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ON STATISTICAL CONVERGENCE OF METRIC VALUED SEQUENCES

ПРО СТАТИСТИЧНУ ЗБІЖНІСТЬ МЕТРИЧНОЗНАЧНИХ ПОСЛІДОВНОСТЕЙ

We study the conditions on the density of a subsequence of a statistical convergent sequence under which this subsequence is also statistical convergent. Some sufficient conditions of this type and almost converse necessary conditions are obtained in the setting of general metric spaces.

Вивчаються умови на щільність підпослідовності статистично збіжної послідовності, за яких ця підпослідовність також є статистично збіжною. Деякі достатні умови такого типу та майже обернені необхідні умови отримано в постановці загальних метричних просторів.

1. Introduction and definitions. Analysis on metric spaces has rapidly developed in present time (see [15, 18]). This development is usually based on some generalizations of the differentiability. The generalizations of the differentiation involve linear structure by means of embeddings of metric spaces in a suitable normed space or by use of geodesics.

A new intrinsic approach to the introduction of the smooth structure for general metric space was proposed by O. Martio and O. Dovgoshey in [10] (see also [1, 3, 4, 7 – 9]). The approach in [10] is completely based on the convergence of the metric valued sequences but it is not apriori clear that the usual convergence is the best possible way to obtain the smooth structure for arbitrary metric space.

The problem of convergence in different ways of a real (or complex) valued divergent sequence goes back to the beginning of nineteenth century. A lot of different convergence methods were defined (Cesaro, Nörlund, Weighted Mean, Abel et al.) and applied to many branches of mathematics. Almost all convergence methods depend on the algebraic structure of the space. It is clear that metric space does not have the algebraic structure in general. However, the notion of statistical convergence is easy to extend for arbitrary metric spaces and this provides a general framework for summability in such spaces [13, 21]. Thus, the studies of statistical convergence give a natural foundation for upbuilding of different tangent spaces to general metric spaces.

The construction of tangent spaces in [3, 4, 7 – 10] is based on the following fundamental fact: “If (x_n) is a convergent sequence in a metric space, then each subsequence $(x_{n(k)})$ of (x_n) is also convergent”. Thus the convergence of subsequence $(x_{n(k)})$ does not depend on the choice of $(x_{n(k)})$. Unfortunately it is not the case for the statistical convergent sequences. The applications of the statistical convergence to the infinitesimal geometry of metric spaces should be based on the complete understanding of the structure of statistical convergent subsequences.

We study the conditions on the density of a subsequence of a statistical convergent sequence under which this subsequence is also statistical convergent. Some sufficient conditions of such type and “almost converse” them necessary conditions are obtained in the setting of general metric spaces.

Let us remember the main definitions. Let (X, d) be a metric space. For convenience denote by $\tilde{X}$ the set of all sequences of points from X.

Definition 1.1. *A sequence* $(x_n) \in \tilde{X}$ *is called convergent to a point* $a \in X$, $\lim_{n\to\infty} x_n = a$, *if for every* $\epsilon > 0$ *there is an* $n_0 = n_0(\epsilon) \in \mathbb{N}$ *such that* $n > n_0$ *implies* $d(x_n, a) < \epsilon$.

Definition 1.2. *A metric valued sequence* $\tilde{x} = (x_n) \in \tilde{X}$ *is* d*-statistical convergent to* $a \in X$ *if*

$$\lim_{n\to\infty} \frac{1}{n} \big| \{k \colon k \le n,\, d(x_k, a) \ge \epsilon\} \big| = 0$$

holds for every $\epsilon > 0$.

Here and later $|B|$ denotes the number of elements of a set B.

The idea of statistical convergence goes back to Zygmund [22]. It was formally introduced by Steinhous [20] and Fast [11]. In recent years, it has become an active research for mathematicians (see, for example, [5, 6, 12–14, 17]).

Definition 1.3 [11] (Dense subset of $\mathbb{N}$)**.** *A set* $K \subseteq \mathbb{N}$ *is called a statistical dense subset of* $\mathbb{N}$ *if*

$$\lim_{n\to\infty} \frac{1}{n} \big|K(n)\big| = 1,$$

where $K(n) = \{k \in K \colon k \le n\}$.

It may be proved that the intersection of two dense subsets is dense. Moreover it is clear that the supersets of dense sets are also dense. Hence the family of all dense sets forms a filter on $\mathbb{N}$. The d -statistical convergence is simply the convergence in (X, d) with respect to this filter.

Definition 1.4 (Dense subsequence)**.** *If* $\big(n(k)\big)$ *is an infinite, strictly increasing sequence of natural numbers and* $\tilde{x} = (x_n) \in \tilde{X}$, *write* $\tilde{x}' = (x_{n(k)})$ *and* $K_{\tilde{x}'} = \big\{n(k) \colon k \in \mathbb{N}\big\}$. *The subsequence* $\tilde{x}'$ *is a dense subsequence of* $\tilde{x}$ *if* $K_{\tilde{x}'}$ *is a dense subset of* $\mathbb{N}$.

In the next definition we introduce an equivalence relation on the set $\tilde{X}$.

Definition 1.5. *Sequences* $\tilde{x} = (x_n) \in \tilde{X}$ *and* $\tilde{y} = (y_n) \in \tilde{X}$ *are statistical equivalent,* $\tilde{x} \asymp \tilde{y}$, *if there is a statistical dense* $M \subseteq \mathbb{N}$ *such that* $x_n = y_n$ *for every* $n \in M$.

2. Convergent sequences and statistical convergent ones. In this section, some basic results on d-statistical convergence will be given for an arbitrary metric space. In particular, it is shown that there is some one-to-one correspondence between metrizable topologies on X and the subsets of $\tilde{X}$ consisting of all statistical convergent sequences.

Let (X, d) be a nonvoid metric space. It is clear that every convergent sequence $(x_n) \in \tilde{X}$ is also d-statistical convergent. Moreover, all statistical convergent sequences are convergent if and only if $|X| = 1$. Nevertheless, we have the following result.

Theorem 2.1. *Let* (X, d_1) *and* (X, d_2) *be two metric spaces with the same underlining set* X. *Then the following statements are equivalent*:

(i) *The set of all* d_1*-statistical convergent sequences coincides the set of all* d_2*-statistical convergent sequences.*

(ii) *The set of all sequences which are convergent in the space* (X, d_1) *coincides the set of all sequences which are convergent in the space* (X, d_2).

(iii) *The metrics* d_1 *and* d_2 *induce one and the same topology on* X.

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) is well known. Since every statistical convergent sequence can be obtained by a variation of values of a suitable convergent sequence outside of a statistical dense set, the implication (ii) $\Rightarrow$ (i) follows.

Suppose now that topologies induced by the metrics d_1 and d_2 are distinct. Then there exist a point $a \in X$ and $\epsilon_0 > 0$ such that either

$$\big\{x \in X\colon d_1(x,a) < \epsilon_0\big\} \not\supseteq \big\{x \in X\colon d_2(x,a) < \delta\big\} \tag{2.1}$$

for all $\delta > 0$ or

$$\big\{x \in X\colon d_2(x,a) < \epsilon_0\big\} \not\supseteq \big\{x \in X\colon d_1(x,a) < \delta\big\}$$

for all $\delta > 0$. We assume, without loss of generality, that (2.1) holds. Then there is a sequence $\tilde{x} = (x_n)$ such that

$$d_2(x_n, a) < \frac{1}{n} \qquad \text{and} \qquad d_1(x_n, a) \geq \epsilon_0 \tag{2.2}$$

for each $n \in \mathbb{N}$. Let us define a new sequence $\tilde{y} = (y_n) \in \tilde{X}$ by the rule

$$y_n = \begin{cases} x_n & \text{if } n \text{ is odd,} \\ a & \text{if } n \text{ is even.} \end{cases}$$

This definition and (2.2) imply the equality

$$\lim_{n\to\infty} \frac{\big|\{k \in \mathbb{N}\colon d_1(y_k, a) \geq \epsilon_0,\ k \leq n\}\big|}{n} = \frac{1}{2}. \tag{2.3}$$

It is clear that the sequence $\tilde{y}$ is d_2-statistical convergent to a. If statement (i) holds, then $\tilde{y}$ is also d_1-statistical convergent. Using Theorem 3.1 (the proof of Theorem 3.1 does not depend on Theorem 2.1, see Section 3 of the paper) we obtain that $\tilde{y}$ is d_1-statistically convergent to the same a. Consequently we have

$$\lim_{n\to\infty} \frac{\big|\{k \in \mathbb{N}\colon d_1(y_k, a) \geq \epsilon_0,\ k \leq n\}\big|}{n} = 0,$$

contrary to (2.3) The implication (i) $\Rightarrow$ (iii) follows.

Theorem 2.1 is proved.

The next simple lemma gives us a tool for a reduction of some questions related to the d-statistical convergence to the case of the statistical convergence in $\mathbb{R}$.

Lemma 2.1. *Let (X,d) be a metric space, $a \in X$ and $\tilde{x} = (x_n) \in \tilde{X}$. Then $\tilde{x}$ is d-statistical convergent to a in X if and only if the sequence $\big(d(x_n, a)\big)$ is statistical convergent to 0 in $\mathbb{R}$.*

The proof follows directly from the definitions.

Theorem 2.2. *Let (X,d) be a metric space, $a \in X$ and let $\tilde{x} = (x_n) \in \tilde{X}$ be a d-statistically convergent to a sequence. There is $\tilde{y} = (y_n) \in \tilde{X}$ such that $\tilde{y} \asymp \tilde{x}$ and $\tilde{y}$ is convergent to a.*

Proof. If $X = \mathbb{R}$ and $d(x,y) = |x - y|$ for all $x, y \in X$, then the theorem is known (see Theorem A in [14] or Lemma 1.1 in [17]). Now let (X,d) be an arbitrary metric space. By Lemma 2.1 $\big(d(x_n, a)\big)$ is statistically convergent to 0. Hence there is a subsequence $\big(d(x_{n(k)}, a)\big)$ of the sequence $\big(d(x_n, a)\big)$ such that $\lim_{k\to\infty} d(x_{n(k)}, a) = 0$ and the set $K = \big\{n(k)\colon k \in \mathbb{N}\big\}$ is a dense subset of $\mathbb{N}$. Define the sequence $\tilde{y} = (y_n) \in \tilde{X}$ as

$$y_n = \begin{cases} x_n & \text{if } n \in K, \\ a & \text{if } n \in \mathbb{N} \setminus K. \end{cases}$$

It is easy to see that $\tilde{y} \asymp \tilde{x}$ and $\lim_{n\to\infty} y_n = a$.

Theorem 2.2 is proved.

3. Statistical convergence of sequences and their subsequences. If a given sequence is d-statistical convergent it is natural to ask how we can check that its subsequence is d-statistical convergent to the same limit.

Theorem 3.1. *Let (X,d) be a metric space, $\tilde{x} = (x_n) \in \tilde{X}$ and let $\tilde{x}' = (x_{n(k)})$ be a subsequence of $\tilde{x}$ such that*

$$\liminf_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{n} > 0.$$

If $\tilde{x}$ is d-statistical convergent to $a \in X$, then $\tilde{x}'$ is also d-statistical convergent to this a.

Proof. Suppose that (x_n) is d-statistical convergent to a. It is clear that

$$\big\{n(k)\colon n(k) \le n, \ \ d(x_{n(k)}, a) \ge \epsilon\big\} \subseteq \big\{m\colon m \le n, \ \ d(x_m, a) \ge \epsilon\big\}$$

for all n. Consequently we have

$$\frac{1}{\big|K_{\tilde{x}'}(n)\big|}\Big|\big\{n(k)\colon n(k) \le n, \ \ d(x_{n(k)}, a) \ge \epsilon\big\}\Big| \le \frac{1}{\big|K_{\tilde{x}'}(n)\big|}\Big|\big\{m\colon m \le n, \ \ d(x_m, a) \ge \epsilon\big\}\Big|. \tag{3.1}$$

The sequence $\tilde{x} = (x_{n(k)})$ is d-statistical convergent if we obtain

$$\limsup_{n\to\infty} \frac{\big|n(k)\colon n(k) \le n, d(x_{n(k)}, a) \ge \epsilon\big|}{\big|K_{\tilde{x}'}(n)\big|} = 0$$

for every $\epsilon > 0$. The last limit relation holds if

$$\limsup_{n\to\infty} \frac{\big|\{m\colon m \le n, d(x_m, a) \ge \epsilon\}\big|}{\big|K_{\tilde{x}'}(n)\big|} = 0. \tag{3.2}$$

To prove (3.2) we can use the inequality

$$\liminf_{n\to\infty} y_n \limsup_{n\to\infty} z_n \le \limsup_{n\to\infty} y_n z_n \tag{3.3}$$

which holds for all sequences of nonnegative real numbers with $0 \ne \liminf_{n\to\infty} y_n \ne \infty$ (see, for example, [2]). Putting in (3.3)

$$y_n = \frac{\big|K_{\tilde{x}'}(n)\big|}{n} \quad \text{and} \quad z_n = \frac{\big|\{m\colon m \le n, d(x_m, a) \ge \epsilon\}\big|}{\big|K_{\tilde{x}'}(n)\big|}$$

we see that

$$y_n z_n = \frac{\big|\{m\colon m \le n, d(x_m, a) \ge \epsilon\}\big|}{n}.$$

Hence we get

$$\liminf_{n\to\infty} \frac{\big|K_{\tilde{x}'}(n)\big|}{n} \limsup_{n\to\infty} \frac{\big|\{m\colon m \le n, d(x_m, a) \ge \epsilon\}\big|}{\big|K_{\tilde{x}'}(n)\big|} \le \limsup_{n\to\infty} \frac{\big|\{m\colon m \le n, d(x_m, a) \ge \epsilon\}\big|}{n}.$$

The last inequality implies (3.2) because (x_n) is d-statistical convergent.

Theorem 3.1 is proved.

Theorem 3.2. *Let (X, d) be a metric space and let $\tilde{x} \in \tilde{X}$. The following statements are equivalent*:

(i) *The sequence $\tilde{x}$ is d-statistical convergent.*

(ii) *Every subsequence $\tilde{x}'$ of $\tilde{x}$ with*

$$\liminf_{n\to\infty} \frac{\left|K_{\tilde{x}'}(n)\right|}{n} > 0$$

is d-statistical convergent.

(iii) *Every dense subsequence $\tilde{x}'$ of $\tilde{x}$ is d-statistical convergent.*

Proof. The implication (i) $\Rightarrow$ (ii) was proved in Theorem 3.1. Since every dense subsequence $\tilde{x}'$ of $\tilde{x}$ satisfies the inequality

$$\liminf_{n\to\infty} \frac{\left|K_{\tilde{x}'}(n)\right|}{n} > 0,$$

we have (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) holds because $\tilde{x}$ is a dense subsequence of it-self.

Theorem 3.2 is proved.

Lemma 3.1. *Let (X, d) be a metric space with $|X| \geq 2$, let $\tilde{x} = (x_n) \in \tilde{X}$ and let $\tilde{x}' = \left(x_{n(k)}\right)$ be an infinite subsequence of $\tilde{x}$ such that*

$$\limsup_{n\to\infty} \frac{\left|K_{\tilde{x}'}(n)\right|}{n} = 0. \tag{3.4}$$

There are a sequence $\tilde{y} \in \tilde{X}$ and a subsequence $\tilde{y}'$ of $\tilde{y}$ such that: $\tilde{x} \asymp \tilde{y}$ and $K_{\tilde{y}'} = K_{\tilde{x}'}$ and $\tilde{y}'$ is not d-statistical convergent.

Proof. Let a and b be two distinct points of X. Define the sequence $\tilde{y} = (y_n) \in \tilde{X}$ by the rule

$$y_n = \begin{cases} x_n & \text{if} \quad n \in \mathbb{N} \setminus K_{\tilde{x}'}, \\ a & \text{if} \quad n = n(k) \in K_{\tilde{x}'} \text{ and } \quad k \quad \text{is odd}, \\ b & \text{if} \quad n = n(k) \in K_{\tilde{x}'} \text{ and } \quad k \quad \text{is even}. \end{cases} \tag{3.5}$$

The set $\mathbb{N} \setminus K_{\tilde{x}'}$ is a statistical dense subset of $\mathbb{N}$. Indeed, the equality

$$n = \left|\{m \in K_{\tilde{x}'} : m \leq n\}\right| + \left|\{m \in \mathbb{N} \setminus K_{\tilde{x}'} : m \leq n\}\right|$$

holds for each $n \in \mathbb{N}$. It implies the inequality

$$\liminf_{n\to\infty} \frac{\left|\{m \in \mathbb{N} \setminus K_{\tilde{x}'} : m \leq n\}\right|}{n} \geq 1 - \limsup_{n\to\infty} \frac{\left|\{m \in K_{\tilde{x}'} : m \leq n\}\right|}{n}. \tag{3.6}$$

Using (3.4) we obtain

$$1 \leq \liminf_{n\to\infty} \frac{\left|\{m \in \mathbb{N} \setminus K_{\tilde{x}'} : m \leq n\}\right|}{n} \leq \limsup_{n\to\infty} \frac{\left|\{m \in \mathbb{N} \setminus K_{\tilde{x}'} : m \leq n\}\right|}{n} \leq 1.$$

Consequently,

$$\lim_{n\to\infty} \frac{\left|\{m \in \mathbb{N} \setminus K_{\tilde{x}'} : m \leq n\}\right|}{n} = 1. \tag{3.7}$$

The equivalence $\tilde{x} \asymp \tilde{y}$ follows.

Define the desired subsequence $\tilde{y}'$ of $\tilde{y}$ as $\tilde{y}' = (y_{n(k)})$. It is easy to see that $\tilde{y}'$ is not d-statistical convergent.

Lemma 3.1 is proved.

Lemma 3.2. *Let (X, d) be a metric space, $a \in X$, $\tilde{x}$ and $\tilde{y}$ belong to $\tilde{X}$ and let $\tilde{x}$ be d-statistical convergent to a. If $\tilde{x} \asymp \tilde{y}$, then $\tilde{y}$ is also d-statistical convergent to a.*

Proof. Suppose that $\tilde{y} \asymp \tilde{x}$. Define a subset M of the set $\mathbb{N}$ as

$$(n \in M) \Leftrightarrow (x_n \neq y_n).$$

Then, by Definition 1.5, $\mathbb{N} \setminus M$ is statistical dense. It implies the equality

$$\lim_{n\to\infty} \frac{\big|\{m \in M \colon m \leq n\}\big|}{n} = 0. \tag{3.8}$$

Let ϵ be a strictly positive number. It follows directly from the definition of the set M that the inclusion

$$\big\{m \in \mathbb{N} \colon m \leq n,\, d(y_m, a) \geq \epsilon\big\} \subseteq \big\{m \in M \colon m \leq n\big\} \cup \big\{m \in \mathbb{N} \colon m \leq n,\, d(x_m, a) \geq \epsilon\big\} \tag{3.9}$$

holds for each $n \in \mathbb{N}$. Using this inclusion and equality (3.8) we obtain

$$\limsup_{n\to\infty} \frac{\big|\{m \in \mathbb{N} \colon m \leq n,\, d(y_m, a) \geq \epsilon\}\big|}{n} \leq$$

$$\leq \limsup_{n\to\infty} \frac{\big|\{m \in M \colon m \leq n\}\big|}{n} + \limsup_{n\to\infty} \frac{\big|\{m \in \mathbb{N} \colon m \leq n,\, d(x_m, a) \geq \epsilon\}\big|}{n} =$$

$$= \limsup_{n\to\infty} \frac{\big|\{m \in \mathbb{N} \colon m \leq n,\, d(x_m, a) \geq \epsilon\}\big|}{n}.$$

Since $\tilde{x}$ is d-statistical convergent to a we have

$$\limsup_{n\to\infty} \frac{\big|\{m \in \mathbb{N} \colon m \leq n,\, d(x_m, a) \geq \epsilon\}\big|}{n} = 0$$

for every $\epsilon > 0$. Consequently the inequality

$$\limsup_{n\to\infty} \frac{\big|\{m \in \mathbb{N} \colon m \leq n,\, d(y_m, a) \geq \epsilon\}\big|}{n} \leq 0 \tag{3.10}$$

holds for every $\epsilon > 0$. Using (3.10) we get

$$0 \leq \liminf_{n\to\infty} \frac{\big|\{m \in \mathbb{N} \colon m \leq n,\, d(y_m, a) \geq \epsilon\}\big|}{n} \leq \limsup_{n\to\infty} \frac{\big|\{m \in \mathbb{N} \colon m \leq n,\, d(y_m, a) \geq \epsilon\}\big|}{n} \leq 0.$$

Hence the limit relation

$$\lim_{n\to\infty} \frac{\big|\{m \in \mathbb{N} \colon m \leq n,\, d(y_m, a) \geq \epsilon\}\big|}{n} = 0$$

holds. The last limit relation holds for every $\epsilon > 0$ if and only if $\tilde{y}$ is d-statistical convergent to a.

Lemma 3.2 is proved.

Theorem 3.3. *Let (X, d) be a metric space with $|X| \geq 2$, $a \in X$, and let $\tilde{x} \in \tilde{X}$ be a d-statistical convergent to a. Then for every infinite subsequence $\tilde{x}'$ of $\tilde{x}$ with*

$$\limsup_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{n} = 0$$

there are a sequence $\tilde{y} \in \tilde{X}$ and a subsequence $\tilde{y}'$ of $\tilde{y}$ such that:

(i) $\tilde{y} \asymp \tilde{x}$ *and* $K_{\tilde{x}'} = K_{\tilde{y}'}$;

(ii) $\tilde{y}$ *is* d*-statistical convergent to* a;

(iii) $\tilde{y}'$ *is not* d*-statistical convergent.*

Proof. By Lemma 3.1 there are $\tilde{y}$ and $\tilde{y}'$ such that (i) and (iii) holds. To prove (ii) note that $\tilde{y} \asymp \tilde{x}$ by (i) and $\tilde{x}$ is d-statistical convergent to a. Consequently, by Lemma 3.2, $\tilde{y}$ is also d-statistical convergent to a.

Theorem 3.3 is proved.

Using this theorem we obtain the following "weak" converse of Theorem 3.1.

Theorem 3.4. *Let (X, d) be a metric space with $|X| \geq 2$ and let $\tilde{x} \in \tilde{X}$ be a d-statistical convergent sequence. Assume $\tilde{x}'$ is a subsequence of $\tilde{x}$ having the following property: if $\tilde{y} \asymp \tilde{x}$ and $\tilde{y}'$ is a subsequence of $\tilde{y}$ such that $K_{\tilde{x}'} = K_{\tilde{y}'}$, then $\tilde{y}'$ is d-statistical convergent. Then the inequality*

$$\limsup_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{n} > 0 \tag{3.11}$$

holds.

Proof. We have either (3.11) or

$$\limsup_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{n} = 0.$$

If the last equality holds then, by Theorem 3.3, there are $\tilde{y}$ and $\tilde{y}'$ such that $\tilde{y} \asymp \tilde{x}$ $K_{\tilde{x}'} = K_{\tilde{y}'}$ and $\tilde{y}'$ is not d-statistical convergent, contrary to the assumption.

Theorem 3.4 is proved.

Similarly we have a "weak" converse of Theorem 3.3.

Theorem 3.5. *Let (X, d) be a metric space, $a \in X$, and let $\tilde{x} \in \tilde{X}$ be a d-statistical convergent to a sequence. Suppose $\tilde{x}' = \left(x_{n(k)}\right)$ is a subsequence of $\tilde{x}$ for which there are $\tilde{y} \in \tilde{X}$ and $\tilde{y}'$ such that conditions* (i) *and* (iii) *of Theorem* 3.3 *hold. Then we have the equality*

$$\liminf_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{n} = 0. \tag{3.12}$$

To prove this result we shall use the next lemma.

Lemma 3.3. *Let (X, d) be a metric space, $\tilde{x}$ and $\tilde{y}$ belong to $\tilde{X}$ and let $\tilde{x} \asymp \tilde{y}$. If K is a subset of $\mathbb{N}$ such that*

$$\liminf_{n\to\infty} \frac{|K(n)|}{n} > 0 \tag{3.13}$$

and if $\tilde{x}' = \left(x_{n(k)}\right)$ and $\tilde{y}' = \left(y_{n(k)}\right)$ are subsequences of $\tilde{x}$ and, respectively, of $\tilde{y}$ such that $K_{\tilde{x}'} = K_{\tilde{y}'} = K$, then the relation $\tilde{y}' \asymp \tilde{x}'$ holds.

Proof. It is sufficient to show that

$$\limsup_{m\to\infty} \frac{\left|\{n(k)\in K\colon x_{n(k)}\neq y_{n(k)},\, n(k)\le m\}\right|}{\left|K(m)\right|}=0. \tag{3.14}$$

Since the inclusion

$$\left\{n(k)\in K\colon x_{n(k)}\neq y_{n(k)},\, n(k)\le m\right\}\subseteq\{n\in\mathbb{N}\colon x_n\neq y_n,\, n\le m\}$$

holds for each $m\in\mathbb{N}$, we have

$$\limsup_{m\to\infty} \frac{\left|\{n(k)\in K\colon x_{n(k)}\neq y_{n(k)},\, n(k)\le m\}\right|}{|K(m)|}\le$$

$$\le\limsup_{m\to\infty}\frac{\left|\{n\in\mathbb{N}\colon x_n\neq y_n,\, n\le m\}\right|}{\left|K(m)\right|}\le\limsup_{m\to\infty}\frac{m}{\left|K(m)\right|}\limsup_{m\to\infty}\frac{\left|\{n\in\mathbb{N}\colon x_n\neq y_n,\, n\le m\}\right|}{m}=$$

$$=\limsup_{m\to\infty}\frac{\left|\{n\in\mathbb{N}\colon x_n\neq y_n,\, n\le m\}\right|}{m}\left(\liminf_{m\to\infty}\frac{\left|K(m)\right|}{m}\right)^{-1}. \tag{3.15}$$

Inequality (3.13) implies that

$$0\le\left(\liminf_{m\to\infty}\frac{\left|K(m)\right|}{m}\right)^{-1}<+\infty. \tag{3.16}$$

Moreover we have

$$\limsup_{m\to\infty}\frac{\left|\{n\in\mathbb{N}\colon x_n\neq y_n,\, n\le m\}\right|}{m}=0$$

because $\tilde{x}\asymp\tilde{y}$. Now (3.14) follows from the last equality, (3.15) and (3.16).

Lemma 3.3 is proved.

***Proof of Theorem* 3.5.** Suppose that

$$\liminf_{n\to\infty}\frac{\left|K_{\tilde{x}'}(n)\right|}{n}>0. \tag{3.17}$$

Let $\tilde{y}\in\tilde{X}$ and let $\tilde{y}'$ be a subsequence of $\tilde{y}$ such that conditions (i) and (iii) of Theorem 3.3 hold. Then we have $K_{\tilde{x}'}=K_{\tilde{y}'}$ and $\tilde{x}\asymp\tilde{y}$. It follows from (3.17) and Lemma 3.3 that $\tilde{x}'\asymp\tilde{y}'$. Moreover, applying Theorem 3.1, we see that $\tilde{x}'$ is d-statistical convergent to a. Since $\tilde{x}'\asymp\tilde{y}'$, Lemma 3.2 shows that $\tilde{y}'$ is also d-statistical convergent to a, contrary to condition (iii) of Theorem 3.3. Hence equality (3.12) holds.

Theorem 3.5 is proved.

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