha^+\right)^2 + \left(\alpha^-\right)^2\right]\left(\frac{\pi - d}{2}\right) + \frac{d}{2} - \alpha^+\alpha^- \cos 2k_n^0 d + \gamma_n + \frac{\xi_n}{n}, \tag{3.13}$$

where δ_n, s_n, $\xi_n \in \ell_2$, b_n, d_n, $\gamma_n \in \ell_\infty$ *and* k_n^0 *are roots of* $\Delta_0(k) := k^3\left[\alpha^+ \cos k\pi - \alpha^- \cos k(2d - \pi)\right]$ *and* $k_n^0 = n + h_n, h_n \in \ell_\infty$.

Proof. Using (3.1), (3.2) and (3.5), we get

$$\Delta(k) = \left(\alpha_1 k^2 + \alpha_2\right)\varphi_0(\pi, k) + \left(\beta_1 k^2 + \beta_2\right)(\Gamma\varphi_0)(\pi, k) +$$
$$+ \left(\alpha_1 k^2 + \alpha_2\right) \mathrm{int}_0^\pi \widetilde{K}_{11}(\pi, t) \sin ktdt +$$
$$+ \beta_1 k^2 + \beta_2 \left[\mathrm{int}_0^\pi \widetilde{K}_{21}(\pi, t) \sin ktdt + \mathrm{int}_0^\pi \widetilde{K}_{22}(\pi, t) \cos ktdt\right] =$$
$$= \left(\alpha_1 k^2 + \alpha_2\right)\left(\alpha^+ \sin k\pi + \alpha^- \sin k(2d - \pi)\right) +$$
$$+ \left(\beta_1 k^2 + \beta_2\right)\left(k\alpha^+ \cos k\pi - k\alpha^- \cos k(2d - \pi)\right) + k^3 O\left(\frac{\exp|\operatorname{Im} k|\pi}{|k|}\right) =$$
$$= \beta_1 \Delta_0(k) + \left(\alpha_1 k^2 + \alpha_2\right)\left(\alpha^+ \sin k\pi + \alpha^- \sin k(2d - \pi)\right) +$$
$$+ \beta_2 k\left(\alpha^+ \cos k\pi - \alpha^- \cos k(2d - \pi)\right) + k^3 O\left(\frac{\exp|\operatorname{Im} k|\pi}{|k|}\right).$$

Denote

$$G_n = \left\{k\colon |k| = |k_n^0| + \frac{\sigma}{2},\ n = 0, \pm1, \pm2, \ldots\right\},$$
$$G_\delta = \left\{k\colon |k - k_n^0| \geq \delta,\ n = 0, \pm1, \pm2, \ldots, \delta > 0\right\},$$

where δ is sufficiently small positive number $\left(\delta \ll \frac{\sigma}{2}\right)$.

Since $|\Delta_0(k)| \geq k^3 C_\delta e^{|\operatorname{Im} k|\pi}$ for $k \in \overline{G_\delta}$ and $k \in G_n$ $|\Delta(k) - \Delta_0(k)| < \frac{C_\delta}{2}|k|^3 e^{|\operatorname{Im} k|\pi}$ for sufficiently large values of n , we get

$$|\Delta_0(k)| > C_\delta k^3 e^{|\operatorname{Im} k|\pi} > |\Delta(k) - \Delta_0(k)|.$$

It follows from that for sufficiently large values of n, functions $\Delta_0(k)$ and $\Delta_0(k) + (\Delta(k) - \Delta_0(k)) = \Delta(k)$ have the same number of zeros counting multiplicities inside contour G_n, according to Rouche's theorem. That is, they have the $(n+1)$ number of zeros: $k_0, k_1, \ldots, k_n$.

Analogously, it is shown by Rouche's theorem that for sufficiently large values of n, function $\Delta(k)$ has a unique of zero inside each circle $|k - k_n^0| < \delta$.

Since δ is sufficiently small number, representing of $k_n = k_n^0 + \varepsilon_n$ is acquired where $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Since numbers k_n are zeros of characteristic function $\Delta(k)$,

$$\Delta(k_n) = \left(\alpha_1 k_n^2 + \alpha_2\right)\left(\alpha^+ \sin k_n\pi + \alpha^- \sin k_n(2d-\pi)\right) +$$
$$+\left(\beta_1 k^2 + \beta_2\right)\left(k_n\alpha^+ \cos k_n\pi - k_n\alpha^- \cos k_n(2d-\pi)\right) + O\left(k_n^2\right).$$

From the last equality, we get

$$\alpha^+ \cos k_n\pi - \alpha^- \cos k_n(2d-\pi) + \frac{\alpha_1}{\beta_1 k_n^2}\left[\alpha^+ \sin k_n\pi + \alpha^- \sin k_n(2d-\pi)\right] +$$
$$+\frac{\alpha_2}{\beta_1 k_n^3}\left[\alpha^+ \sin k_n\pi + \alpha^- \sin k_n(2d-\pi)\right] +$$
$$+\frac{\beta_2}{\beta_1 k_n^2}\left[\alpha^+ \cos k_n\pi - \alpha^- \cos k_n(2d-\pi)\right] + O\left(\frac{1}{k_n}\right) = 0.$$

If we write $k_n^0 + \varepsilon_n$ instead of k_n and use $\Delta_0\left(k_n^0 + \varepsilon_n\right) = \dot{\Delta}_0\left(k_n^0\right)\varepsilon_n + o(\varepsilon_n)$ and also the study [5] (see also [6]) is used then we get that $k_n^0 = n + h_n$ where $\sup\limits_n |h_n| < M$. Therefore

$$\varepsilon_n = \frac{d_n}{n} + \frac{\delta_n}{n}, \qquad \delta_n \in \ell_2, \quad d_n \in \ell_\infty.$$

Hence for the eigenvalues k_n of the problem L, asymptotic formula (3.10) is true. Now, let's try to find the asymptotic formula for the eigenfunction:

$$\varphi(x,k_n) = \alpha^+ \sin k_n x + \alpha^- \sin k_n(2d-x) + \text{int}_0^x \widetilde{K}_{11}(x,t)\sin k_n t dt =$$
$$= \alpha^+ \sin\left(k_n^0 + \varepsilon_n\right)x + \alpha^- \sin\left(k_n^0 + \varepsilon_n\right)(2d-x) +$$
$$-\frac{1}{k_n^0 + \varepsilon_n}\text{int}_0^x \widetilde{K}_{11}(x,t) d\left(\cos\left(k_n^0 + \varepsilon_n\right)\right) t\, dt =$$
$$= \alpha^+ \sin k_n^0 x + \alpha^- \sin k_n^0(2d-x) -$$
$$-\frac{1}{k_n^0 + \varepsilon_n}\left[\widetilde{K}_{11}(x,t)\cos k_n^0 t\right]\left(\big|_0^{2d-x-0} + \big|_{2d-x+0}^x\right) +$$
$$+\frac{1}{k_n^0 + \varepsilon_n}\text{int}_0^x \widetilde{K}'_{11t}(x,t)\cos k_n^0 t\, dt.$$

Since

$$\widetilde{K}_{11}(x,x) = \frac{\alpha^+}{2}u(x), \qquad \widetilde{K}_{11}(x, 2d-x+0) - \widetilde{K}_{11}(x, 2d-x-0) = \frac{\alpha^-}{2}u(x),$$

and

$$\text{int}_0^x \widetilde{K}'_{11t}(x,t)\cos k_n^0 t dt \in \ell_2$$

it is obtained that

$$\varphi(x,k_n) = \alpha^+ \sin k_n^0 x + \alpha^- \sin k_n^0(2d-x) +$$
$$+\frac{\alpha^- \cos k_n^0(2d-x) - \alpha^+ \cos k_n^0 x}{2k_n^0}u(x) + \frac{b_n}{n} + \frac{s_n}{n}, \quad s_n \in \ell_2, \quad b_n \in \ell_\infty.$$

Then we get the asymptotic formula (3.12). Finally, in order to show (3.13) is true, using (3.1) and (3.5), we get

$$a_n = \text{int}_0^\pi \varphi^2 (x, k_n)\, dx + \frac{1}{\rho} \left[\alpha_1 \varphi(\pi, k_n) + \beta_1 (\Gamma\varphi)(\pi, k_n)\right]^2 =$$

$$= \text{int}_0^d \left[\sin^2 k_n x dx + \left(\text{int}_0^x \widetilde{K}_{11}(x,t) \sin k_n t dt\right)^2\right] dx+$$

$$+2\text{int}_0^d \sin k_n x \text{int}_0^x \widetilde{K}_{11}(x,t) \sin k_n t dt dx+$$

$$+\text{int}_d^\pi \left[\left(\alpha^+\right)^2 \sin^2 k_n x + \left(\alpha^-\right)^2 \sin^2 k_n (2d - x) + \left(\text{int}_d^x \widetilde{K}_{11}(x,t) \sin k_n t dt\right)^2\right] dx+$$

$$+2\alpha^+\alpha^- \text{int}_d^\pi \sin k_n x \sin k_n (2d - x) dx+$$

$$+2\alpha^+ \text{int}_d^\pi \sin k_n x \text{int}_d^x \widetilde{K}_{11}(x,t) \sin k_n t dt dx+$$

$$+2\alpha^- \text{int}_d^\pi \sin k_n (2d - x) \text{int}_d^x \widetilde{K}_{11}(x,t) \sin k_n t dt dx+$$

$$+\frac{1}{\rho} \left[\alpha_1 \varphi(\pi, k_n) + \beta_1 \left(\Gamma\varphi\right)(\pi, k_n)\right]^2 =$$

$$= \frac{\pi - d}{2} \left[\left(\alpha^+\right)^2 + \left(\alpha^-\right)^2\right] + \frac{d}{2} - \alpha^+\alpha^- \cos 2k_n^0 d + \gamma_n + \frac{\xi_n}{n}.$$

Theorem 3.1 is proved.

4. Inverse problem. In the present section, we study the inverse problems recovering the boundary value problem L from the spectral data. We consider three statements of the inverse problem of the reconstruction of the boundary-value problem L from the Weyl function, from the spectral data $\{k_n, a_n\}_{n \geq 0}$ and from two spectra $\{k_n, \mu_n\}_{n \geq 0}$. These inverse problems are generalizations of the well-known inverse problems for Sturm – Liouville operator [22, 42].

Let $\Phi(x,k)$ be solution of (2.2) under the condititons $U(\Phi) = \Phi(0,k) = 1$ and $V(\Phi) = \left(\alpha_1 k^2 + \alpha_2\right) \Phi(\pi,k) + \left(\beta_1 k^2 + \beta_2\right) (\Gamma\Phi)(\pi,k) = 0$ and under the jump condititons (2.4). Also $C(x,k)$ be solution of (2.2) under the condititons $C(0,k) = 1$ and $(\Gamma C)(0,k) = 0$ and under the jump condititons (2.4). Then function $\psi(x,k)$ can be represented as follows:

$$\psi(x,k) = \frac{1}{k} \left(\Gamma\psi\right)(0,k)\, \varphi(x,k) + \Delta(k) C(x,k)$$

or

$$\frac{1}{\Delta(k)} \psi(x,k) = \frac{\left(\Gamma\psi\right)(0,k)}{k\Delta(k)} \varphi(x,k) + C(x,k). \tag{4.1}$$

Denote

$$M(k) := \frac{\left(\Gamma\psi\right)(0,k)}{k\Delta(k)}. \tag{4.2}$$

It is clear that

$$\Phi(x,k) = M(k) \varphi(x,k) + C(x,k). \tag{4.3}$$

The function $\Phi(x,k)$ is called the Weyl solution and the function $M(k)$ is called the Weyl function for the boundary-value problem L.

The Weyl solution and Weyl function are meromorphic functions with respect to k having poles in the spectrum of the problem L.

It follows from (4.1) and (4.2) that

$$\Phi(x,k) = \frac{\psi(x,k)}{\Delta(k)} \quad \text{and} \quad (\Gamma\Phi)(0,k) = \frac{(\Gamma\psi)(0,k)}{k\Delta(k)} = M(k). \tag{4.4}$$

Note that, by virtue of equalities $\langle C(x,k), \varphi(x,k)\rangle \equiv 1$, (4.2) and (4.3) we have

$$\langle \Phi(x,k), \varphi(x,k)\rangle \equiv k, \qquad \langle \psi(x,k), \varphi(x,k)\rangle \equiv k\Delta(k). \tag{4.5}$$

Theorem 4.1. *The following representation holds*;

$$M(k) = \frac{1}{a_0\,(k-k_0)} + \sum_{n=1}^{\infty}{}' \left\{ \frac{1}{a_n\,(k-k_n)} + \frac{1}{a_n^0 k_n^0} \right\}. \tag{4.6}$$

Proof. Let's write a representation solution of $\psi(x,k) = -(\beta_1 k^2 + \beta_2) \times \times C(x,k) + \left(\alpha_1 k^2 + \alpha_2\right) S(x,k)$ as a representation solution of $\varphi(x,k)$:

for $x > d$

$$\psi(x,k) = -\left(\beta_1 k^2 + \beta_2\right)\cos k\,(\pi - x) + \left(\alpha_1 k^2 + \alpha_2\right)\sin k\,(\pi - x) + \\ + \mathrm{int}_0^{\pi - x}\widetilde{N}_{11}(x,t)\left[-\left(\beta_1 k^2 + \beta_2\right)\cos kt + \left(\alpha_1 k^2 + \alpha_2\right)\sin kt\right]dt,$$

$$(\Gamma\psi)(x,k) = -k\left[\left(\beta_1 k^2 + \beta_2\right)\sin k\,(\pi - x) + \left(\alpha_1 k^2 + \alpha_2\right)\cos k\,(\pi - x)\right] - \\ - b(x)\left[\left(\beta_1 k^2 + \beta_2\right)\cos k\,(\pi - x) - \left(\alpha_1 k^2 + \alpha_2\right)\sin k\,(\pi - x)\right] + \\ + \mathrm{int}_0^{\pi - x}\widetilde{N}_{21}(x,t)\left[-\left(\beta_1 k^2 + \beta_2\right)\cos kt + \left(\alpha_1 k^2 + \alpha_2\right)\sin kt\right]dt + \\ + k\mathrm{int}_0^{\pi - x}\widetilde{N}_{22}(x,t)\left[\left(\beta_1 k^2 + \beta_2\right)\sin kt + \left(\alpha_1 k^2 + \alpha_2\right)\cos kt\right]dt;$$

for $x < d$

$$\psi(x,k) = \alpha^+\left[-\left(\beta_1 k^2 + \beta_2\right)\cos k\,(\pi - x) + \left(\alpha_1 k^2 + \alpha_2\right)\sin k\,(\pi - x)\right] + \\ + \alpha^-\left[-\left(\beta_1 k^2 + \beta_2\right)\cos k\,(\pi - 2d + x) + \left(\alpha_1 k^2 + \alpha_2\right)\sin k\,(\pi - 2d + x)\right] + \\ + \mathrm{int}_0^{\pi - x}\widetilde{N}_{11}(x,t)\left[-\left(\beta_1 k^2 + \beta_2\right)\cos kt + \left(\alpha_1 k^2 + \alpha_2\right)\sin kt\right]dt,$$

$$(\Gamma\psi)(x,k) = -k\alpha^+\left[\left(\beta_1 k^2 + \beta_2\right)\sin k\,(\pi - x) + \left(\alpha_1 k^2 + \alpha_2\right)\cos k\,(\pi - x)\right] + \\ + k\alpha^-\left[\left(\beta_1 k^2 + \beta_2\right)\cos k\,(\pi - 2d + x) - \left(\alpha_1 k^2 + \alpha_2\right)\sin k\,(\pi - 2d + x)\right] + \\ + b(x)\alpha^+\left[-\left(\beta_1 k^2 + \beta_2\right)\cos k\,(\pi - x) + \left(\alpha_1 k^2 + \alpha_2\right)\sin k\,(\pi - x)\right] + \\ + b(x)\alpha^-\left[\left(\beta_1 k^2 + \beta_2\right)\cos k\,(\pi - 2d + x) - \left(\alpha_1 k^2 + \alpha_2\right)\sin k\,(\pi - 2d + x)\right] + \\ + \mathrm{int}_0^{\pi - x}\widetilde{N}_{21}(x,t)\left[-\left(\beta_1 k^2 + \beta_2\right)\cos kt + \left(\alpha_1 k^2 + \alpha_2\right)\sin kt\right]dt + \\ + k\mathrm{int}_0^{\pi - x}\widetilde{N}_{22}(x,t)\left[\left(\beta_1 k^2 + \beta_2\right)\sin kt + \left(\alpha_1 k^2 + \alpha_2\right)\cos kt\right]dt,$$

where $\widetilde{N}_{ij}(x,t) = N_{ij}(x,t) - N_{ij}(x,-t)$, $i, j = 1, 2$. In the case of $C = 0$ and $q(x) \equiv 0$, denote the solutions with $\psi_{01}(x,k)$ and $\psi_{02}(x,k)$, so we have

$$\psi_0(x,k) = \Psi_{01}(x,k) + f_1,$$

$$(\Gamma\psi_0)(x,k) = \Psi_{02}(x,k) + f_2,$$

where

$$f_1 = \text{int}_0^{\pi-x} \widetilde{N}_{11}(x,t)\left[-\left(\beta_1 k^2+\beta_2\right)\cos kt + \left(\alpha_1 k^2+\alpha_2\right)\sin kt\right]dt,$$

$$f_2 = b(x)\alpha^+\left[-\left(\beta_1 k^2+\beta_2\right)\cos k(\pi-x) + \left(\alpha_1 k^2+\alpha_2\right)\sin k(\pi-x)\right]+$$

$$+b(x)\alpha^-\left[-\left(\beta_1 k^2+\beta_2\right)\cos k(\pi-2d+x) + \left(\alpha_1 k^2+\alpha_2\right)\sin k(\pi-2d+x)\right]+$$

$$+\,\text{int}_0^{\pi-x}\,\widetilde{N}_{21}(x,t)\left[-\left(\beta_1 k^2+\beta_2\right)\cos kt + \left(\alpha_1 k^2+\alpha_2\right)\sin kt\right]dt+$$

$$+k\text{int}_0^{\pi-x}\widetilde{N}_{22}(x,t)\left[\left(\beta_1 k^2+\beta_2\right)\sin kt + \left(\alpha_1 k^2+\alpha_2\right)\cos kt\right]dt.$$

On the other hand, we can write

$$M(k) - M_0(k) = \frac{(\Gamma\psi)(0,k)}{k\psi(0,k)} - \frac{(\Gamma\psi_0)(0,k)}{k\psi_0(0,k)} = \frac{f_2}{k\Delta(k)} - \frac{f_1}{\Delta(k)}M_0(k).$$

Since $\lim\limits_{|k|\to\infty} e^{-|\operatorname{Im} k|\pi}|f_i(k)| = 0$ and $\Delta(k) > C_\delta e^{|\operatorname{Im} k|\pi}$ for $k \in G_\delta$, the equality

$$\frac{f_2}{k\Delta(k)} - \frac{f_1}{\Delta(k)}M_0(k)$$

yields

$$\lim_{|k|\to\infty k\in G_\delta}\sup \left|M(k) - M_0(k)\right| = 0. \tag{4.7}$$

Weyl function $M(k)$ is meromorphic with respect to poles k_n. Using (3.4), (4.1) and Lemma 3.2, we calculate that

$$\operatorname*{Re}_{k=k_n} s M(k) = \frac{(\Gamma\psi)(0,k_n)}{k_n\dot{\Delta}(k_n)} = -\frac{1}{a_n},$$

$$\operatorname*{Re}_{k=k_n^0} s M_0(k) = \frac{(\Gamma\psi_0)\left(0,k_n^0\right)}{k_n^0\dot{\Delta}(k_n^0)} = -\frac{1}{a_n^0}. \tag{4.8}$$

Consider the contour integral

$$I_n(k) = \frac{1}{2\pi i}\text{int}_{\Gamma_n}\frac{M(\mu) - M_0(\mu)}{k-\mu}d\mu, \quad k \in \operatorname{int}\Gamma_n.$$

By virtue of (4.7) , we have $\lim\limits_{n\to\infty} I_n(k) = 0$. On the other hand, the residue theorem and (4.8) yield

$$I_n(k) = -M(k) + M_0(k) + \sum_{k_n\in int\Gamma_n}\frac{1}{a_n(k-k_n)} - \sum_{k_n^0\in int\Gamma_n}\frac{1}{a_n^0(k-k_n^0)}.$$

Therefore as $n\to\infty$ we get

$$M(k) = M_0(k) + \sum_{n=-\infty}^{+\infty}\frac{1}{a_n(k-k_n)} + \sum_{n=-\infty}^{+\infty}\frac{1}{a_n^0(k-k_n^0)}.$$

It follows from the form of the function $M_0(k)$ that

$$M_0(k) = \frac{1}{a_n^0 k} + \sum_{n=-\infty}^{+\infty}{}' \frac{1}{a_n^0} \left(\frac{1}{k - k_n^0} + \frac{1}{k_n^0} \right).$$

From the last two equalities yield (4.6).

Theorem 4.6 is proved.

Let us formulate a theorem on the uniqueness of a solution of the inverse problem with the use of the Weyl function. For this purpose, parallel with L, we consider the boundary-value problem $\widetilde{L}$ of the same form but with different potential $\widetilde{q}(x)$. It is asumed in what follows that if a certain symbol α denotes an object related to the problem L, then $\widetilde{\alpha}$ denotes the corresponding object related to the problem $\widetilde{L}$.

Theorem 4.2. *If $M(k) = \widetilde{M}(k)$ then $L = \widetilde{L}$. Thus the specification of the Weyl function uniquely determines the operator.*

Proof. Let us define the matrix $P(x,k) = [P_{jk}(x,k)]_{j,k=1,2}$ by the formula

$$P(x,k) \begin{pmatrix} \widetilde{\varphi} & \widetilde{\Phi} \\ \Gamma\widetilde{\varphi} & \Gamma\widetilde{\Phi} \end{pmatrix} = \begin{pmatrix} \varphi & \Phi \\ \Gamma\varphi & \Gamma\Phi \end{pmatrix}. \tag{4.9}$$

Using (4.9) and (4.5) we calculate

$$\begin{aligned}
P_{11}(x,k) &= -\frac{1}{k} \left[\varphi(x,k) \left(\Gamma\widetilde{\Phi} \right)(x,k) - \Phi(x,k) \left(\Gamma\widetilde{\varphi} \right)(x,k) \right], \\
P_{12}(x,k) &= -\frac{1}{k} \left[\Phi(x,k)\widetilde{\varphi}(x,k) - \varphi(x,k)\widetilde{\Phi}(x,k) \right], \\
P_{21}(x,k) &= -\frac{1}{k} \left[\left(\Gamma\varphi \right)(x,k) \big(\Gamma\widetilde{\Phi} \big)(x,k) - \left(\Gamma\Phi \right)(x,k) \big(\Gamma\widetilde{\varphi} \big)(x,k) \right], \\
P_{22}(x,k) &= -\frac{1}{k} \left[\left(\Gamma\Phi \right)(x,k)\widetilde{\varphi}(x,k) - (\Gamma\varphi)(x,k)\widetilde{\Phi}(x,k) \right]
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
\varphi(x,k) &= P_{11}(x,k)\widetilde{\varphi}(x,k) + P_{12}(x,k) \left(\Gamma\widetilde{\varphi} \right)(x,k), \\
(\Gamma\varphi)(x,k) &= P_{21}(x,k)\widetilde{\varphi}(x,k) + P_{22}(x,k) \left(\Gamma\widetilde{\varphi} \right)(x,k), \\
\Phi(x,k) &= P_{11}(x,k)\widetilde{\Phi}(x,k) + P_{12}(x,k) \big(\Gamma\widetilde{\Phi} \big)(x,k), \\
\left(\Gamma\Phi \right)(x,k) &= P_{21}(x,k)\widetilde{\Phi}(x,k) + P_{22}(x,k) \big(\Gamma\widetilde{\Phi} \big)(x,k).
\end{aligned} \tag{4.11}$$

It follows from (4.10), (4.2) and (4.5)

$$\begin{aligned}
P_{11}(x,k) &= 1 + \frac{1}{k\Delta(k)} \Big[\varphi(x,k) \big(\big(\Gamma\widetilde{\Psi} \big)(x,k) - \left(\Gamma\Psi \right)(x,k) \big) - \\
&\qquad - \Psi(x,k) \big(\big(\Gamma\widetilde{\varphi} \big)(x,k) - (\Gamma\varphi)(x,k) \big) \Big], \\
P_{12}(x,k) &= \frac{1}{k\Delta(k)} \left[\Psi(x,k)\widetilde{\varphi}(x,k) - \varphi(x,k)\widetilde{\Psi}(x,k) \right], \\
P_{21}(x,k) &= \frac{1}{k\Delta(k)} \left[(\Gamma\varphi)(x,k) \left(\Gamma\widetilde{\Psi} \right)(x,k) - \left(\Gamma\Psi \right)(x,k) \left(\Gamma\widetilde{\varphi} \right)(x,k) \right],
\end{aligned}$$

$$P_{22}(x,k) = 1 + \frac{1}{k\Delta(k)}\Big[\big(\Gamma\Psi\big)(x,k)\big(\widetilde{\varphi}(x,k) - \varphi(x,k)\big) -$$
$$- \big(\Gamma\varphi\big)(x,k)\big(\widetilde{\Psi}(x,k) - \Psi(x,k)\big)\Big].$$

According to (4.10) and (4.2), for each fixed x, the functions $P_{jk}(x,k)$ are meromorphic in k with poles in the points k_n and $\widetilde{k}_n$. It follows from the representations of the solutions $\Psi(x,k)$ and $\varphi(x,k)$ that

$$\lim_{\substack{k\to\infty\\ k\in G_\delta}} \max_{0\le x\le\pi} |P_{11}(x,k) - 1| = \lim_{\substack{k\to\infty\\ k\in G_\delta}} \max_{0\le x\le\pi} |P_{12}(x,k)| =$$
$$= \lim_{\substack{k\to\infty\\ k\in G_\delta}} \max_{0\le x\le\pi} |P_{22}(x,k) - 1| = \lim_{\substack{k\to\infty\\ k\in G_\delta}} \max_{0\le x\le\pi} |P_{21}(x,k)| = 0. \qquad (4.12)$$

According to (4.2) and (4.3) we have

$$P_{11}(x,k) = -\frac{1}{k}\Big[\varphi(x,k)\big(\Gamma\widetilde{C}\big)(x,k) - C(x,k)\big(\Gamma\widetilde{\varphi}\big)(x,k) +$$
$$+ \big(\widetilde{M}(k) - M(k)\big)\varphi(x,k)\big(\Gamma\widetilde{\varphi}\big)(x,k)\Big],$$
$$P_{12}(x,k) = -\frac{1}{k}\Big[\widetilde{\varphi}(x,k)C(x,k) - \widetilde{C}(x,k)\varphi(x,k) +$$
$$+ \big(M(k) - \widetilde{M}k)\big)\varphi(x,k)\widetilde{\varphi}(x,k)\Big],$$
$$P_{21}(x,k) = -\frac{1}{k}\Big[\big(\Gamma\varphi\big)(x,k)\big(\Gamma\widetilde{C}\big)(x,k) - (\Gamma C)(x,k)\big(\Gamma\widetilde{\varphi}\big)(x,k)\Big] - \qquad (4.13)$$
$$- \frac{1}{k}\Big[\big(\widetilde{M}(k) - M(k)\big)(\Gamma\varphi)(x,k)\big(\Gamma\widetilde{\varphi}\big)(x,k)\Big],$$
$$P_{22}(x,k) = -\frac{1}{k}\Big[\widetilde{\varphi}(x,k)(\Gamma C)(x,k) - \widetilde{C}(x,k)(\Gamma\varphi)(x,k) +$$
$$+ \big(M(k) - \widetilde{M}(k)\big)(\Gamma\varphi)(x,k)\widetilde{\varphi}(x,k)\Big].$$

Thus if $M(k) = \widetilde{M}(k)$ then the functions $P_{jk}(x,k)$ are entire in k for each fixed x. Together with (4.12) we get that

$$P_{11}(x,k) \equiv 1, \qquad P_{12}(x,k) \equiv 0, \qquad P_{21}(x,k) \equiv 0, \qquad P_{22}(x,k) \equiv 1.$$

Substituting into (4.11) we get

$$\varphi(x,k) \equiv \widetilde{\varphi}(x,k), \qquad (\Gamma\varphi)(x,k) \equiv (\Gamma\widetilde{\varphi})(x,k),$$
$$\Phi(x,k) \equiv \widetilde{\Phi}(x,k), \qquad (\Gamma\Phi)(x,k) \equiv \Big(\Gamma\widetilde{\Phi}\Big)(x,k)$$

for all x and k. Consequently $L = \widetilde{L}$.

Theorem 4.2 is proved.

Theorem 4.3. *If $k_n = \widetilde{k}_n$, $a_n = \widetilde{a}_n$, $n \ge 0$ then $L = \widetilde{L}$. Thus, the specification of the spectral data $\big\{k_n, \alpha_n\big\}_{n\ge0}$ uniquely determines the operator.*

Proof. We have

$$M(k)=\frac{1}{a_0\,(k-k_0)}+{\sum_{n=1}^{\infty}}'\left\{\frac{1}{a_n\,(k-k_n)}+\frac{1}{a_n^0k_n^0}\right\},$$
$$\widetilde{M}(k)=\frac{1}{\widetilde{a}_0\left(\widetilde{k}-\widetilde{k}_0\right)}+{\sum_{n=1}^{\infty}}'\left\{\frac{1}{\widetilde{a}_n\left(\widetilde{k}-\widetilde{k}_n\right)}+\frac{1}{\widetilde{a}_n^0\widetilde{k}_n^0}\right\}. \tag{4.14}$$

Under the hypothesis of the theorem and in view of (4.13), we get that $M(k)=\widetilde{M}(k)$ and consequently by Theorem 3.1, $L=\widetilde{L}$.

Theorem 4.4. *If* $k_n=\widetilde{k}_n$, $\mu_n=\widetilde{\mu}_n$, $n\geq 0$, *then* $L=\widetilde{L}$.

Proof. From these properties of functions $\Delta(k)$ and $\widetilde{\Delta}(k)$, it is clear that $\lim\limits_{k\to\infty}\dfrac{\Delta(k)}{\widetilde{\Delta}(k)}=1$, $k_n=\widetilde{k}_n$, and functions of $\Delta(k)$, $\widetilde{\Delta}(k)$ are analytic functions. From the uniqueness theorem of analytic functions, $\Delta(k)=\widetilde{\Delta}(k)$. From Lemma 3.2, we have $\widetilde{\psi}(x,\widetilde{k}_n)=\widetilde{\gamma}_n\widetilde{\varphi}(x,\widetilde{k}_n)=\widetilde{\gamma}_n\widetilde{\varphi}(x,k_n)$ and $\widetilde{\Psi}(x,\widetilde{k}_n)=\widetilde{\Psi}(x,k_n)=\gamma_n\widetilde{\varphi}(x,k_n)$. It follows that $\gamma_n=\widetilde{\gamma}_n$ and so $a_n=\widetilde{a}_n$. Consequently by Theorem 4.1, $L=\widetilde{L}$.

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