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COMMON FIXED POINTS AND INVARIANT APPROXIMATION OF R-SUBWEAKLY COMMUTING MAPS IN CONVEX METRIC SPACES

СПІЛЬНІ НЕРУХОМІ ТОЧКИ ТА ІНВАРІАНТНЕ НАБЛИЖЕННЯ R-СУБСЛАБКО КОМУТУЮЧИХ ВІДОБРАЖЕНЬ В ОПУКЛИХ МЕТРИЧНИХ ПРОСТОРАХ

Sufficient conditions for the existence of a common fixed point of R-subweakly commuting mappings in the framework of a convex metric space are derived. As applications, various best approximation results for this class of mappings are obtained that generalize various results known in the literature.

Встановлено достатні умови існування спільної нерухомої точки R-субслабко комутуючих відображень у рамках опуклого метричного простору. Як застосування, одержано різні результати щодо найкращих наближень для згаданого класу відображень, які узагальнюють інші відомі з літератури результати.

1. Introduction and preliminaries. Applying fixed point theorems, many interesting and useful results have been proved in approximation theory (see, e.g., [1, 2, 7, 9 – 11] and the references cited therein). This paper deals with the common fixed points of R-subweakly commuting mappings in the framework of convex metric spaces. We also establish results on invariant approximation for this class of mappings. The results proved in the paper generalize and extend some of the results of [1, 2, 4, 7, 9, 11, 13].

To begin with, we recall some definitions and known facts to be used in the sequel.

For a metric space (X, d), a continuous mapping $W\colon X \times X \times [0, 1] \to X$ is said to be a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

for all $u \in X$. A metric space (X, d) with a convex structure is called a **convex metric space** [12].

A subset M of a convex metric space (X, d) is said to be a **convex set** [12] if $W(x, y, \lambda) \in M$ for all $x, y \in M$ and $\lambda \in [0, 1]$. A set M is said to be p**-starshaped** [3] where $p \in M$, provided $W(x, p, \lambda) \in M$ for all $x \in M$ and $\lambda \in [0, 1]$, i. e., if the segment $[p, x] = \{W(x, p, \lambda)\colon 0 \leq \lambda \leq 1\}$ joining p to x is contained in M for all $x \in M$. M is said to be **starshaped** if it is p-starshaped for some $p \in M$.

Clearly, each convex set M is starshaped but converse is not true.

A convex metric space (X, d) is said to satisfy **Property (I)** [3] if for all $x, y, q \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y).$$

A normed linear space X and each of its convex subsets are simple examples of convex metric spaces with W given by $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$ for $x, y \in X$ and $0 \leq \lambda \leq 1$. There are many convex metric spaces which are not normed linear spaces (see [3, 12]). Property (I) is always satisfied in a normed linear space.

For a non-empty subset K of a metric space (X,d) and $x \in X$, an element $y \in K$ is said to be a **best approximant** to x or a **best K-approximant** to x if $d(x,y) = = d(x,K) \equiv \inf\{d(x,k)\colon k \in K\}$. The set of all such $y \in K$ is denoted by $P_K(x)$.

For a convex subset K of a convex metric space (X,d), a mapping $g\colon K \to X$ is said to be **affine** if for all $x,y \in K$, $g(W(x,y,\lambda)) = W(gx,gy,\lambda)$ for all $\lambda \in [0,1]$.

g is said to be **affine with respect to** $p \in K$ if $g(W(x,p,\lambda)) = W(gx,gp,\lambda)$ for all $x \in K$ and $\lambda \in [0,1]$.

Suppose (X,d) is a metric space, M a nonempty subset of X, and S,T,f,g are self mappings of M. T is said to be an (f,g)**-contraction** if there exists $k \in [0,1)$ such that $d(Tx,Ty) \le kd(fx,gy)$, ((f,g)**-nonexpansive** if $d(Tx,Ty) \le d(fx,gy)$) for all $x,y \in M$. If $f = g$ then T is said to be an f**-contraction** (f**-nonexpansive**). A point $x \in M$ is a common fixed (coincidence) point of S and T if $x = Sx = Tx$ ($Sx = Tx$). The pair (S,T) is said to be **(a) commuting** on M if $STx = TSx$ for all $x \in M$ **(b)** R**-weakly commuting** [8] on M if there exists a real number $R > 0$ such that $d(TSx,STx) \le Rd(Tx,Sx)$ for all $x \in M$ **(c) compatible** [5] if $\lim d(TSx_n,STx_n) = 0$ whenever (x_n) is a sequence such that $\lim Tx_n = \lim Sx_n = = t$ for some t in M **(d) weakly compatible** [6] if they commute at their coincidence points, i.e., if $STx = TSx$ whenever $Sx = Tx$.

Suppose (X,d) is a convex metric space, M is starshaped with respect to q, where q a fixed point of S, and is both T- and S- invariant. T and S are called **(e)** R**-subcommuting** [10] on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx,STx) \le (R/\lambda)\operatorname{dist}(Sx,W(Tx,q,\lambda))$, $\lambda \in (0,1]$ **(f)** R**-subweakly commuting** [9] on M if for all $x \in M$ and $\lambda \in [0,1]$, there exists a real number $R > 0$ such that $d(TSx,STx) \le R\operatorname{dist}(Sx,W(Tx,q,\lambda))$.

Clearly Compatible maps are weakly compatible but the converse need not be true (see [6]). Commuting mappings are R-subweakly commuting, but the converse may not be true (see [9]). It is well known that R-subweakly commuting maps are R-weakly commuting and R-weakly commuting are compatible but not conversely (see [5, 9]). R-subcommuting and R-subweakly commuting maps are weakly compatible but their converses do not hold (see [9 – 11]).

Throughout, we shall write $\overline{M}$ for closure of the set M, $F(S)$ for the set of fixed points of a mapping S, $F(S,T)$ ($C(S,T)$) for the set of fixed points (coincidence points) of mappings S and T.

2. Main results. The following four lemmas will be used in proving results of this paper.

Lemma A [11]. *Let M be a closed subset of a metric space (X,d), and let T,S be R-weakly commuting self mappings of M such that $T(M) \subseteq S(M)$. Suppose there exists $k \in [0,1)$ such that*

$$d(Tx,Ty) \le k\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}[d(Sx,Ty)+d(Sy,Tx)]\right\},$$

for all $x,y \in M$. If $\overline{T(M)}$ is complete and T is continuous, then $M \cap F(T) \cap F(S)$ is a singleton.

Lemma B [10]. *Let M be a closed subset of a metric space (X,d), and let S,T be R-weakly commuting self mappings of M such that $T(M) \subseteq S(M)$. Suppose T is an S-contraction. If $\overline{T(M)}$ is complete and T is continuous, then $F(T) \cap F(S)$ is a singleton.*

Lemma C [1]. *Let (X,d) be a convex metric space, $M \subset X$ and $x_0 \in X$. Then $P_M(x_0) \subset \partial M \cap M$.*

Lemma D [5]. *Let A, B, S and T be self mappings of a complete metric space (X,d). Suppose that S, T are continuous, the pairs (A,S) and (B,T) are compatible pairs, and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If there exists $r \in (0,1)$ such that*

$$d(Ax,By) \le r \max\left\{ d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(Ax,Ty)+d(Sx,By)] \right\},$$

for all $x, y \in X$, then there is a unique point z in X such that $Az = Bz = Sz = Tz = z$.

For continuous self mappings on closed subsets of convex metric spaces, we have the following result.

Theorem 1. *Let M be a closed subset of a convex metric space (X,d) with Property* (I) *and let T, S be continuous self mappings on M such that $T(M) \subseteq S(M)$. Suppose S is affine, $p \in F(S)$, M is starshaped with respect to p, and $\overline{T(M)}$ is compact. If T and S are R-subweakly commuting and satisfy*

$$d(Tx,Ty) \le \max\Big\{ d(Sx,Sy), \operatorname{dist}(Sx, W(Tx,p,\lambda)), \operatorname{dist}(Sy, W(Ty,p,\lambda)),$$

$$\frac{1}{2}[\operatorname{dist}(Sx, W(Ty,p,\lambda)) + \operatorname{dist}(Sy, W(Tx,p,\lambda))]\Big\},$$

for all $x, y \in M$, $\lambda \in [0,1)$, then $M \cap F(T) \cap F(S) \ne \emptyset$.

Proof. For each n, define $T_n \colon M \to M$ by $T_n x = W(Tx,p,\lambda_n)$, $x \in M$ where (λ_n) is a sequence in $(0,1)$ such that $\lambda_n \to 1$. Since M is starshaped with respect to p, S is affine with respect to p and $T(M) \subseteq S(M)$, we have

$$T_n(x) = W(Tx,p,\lambda_n) = W(Tx,Sp,\lambda_n) \in S(M)$$

and so $T_n(M) \subseteq S(M)$ for each n. Consider

$$\begin{aligned} d(T_nSx, ST_nx) &= d(W(TSx,p,\lambda_n), SW(Tx,p,\lambda_n)) \\ &= d(W(TSx,p,\lambda_n), W(STx,Sp,\lambda_n)), \quad S \text{ is affine} \\ &= d(W(TSx,p,\lambda_n), W(STx,p,\lambda_n)), \quad p \in F(S) \\ &\le \lambda_n d(TSx,STx), \ \text{Property (I)} \\ &\le \lambda_n R \operatorname{dist}(Sx, W(Tx,p,\lambda)), \end{aligned}$$

for each n since $\lambda_n \in (0,1)$, we obtain

$$d(T_nSx, ST_nx) \le \lambda_n R \operatorname{dist}(Sx, W(Tx,p,\lambda_n)) \le \lambda_n R d(Sx, T_nx)$$

for all $x \in M$. This shows that T_n and S are $\lambda_n R$-weakly commuting for each n. Also

$$d(T_nx, T_ny) = d(W(Tx,p,\lambda_n), W(Ty,p,\lambda_n))$$

$$\leq \lambda_n d(Tx, Ty), \text{ Property (I)}$$

$$\leq \lambda_n \max \left\{ d(Sx, Sy), \operatorname{dist}(Sx, W(Tx, p, \lambda)), \operatorname{dist}(Sy, W(Ty, p, \lambda)), \right.$$

$$\left. \frac{1}{2}[\operatorname{dist}(Sx, W(Ty, p, \lambda)) + \operatorname{dist}(Sy, W(Tx, p, \lambda))] \right\},$$

for each n since $\lambda_n \in (0, 1)$, we obtain

$$d(T_n x, T_n y) \leq \lambda_n \max \left\{ d(Sx, Sy), \operatorname{dist}(Sx, W(Tx, p, \lambda_n)), \operatorname{dist}(Sy, W(Ty, p, \lambda_n)), \right.$$

$$\left. \frac{1}{2}[\operatorname{dist}(Sx, W(Ty, p, \lambda_n)) + \operatorname{dist}(Sy, W(Tx, p, \lambda_n))] \right\}$$

$$\leq \lambda_n \max \left\{ d(Sx, Sy), d(Sx, T_n x), d(Sy, T_n y), \right.$$

$$\left. \frac{1}{2}[d(Sx, T_n y) + d(Sy, T_n x)] \right\}$$

for all $x, y \in M$. Now by Lemma A, there exists some $x_n \in M$ such that $F(T_n) \cap \cap F(S) = \{x_n\}$ for each n. The compactness of $\overline{T(M)}$ implies the existence of a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} \to y \in M$. By the continuity of T and S, we have $y \in F(T) \cap F(S)$. Hence $M \cap F(T) \cap F(S) \neq \emptyset$.

***Corollary* 1.1** ([11], Theorem 2.2)**.** *Let M be a closed subset of a normed linear space X, and let T, S be continuous self mappings on M such that $T(M) \subseteq S(M)$. Suppose S is linear, $p \in F(S)$, M is starshaped with respect to p, and $\overline{T(M)}$ is compact. If T and S are R-subweakly commuting and satisfy*

$$\|Tx - Ty\| \leq \max \left\{ \|Sx - Sy\|, \operatorname{dist}(Sx, [Tx, p]), \operatorname{dist}(Sy, [Ty, p]), \right.$$

$$\left. \frac{1}{2}\big[\operatorname{dist}(Sx, [Ty, p]) + \operatorname{dist}(Sy, [Tx, p])\big] \right\},$$

for all $x, y \in M$, then $M \cap F(T) \cap F(S) \neq \emptyset$.

***Corollary* 1.2.** *Let M be a closed subset of a normed linear space X and let T, S be continuous self mappings on M such that $T(M) \subseteq S(M)$. Suppose S is linear, $p \in F(S)$, M is starshaped with respect to p, and $\overline{T(M)}$ is compact. If T and S are commuting and satisfy*

$$\|Tx - Ty\| \leq \max \left\{ \|Sx - Sy\|, \operatorname{dist}(Sx, [Tx, p]), \operatorname{dist}(Sy, [Ty, p]), \right.$$

$$\frac{1}{2}\big[\mathrm{dist}(Sx,[Ty,p])+\mathrm{dist}(Sy,[Tx,p])\big]\Bigg\},$$

for all $x,y\in M$, *then* $M\cap F(T)\cap F(S)\neq\emptyset$.

Using Lemma B, we prove the following theorem.

Theorem 2. *Let* M *be a closed subset of a convex metric space* (X,d) *with Property* (I), *and let* T,S *be continuous self mappings on* M *such that* $T(M)\subseteq S(M)$. *Suppose* S *is affine,* $p\in F(S)$, M *is starshaped with respect to* p, *and* $\overline{T(M)}$ *is compact. If* T,S *are* R*-subweakly commuting and* T *is* S*-nonexpansive on* M, *then* $M\cap F(T)\cap F(S)\neq\emptyset$.

Proof. Proceeding as in Theorem 1, we have

$$d(T_nSx,ST_nx)\leq\lambda_nR\,\mathrm{dist}\,(Sx,W(Tx,p,\lambda_n))\leq\lambda_nRd(Sx,T_nx)$$

for all $x\in M$. This shows that T_n and S are λ_nR-weakly commuting for each n. Also

$$d(T_nx,T_ny)=d(W(Tx,p,\lambda_n),W(Ty,p,\lambda_n))\leq\lambda_nd(Tx,Ty)\leq\lambda_nd(Sx,Sy).$$

Thus each T_n is S-contraction. Since $\overline{T(M)}$ is compact, by Lemma B, there exists some $x_n\in M$ such that $F(T_n)\cap F(S)=\{x_n\}$ for each n. Since $(T(x_n))$ is a sequence in $T(M)$, there exists a subsequence $(T(x_{n_i}))$ with $T(x_{n_i})\to x_0\in\overline{T(M)}$. Since $x_{n_i}=$ $=T_{n_i}x_{n_i}=W(Tx_{n_i},p,\lambda_{n_i})\to x_0$, the continuity of T and S imply $x_0\in F(T,S)$. Hence the result.

***Corollary* 2.1.** *Let* M *be a closed subset of a normed linear space* X, *and let* T,S *be continuous self mappings on* M *such that* $T(M)\subseteq S(M)$. *Suppose* S *is linear,* $p\in F(S)$, M *is starshaped with respect* p, *and* $\overline{T(M)}$ *is compact. If* T *and* S *are* R*-subweakly commuting and* T *is* S*-nonexpansive on* M, *then* $M\cap F(T)\cap F(S)\neq\emptyset$.

***Remark* 1.** Theorems 1, 2 and their Corollaries 1.1, 1.2, and 2.1 generalize and extend the corresponding results of [1, 4, 9, and 11].

For a real number $R>0$, let $D_M^{R,S}(x_0)=P_M(x_0)\cap G_M^{R,S}(x_0)$, where $G_M^{R,S}(x_0)=$ $=\{x\in M\colon d(Sx,x_0)\leq(2R+1)dist(x_0,M)\}$.

Applying Lemma C and Theorem 1, we prove the following theorem.

Theorem 3. *Let* T *and* S *be self mappings of a convex metric space* (X,d) *with Property* (I), $x_0\in F(T,S)$ *and* M *be a subset of* X *such that* $T(\partial M\cap M)\subset M$. *Suppose* S *is affine on* $D_M^{R,S}(x_0)$, $p\in F(S)$, $D_M^{R,S}(x_0)$ *is closed and starshaped with respect to* p, $\overline{T(D_M^{R,S}(x_0))}$ *is compact, and* $S(D_M^{R,S}(x_0))=D_M^{R,S}(x_0)$. *If* T, S *are* R*-subweakly commuting on* $D_M^{R,S}(x_0)$ *and satisfy*

$$d(Tx,Ty)\leq\begin{cases}d(Sx,Sx_0), & if\quad y=x_0,\\ Q(x,y), & if\quad y\in D_M^{R,S}(x_0),\end{cases}$$

where

$$Q(x,y)=\max\Bigg\{d(Sx,Sy),\mathrm{dist}(Sx,W(Tx,p,\lambda)),\mathrm{dist}(Sy,W(Ty,p,\lambda)),$$

$$\frac{1}{2}\big[\mathrm{dist}(Sx,W(Ty,p,\lambda))+\mathrm{dist}(Sy,W(Tx,p,\lambda))\big]\Bigg\},$$

for all $x \in D_M^{R,S}(x_0) \cup \{x_0\}$, *and* $\lambda \in [0,1)$, *then* $P_M(x_0) \cap F(T) \cap F(S) \neq \emptyset$.

Proof. Let $x \in D_M^{R,S}(x_0)$. Then by Lemma C, $x \in \partial M \cap M$ and so $Tx \in M$ as $T(\partial M \cap M) \subset M$. Since $d(Tx, x_0) = d(Tx, Tx_0) \leq d(Sx, Sx_0) = d(Sx, x_0) = \operatorname{dist}(x_0, M)$, we obtain $Tx \in P_M(x_0)$. From the R-subweak commutativity of T and S it follows that

$$\begin{aligned} d(STx, x_0) &= d(STx, Tx_0) \\ &\leq d(STx, TSx) + d(TSx, Tx_0) \\ &\leq Rd(Tx, Sx) + d(S^2x, Sx_0) \\ &\leq R[d(Tx, Tx_0) + d(Tx_0, Sx)] + d(S^2x, Sx_0) \\ &\leq R[\operatorname{dist}(x_0, M) + \operatorname{dist}(x_0, M)] + \operatorname{dist}(x_0, M) \\ &\leq (2R+1)\operatorname{dist}(x_0, M). \end{aligned}$$

This implies that $Tx \in G_M^{R,S}(x_0)$. Consequently, $Tx \in D_M^{R,S}(x_0)$ and so $T(D_M^{R,S}(x_0)) \subseteq D_M^{R,S}(x_0) = S(D_M^{R,S}(x_0))$. Now, Theorem 1 guarantees that $P_M(x_0) \cap F(T) \cap F(S) \neq \emptyset$.

***Corollary* 3.1** ([11], Theorem 2.5)**.** *Let T and S be self mappings of a normed linear space X, $x_0 \in F(T,S)$ and M be a subset of X such that $T(\partial M \cap M) \subset M$. Suppose S is linear on $D_M^{R,S}(x_\circ)$, $p \in F(S)$, $D_M^{R,S}(x_0)$ is closed and starshaped with respect to p, $\overline{T(D_M^{R,S}(x_0))}$ is compact, and $S(D_M^{R,S}(x_0)) = D_M^{R,S}(x_0)$. If T and S are R-subweakly commuting on $D_M^{R,S}(x_0)$ and satisfy*

$$\|Tx - Ty\| \leq \begin{cases} \|Sx - Sy\|, & if \quad y = x_0, \\ Q(x,y), & if \quad y \in D_M^{R,S}(x_0), \end{cases}$$

where

$$Q(x,y) = \max\left\{ \|Sx - Sy\|, \operatorname{dist}(Sx, [Tx,p]), \operatorname{dist}(Sy, [Ty,p]), \right.$$
$$\left. \frac{1}{2}\big[\operatorname{dist}(Sx, [Ty,p]) + \operatorname{dist}(Sy, [Tx,p])\big] \right\},$$

for all $x \in D_M^{R,S}(x_0) \cup \{x_0\}$, then $P_M(x_0) \cap F(T) \cap F(S) \neq \emptyset$.

***Corollary* 3.2** ([7], Theorem 2.3)**.** *Let T and S be self mappings of a convex metric space (X, d) with Property* (I), *$x_0 \in F(T,S)$ and M be a subset of X such that $T(\partial M \cap M) \subset M$ and $p \in F(S) \cap M$. Suppose T and S are R-subweakly commuting on $D_M^{R,S}(x_0)$, T is S-nonexpansive on $D_M^{R,S}(x_0) \cup \{x_0\}$ and S is affine on $D_M^{R,S}(x_0)$. If $D_M^{R,S}(x_0)$ is closed and starshaped with respect to p, $\overline{T(D_M^{R,S}(x_0))}$ is compact, $S(D_M^{R,S}(x_0)) = D_M^{R,S}(x_0)$, and T is continuous, then $P_M(x_0) \cap F(T) \cap F(S)$ is nonempty.*

Theorem 4. *Let T and S be self mappings of a convex metric space (X, d) with Property* (I), *$x_0 \in F(T, S)$ and M be a subset of X such that $T(\partial M \cap M) \subset \subset S(M) \subset M$. Suppose S is affine on $D_M^{R,S}(x_0)$, $p \in F(S)$, $D_M^{R,S}(x_0)$ is closed and starshaped with respect to p, $\overline{T(D_M^{R,S}(x_0))}$ is compact, and $S(G_M^{R,S}(x_0)) \cap D_M^{R,S}(x_0) = = S(D_M^{R,S}(x_0)) \subset D_M^{R,S}(x_0)$. If T and S are R-subweakly commuting and continuous on $D_M^{R,S}(x_0)$ and satisfy*

$$d(Tx, Ty) \leq \begin{cases} d(Sx, Sx_0), & if \quad y = x_0, \\ Q(x, y), & if \quad y \in D_M^{R,S}(x_0), \end{cases}$$

where

$$Q(x, y) = \max \Big\{ d(Sx, Sy), \operatorname{dist}(Sx, W(Tx, p, \lambda)), \operatorname{dist}(Sy, W(Ty, p, \lambda)),$$

$$\frac{1}{2}\big[\operatorname{dist}(Sx, W(Ty, p, \lambda)) + \operatorname{dist}(Sy, W(Tx, p, \lambda))\big] \Big\},$$

for all $x \in D_M^{R,S}(x_0) \cup \{x_0\}$, and $\lambda \in [0, 1)$, then $P_M(x_0) \cap F(T) \cap F(S) \neq \emptyset$.

Proof. Let $x \in D_M^{R,S}(x_0)$. Then as in Theorem 3, $Tx \in D_M^{R,S}(x_0)$, i.e., $T(D_M^{R,S}(x_0)) \subseteq D_M^{R,S}(x_0)$. Also, $d(W(x, x_0, k), x_0) < kd(x, x_0) + (1-k)d(x_0, x_0) = = kd(x, x_0) < \operatorname{dist}(x_0, M)$ for all $k \in (0, 1)$. Then by Lemma C, $x \in \partial M \cap \cap M$ and so $T(D_M^{R,S}(x_0)) \subseteq T(\partial M \cap M) \subset S(M)$ and so we can choose $y \in M$ such that $Tx = Sy$. Since $Sy = Tx \in P_M(x_0)$, it follows that $y \in G_M^{R,S}(x_0)$. Consequently, $T(D_M^{R,S}(x_0)) \subseteq S(G_M^{R,S}(x_0)) \subset P_M(x_0)$. Therefore, $T(D_M^{R,S}(x_0)) \subseteq \subseteq S(G_M^{R,S}(x_0)) \cap D_M^{R,S}(x_0) = S(D_M^{R,S}(x_0)) \subset D_M^{R,S}(x_0)$. So, Theorem 1 guarantees that $P_M(x_0) \cap F(T) \cap F(S) \neq \emptyset$.

***Corollary* 4.1** ([11], Theorem 2.6)**.** *Let T and S be self mappings of a normed linear space, $x_0 \in F(T, S)$ and M be a subset of X such that $T(\partial M \cap M) \subset S(M) \subset M$. Suppose S is linear on $D_M^{R,S}(x_0)$, $p \in F(S)$, $D_M^{R,S}(x_0)$ is closed and starshaped with respect to p, $\overline{T(D_M^{R,S}(x_0))}$ is compact, and $S(G_M^{R,S}(x_0)) \cap D_M^{R,S}(x_0) = S(D_M^{R,S}(x_0)) \subset \subset D_M^{R,S}(x_0)$. If T and S are R-subweakly commuting and continuous on $D_M^{R,S}(x_0)$ and satisfy*

$$\|Tx - Ty\| \leq \begin{cases} \|Sx - Sy\|, & if \quad y = x_0, \\ Q(x, y), & if \quad y \in D_M^{R,S}(x_0), \end{cases}$$

where

$$Q(x, y) = \max \Big\{ \|Sx - Sy\|, \operatorname{dist}(Sx, [Tx, p]), \operatorname{dist}(Sy, [Ty, p]),$$

$$\frac{1}{2}\big[\operatorname{dist}(Sx, [Ty, p]) + \operatorname{dist}(Sy, [Tx, p])\big] \Big\},$$

for all $x \in D_M^{R,S}(x_0) \cup \{x_0\}$, then $P_M(x_0) \cap F(T) \cap F(S) \neq \emptyset$.

***Corollary* 4.2** ([7], Theorem 2.4)**.** *Let T and S be self mappings of a convex metric space (X, d) with Property* (I), *$x_0 \in F(T, S)$ and M be a subset of X such that*

$T(\partial M \cap M) \subset S(M) \subset M$ and $p \in F(S) \cap M$. Suppose T and S are R-subweakly commuting on $D_M^{R,S}(x_0)$, T is S-nonexpansive on $D_M^{R,S}(x_0) \cup \{x_0\}$ and S is affine on $D_M^{R,S}(x_0)$. If $D_M^{R,S}(x_0)$ is closed and starshaped with respect to, $\overline{T(D_M^{R,S}(x_0))}$ is compact, $S(M) \cap D_M^{R,S}(x_0) \subset S(D_M^{R,S}(x_0)) \subset D_M^{R,S}(x_0)$, and T is continuous, then $P_M(x_0) \cap F(T) \cap F(S)$ is nonempty.

***Remark* 2.** Theorems 3 and 4 remain valid when $D_M^{R,S}(x_0) = P_M(x_0)$. If $S(P_M(x_0)) \subset P_M(x_0)$, then $P_M(x_0) \subset G_M^{R,S}(x_0)$ and so $D_M^{R,S}(x_0) = P_M(x_0)$. Consequently, Theorem 3 contains the following result as a special case.

Theorem 5 ([2], Theorem 6)**.** *Let T and S be self mappings of a convex metric space (X, d) with Property* (I), *$x_0 \in F(T, S)$ and M be a subset of X such that $T(\partial M) \subseteq M$. Suppose T is S-nonexpansive on $P_M(x_0) \cup \{x_0\}$, S is affine and continuous on $P_M(x_0)$ and $STx = TSx$ for all x in $P_M(x_0)$. If $P_M(x_0)$ is nonempty, compact and starshaped with respect to p, $p \in F(S)$, and if $S(P_M(x_0)) = P_M(x_0)$, then $P_M(x_0) \cap F(T) \cap F(S)$ is nonempty.*

As an application of Lemma D, we obtain the following theorem.

Theorem 6. *Let M be a nonempty subset of a convex metric space (X, d) with Property* (I) *and T, f and g be continuous self maps of M. Suppose that M is starshaped with respect to q, f and g are affine with $q \in F(f) \cap F(g)$, $\overline{T(M)} \subset f(M) \cap g(M)$, and $\overline{T(M)}$ is compact. If the pairs (T, f) and (T, g) are R-subweakly commuting and satisfy*

$$d(Tx, Ty) \leq \max \Bigg\{ d(fx, gy), \operatorname{dist}(fx, W(Tx, q, \lambda)), \operatorname{dist}(gy, W(Ty, q, \lambda)),$$

$$\frac{1}{2}\big[\operatorname{dist}(fx, W(Ty, q, \lambda)) + \operatorname{dist}(gy, W(Tx, q, \lambda))\big] \Bigg\},$$

for all $x, y \in M$, and $\lambda \in [0, 1)$, then $F(T) \cap F(f) \cap F(g)$ is nonempty.

Proof. For each n, define $T_n \colon M \to M$ by $T_n x = W(Tx, q, \lambda_n)$, $x \in D$ where (λ_n) is a sequence in $(0, 1)$ such that $\lambda_n \to 1$. Since M is starshaped with respect to q, f and g are affine with respect to q and $\overline{T(M)} \subset f(M) \cap g(M)$, we have

$$\begin{aligned} d(T_n fx, fT_n x) &= d(W(Tfx, q, \lambda_n), fW(Tx, q, \lambda_n)) \\ &= d(W(Tfx, q, \lambda_n), W(fTx, fq, \lambda_n)), f \text{ is affine} \\ &= d(W(Tfx, q, \lambda_n), W(fTx, q, \lambda_n)), q \in F(f) \\ &\leq \lambda_n d(Tfx, fTx), \text{ Property (I)} \\ &\leq \lambda_n R \operatorname{dist}(fx, W(Tx, q, \lambda)) \end{aligned}$$

for each n since $\lambda_n \in (0, 1)$, we obtain

$$d(T_n fx, fT_n x) \leq \lambda_n R \operatorname{dist}(fx, W(Tx, q, \lambda_n)) \leq \lambda_n R d(fx, T_n x)$$

for all $x \in M$. This shows that T_n and f are $\lambda_n R$-weakly commuting for each n, similarly T_n and g are $\lambda_n R$-weakly commuting for each n. Also

$$d(T_nx,T_ny) = d(W(Tx,q,\lambda_n),W(Ty,q,\lambda_n))$$

$$\leq \lambda_n d(Tx,Ty)$$

$$\leq \lambda_n \max\left\{d(fx,gy),\mathrm{dist}(fx,W(Tx,q,\lambda)),\mathrm{dist}(gy,W(Ty,q,\lambda)),\right.$$

$$\left.\frac{1}{2}[\mathrm{dist}(fx,W(Ty,q,\lambda))+\mathrm{dist}(gy,W(Tx,q,\lambda))]\right\}$$

for each n since $\lambda_n \in (0,1)$, we obtain

$$d(T_nx,T_ny) \leq \lambda_n \max\left\{d(fx,gy),\mathrm{dist}(fx,W(Tx,q,\lambda_n)),\mathrm{dist}(gy,W(Ty,q,\lambda_n)),\right.$$

$$\left.\frac{1}{2}[\mathrm{dist}(fx,W(Ty,q,\lambda_n))+\mathrm{dist}(gy,W(Tx,q,\lambda_n))]\right\}$$

$$\leq \lambda_n \max\left\{d(fx,gy),d(fx,T_nx),d(gy,T_ny),\right.$$

$$\left.\frac{1}{2}[d(fx,T_ny)+d(gy,T_nx)]\right\}$$

for all $x,y \in M$. Now by Lemma D, for each $n \geq 1$, there exists some $x_n \in M$ such that $x_n = fx_n = gx_n = T_nx_n = W(Tx,q,\lambda_n)$. The compactness of $\overline{T(M)}$ implies the existence of a subsequence (x_{n_i}) of (x_n) such that $Tx_{n_i} \to y \in \overline{T(M)}$. Now $x_{n_i} = fx_{n_i} = gx_{n_i} = T_{n_i}x_{n_i} = W(Tx_{n_i},q,\lambda_{n_i}) \to y$ and also $y \in f(M)\cap g(M)$ by $\overline{T(M)} \subset f(M)\cap g(M)$. It follows from the continuity of T, f and g that $Tx_{n_i} \to Ty$, $fx_{n_i} \to fy$ and $gx_{n_i} \to gy$ respectively. So, we get $y = Ty = fy = gy$. Hence $F(T)\cap F(f)\cap F(g) \neq \emptyset$.

***Corollary* 6.1** ([13], Theorem 1)**.** *Let M be a nonempty subset of a normed linear space X and T, f and g be continuous self maps of M. Suppose that M is starshaped with respect to q, f and g are affine with $q \in F(f)\cap F(g)$, $\overline{T(M)} \subset f(M)\cap g(M)$, and $\overline{T(M)}$ is compact. If the pairs (T,f) and (T,g) are R-subweakly commuting and satisfy*

$$\|Tx-Ty\| \leq \max\left\{\|fx-gy\|,\mathrm{dist}(fx,[Tx,q]),\mathrm{dist}(gy,[Ty,q]),\right.$$

$$\left.\frac{1}{2}\left[\mathrm{dist}(fx,[Ty,q])+\mathrm{dist}(gy,[Tx,q])\right]\right\},$$

for all $x,y \in M$, then $F(T)\cap F(f)\cap F(g)$ is nonempty.

Theorem 7. *Let M be a nonempty subset of a convex metric space (X,d) with Property* (I) *and T, f and g be self maps of M. Suppose that M is starshaped with* *respect to* q, f *and* g *are affine,* $\overline{T(M)} \subset f(M) \cap g(M)$, *and* $\overline{T(M)}$ *is compact. If the pairs* (T, f) *and* (T, g) *are R-subweakly commuting,* T *is* (f, g)*-nonexpansive, and either* T *or* f *or* g *is continuous, then* $F(T) \cap F(f) \cap F(g)$ *is nonempty.*

Proof. Proceeding as in Theorem 6, we see that for each n, there exists $x_n \in M$ such that $x_n = fx_n = gx_n = T_n x_n = W(Tx, q, \lambda_n)$. The compactness of $\overline{T(M)}$ implies the existence of a subsequence (x_{n_i}) of (x_n) such that $Tx_{n_i} \to y \in \overline{T(M)}$. Now $x_{n_i} = fx_{n_i} = gx_{n_i} = T_{n_i} x_{n_i} = W(Tx_{n_i}, q, \lambda_{n_i}) \to y$ and also $y \in f(M) \cap g(M)$ by $\overline{T(M)} \subset f(M) \cap g(M)$. Hence there exists $u \in M$ such that $y = fu = gu$. Consider

$$d(Tu, Tx_{n_i}) \leq d(fu, gx_{n_i}) = d(y, gx_{n_i}) \to 0,$$

therefore $Tx_{n_i} \to Tu = y$ i.e. $y = Tu = fu = gu$. As R-subweak commutativity of (T, f) and (T, g) imply weak compatibility, $fy = fTu = Tfu = Ty = Tgu = gTu = gy$. It follows from the continuity of either T or f or g that $Tx_{n_i} \to Ty$ or $fx_{n_i} \to fy$ or $gx_{n_i} \to gy$. Hence $y = Ty = fy = gy$.

***Corollary* 7.1** ([13], Theorem 2)**.** *Let* M *be a nonempty subset of a normed linear space* X *and* T, f *and* g *be self maps of* M. *Suppose that* M *is starshaped with respect to* q, f *and* g *are affine,* $\overline{T(M)} \subset f(M) \cap g(M)$, *and* $\overline{T(M)}$ *is compact. If the pairs* (T, f) *and* (T, g) *are R-subweakly commuting,* T *is* (f, g)*-nonexpansive, and either* T *or* f *or* g *is continuous, then* $F(T) \cap F(f) \cap F(g)$ *is nonempty.*

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