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E. Nadaraya, P. Babilua (Tbilisi State Univ., Georgia),
G. Sokhadze (A. Tsereteli Kutaisi State Univ., Georgia)

ON AN INTEGRAL SQUARE DEVIATION MEASURE WITH THE GENERALIZED WEIGHT OF THE ROSENBLATT – PARZEN PROBABILITY DENSITY ESTIMATOR

ПРО МІРУ ІНТЕГРАЛЬНОГО КВАДРАТИЧНОГО ВІДХИЛЕННЯ ІЗ УЗАГАЛЬНЕНОЮ ВАГОЮ ДЛЯ ОЦІНКИ ЩІЛЬНОСТІ РОЗПОДІЛУ ЙМОВІРНОСТЕЙ РОЗЕНБЛАТТА – ПАРЗЕНА

The limit distribution of an integral square deviation with the weight of “delta-functions” of the Rosenblatt – Parzen probability density estimator is defined. Also, the limit power of the goodness-of-fit test constructed by means of this deviation is investigated.

Встановлено граничний розподіл інтегрального квадратичного відхилення з вагою типу дельта-функцій для оцінки щільності розподілу ймовірностей Розенблатта – Парзена. Також досліджено граничну потужність критерію, побудованого за допомогою цього відхилення.

It is well known that the limit distributions of some global measures of deviation of estimates $f_n(x)$ of a density $f(x)$, for example, the integral quadratic deviation constructed by means of the so-called weight function $W(x)$ not depending on n were studied in the works of P. Bickel and M. Rosenblatt [1], M. Rosenblatt [2], E. Nadaraya [3], P. Hall [4] and others.

In T. Tony Cai and Mark G. Low [5], the theory of obtaining the asymptotic behavior of the mean square error

$$R(f_n, f; W_n) = E \int W_n(x)\big(f_n(x) - f(x)\big)^2\, dx, \qquad (1)$$

is developed, where $W_n(x) = a_n W(a_n(x - \ell_0))$, $\{a_n\}$ is a sequence of positive integers, $W(x) \geq 0$ is a Borel-measurable function and ℓ_0 is some fixed point.

If in (1) we take $W(x) = \dfrac{1}{2}\, I\,(-1 \leq x \leq 1)$, and pass to the limit as $a_n \to \infty$ for fixed n, then, roughly speaking,

$$R(f_n, f; W_n) \simeq E\big(f_n(\ell_0) - f(\ell_0)\big)^2,$$

i.e., we come to the mean square error of the nonparametric estimate of the density $f_n(x)$ at the point ℓ_0.

If however in (1) we take $a_n \equiv 1$ for all n, $\ell_0 = 0$ and assume that $W(x) \geq 0$ is an arbitrary bounded function, then

$$R(f_n, f; W) = E\|f_n - f\|^2_{L_2(W)},$$

i.e., we obtain a usual integral mean square error of the estimate $f_n(x)$. Therefore the value $R(f_n, f; W_n)$ can be regarded as a generalization of the measure of density estimation precision covering a mean square deviation of the estimate of the density

$f_n(x)$ at the fixed point and an integral mean square deviation. Hence it is natural to pose the question on the limit distribution of the value $\|f_n - f\|^2_{L_2(W_n)}$, $W_n(x) = a_n W(a_n(x - \ell_0))$.

Let us give the corresponding result for the case where $f_n(x)$ is the nonparametric estimate of the Rosenblatt – Parsen density distribution and $a_n \to \infty$ as $n \to \infty$. The case $a_n \to a_0 < \infty$ is of no interest since it follows from the results of [1 – 4].

Let $X_1, X_2, \ldots, X_n$ be independent, equally distributed random values having the unknown density function of $f(x)$. Assume that the sought density $f(x) \in L_2(W_n)$ ($W_n(x)$ is a weight function) and consider the ways of empirical approximation of this density when measuring the error value in the metric $L_2(W_n)$ of the following form:

$$f_n(x) = \frac{\lambda_n}{n} \sum_{i=1}^{n} K\big(\lambda_n(x - X_i)\big),$$

where $K(x)$ is a function belonging to the class of functions

$$H = \Bigg\{K\colon\ K(x) \geq 0, \quad \int K(x) = 1, \quad K(-x) = K(x),$$

$$\sup_{x \in (-\infty, \infty)} K(x) < \infty, \quad x^2 K(x) \in L_1(-\infty, \infty)\Bigg\},$$

and $\{\lambda_n\}$ is a sequence of numbers converging to infinity.

Denote by F the set of bounded functions on $(-\infty, \infty)$ having bounded derivatives up to second order inclusive.

In this paper we consider the problem of finding the limit distribution of the functional

$$U_n = \frac{n}{\lambda_n} \int \big(f_n(x) - f(x)\big)^2 W_n(x)\, dx.$$

We also study the properties of the power of the goodness-of-fit test constructed by means of the statistic U_n.

1. Limit distribution of U_n. We will need the following notation:

$$U_n^{(1)} = n \int \big(f_n(x) - Ef_n(x)\big)^2 W_n(x)\, dx, \quad \Delta_n(f) = EU_n^{(1)},$$

$$\alpha_n(x, y) = \lambda_n \Big[K\big(\lambda_n(x - y)\big) - EK\big(\lambda_n(x - X_1)\big)\Big],$$

$$\sigma_n^2(f) = 2 \iint \big(E\alpha_n(u_1, X_1)\alpha_n(u_2, X_1)\big)^2 W_n(u_1) W_n(u_2)\, du_1\, du_2,$$

$$W_n(x) = a_n W\big(a_n(x - \ell_0)\big),$$

$$\eta_{ij}^{(n)} = \frac{2}{n\sigma_n(f)} \int \alpha_n(x, X_i)\alpha_n(x, X_j) W_n(x)\, dx,$$

$$\xi_j^{(n)} = \sum_{i=1}^{j-1} \eta_{ij}^{(n)}, \quad j = 2, \ldots, n,$$

$$\xi_1^{(n)} = 0, \quad \xi_j^{(n)} = 0, \quad j > n,$$

$$Y_k^{(n)} = \sum_{i=1}^{k} \xi_i^{(n)}, \quad \mathcal{F}_k = \sigma(\omega\colon X_1, \ldots, X_k),$$

where $\mathcal{F}_k$ is the σ-algebra generated by random values $X_1, X_2, \ldots, X_k$ and $\mathcal{F}_0 = \{\varnothing, \Omega\}$.

In the sequel, for the sake of simplicity, instead of $\xi_j^{(n)}$, $\eta_{ij}^{(n)}$ and $Y_j^{(n)}$ we will write respectively ξ_j, η_{ij} and Y_j.

Lemma 1. *A stochastic sequence $(Y_j, \mathcal{F}_j)_{j\ge 1}$ is a martingale, while a sequence $(\xi_j, \mathcal{F}_j)_{j\ge 1}$ is a difference-martingale.*

The proof follows from the representation

$$E\big(Y_{j+1} \mid \mathcal{F}_j\big) = E\left(\sum_{i=1}^{j+1} \xi_i \mid \mathcal{F}_j\right) =$$

$$= E\left(\sum_{i=1}^{j} \xi_i \mid \mathcal{F}_j\right) + E\big(\xi_{j+1} \mid \mathcal{F}_j\big) = Y_j \quad \text{a.s.},$$

since $E(\xi_{j+1} \mid \mathcal{F}_j) = 0$ and for all $j \ge 1$ we have $E|Y_j| < \infty$.

Furthermore, since $\xi_{j+1} = Y_{j+1} - Y_j$ and $E(\xi_{j+1} \mid \mathcal{F}_j) = 0$ a.s., $(\xi_j, \mathcal{F}_j)_{j\ge 1}$ is a difference-martingale.

Lemma 2. *Let $K(x) \in H$, $f(x) \in F$, $W(x)$ be bounded and $W(x) \in L_2(R)$. If $\lambda_n \to \infty$, $a_n \to \infty$ and $a_n/\lambda_n \to 0$ as $n \to \infty$, then*

$$(\lambda_n a_n)^{-1} \sigma_n^2(f) \longrightarrow \sigma^2(f) = 2 f^2(\ell_0) \int K_0^2(z)\, dz \int W^2(v)\, dv,$$

where $K_0 = K * K$, $f(\ell_0) \neq 0$.

Proof. We have

$$\sigma_n^2(f) = 2\lambda_n^4 \iint \left[\lambda_n^{-1} \int K(t) K\big(\lambda_n(u_2 - u_1) - t\big) f\left(u_1 - \frac{t}{\lambda_n}\right) dt - \right.$$

$$\left. -\lambda_n^{-2} \int K(t_1) f\left(u_1 - \frac{t_1}{\lambda}\right) t_1 \int K(t_2) f\left(u_2 - \frac{t_2}{\lambda_n}\right) dt_2 \right]^2 \times$$

$$\times a_n^2 W\big(a_n(u_1 - \ell_0)\big) W\big(a_n(u_2 - \ell_0)\big)\, du_1\, du_2. \tag{2}$$

Next performing the change of variables in (2) we obtain

$$\sigma_n^2(f) = I_{n1} + I_{n2} + I_{n3},$$

where

$$I_{n1} = 2\lambda_n a_n^2 \iint \left[\int K(t) K(z - t) f\left(u_1 - \frac{t}{\lambda_n}\right) dt\right]^2 \times$$

$$\times W\big(a_n(u_1 - \ell_0)\big) W\left(a_n\left(u_1 + \frac{z}{\lambda_n} - \ell_0\right)\right) du_1\, dz,$$

$$I_{n2} = -4 a_n^2 \iint \left[\int K(t) K(z - t) f\left(u_1 - \frac{t}{\lambda_n}\right) dt \times\right.$$

$$\times \int K(t_1) f\left(u_1 - \frac{t_1}{\lambda_n}\right) dt_1 \int K(t_2) f\left(u_1 + \frac{z}{\lambda_n} - \frac{t_2}{\lambda_n}\right) dt_2\Bigg]\times$$

$$\times W\big(a_n(u_1 - \ell_0)\big) W\left(a_n\left(u_1 + \frac{z}{\lambda_n} - \ell_0\right)\right) du_1\, dz,$$

$$I_{n3} = 2\lambda_n^{-1} a_n^2 \iint \left[\int K(t_1) f\left(u_1 - \frac{t_1}{\lambda_n}\right) dt_1 \times\right.$$

$$\left.\times \int K(t_2) f\left(u_1 + \frac{z}{\lambda_n} - \frac{t_2}{\lambda_n}\right) dt_2\right]^2 \times$$

$$\times W\left(a_n(u_1 - \ell_0)\right) W\left(a_n\left(u_1 + \frac{z}{\lambda_n} - \ell_0\right)\right) du_1\, dz.$$

It is easy to see that

$$I_{n2} \le 4c_1 a_n^2 \int \left[\int K(t) K(z-t)\, dt \times\right.$$

$$\left.\times \left(\int K(t_1)\, dt_1\right)^2 \int W\left(a_n(u_1 - \ell_0)\right) du_1\right] dz \le$$

$$\le c_2 a_n,$$

$$I_{n3} \le 2c_3 \lambda_n^{-1} a_n^2 \int \left[W\left(a_n(u_1 - \ell_0)\right) \times\right.$$

$$\left.\times \int W\left(a_n\left(u_1 - \frac{z}{\lambda_n} - \ell_0\right)\right) dz\right] du_1 \le c_4.$$

Therefore,

$$\sigma_n^2(f) = I_{n1} + O(a_n) + O(1),$$

and also

$$I_{n1} = 2\lambda_n a_n^2 \iint f^2(u_1) K_0^2(z) W\left(a_n(u_1 - \ell_0)\right) \times$$

$$\times W\left(a_n\left(u_1 + \frac{z}{\lambda_n} - \ell_0\right)\right) du_1\, dz + A_{n1} + A_{n2},$$

$$A_{n1} = 2\lambda_n a_n^2 \iint \left[\int K(t) K(z-t) \left(f\left(u_1 - \frac{t}{\lambda_n}\right) - f(u_1)\right) dt\right]^2 \times$$

$$\times W\left(a_n(u_1 - \ell_0)\right) W\left(a_n\left(u_1 + \frac{z}{\lambda_n} - \ell_0\right)\right) du_1\, dz \le$$

$$\le c_5 \frac{a_n}{\lambda_n} \int t^2 K(t)\, dt = O\left(\frac{a_n}{\lambda_n}\right),$$

$$A_{n2} \le c_6 \lambda_n a_n^2 \iint \left[\int K(t_1) K(z-t_1) \frac{|t_1|}{\lambda_n}\, dt_1 \int K(t_2) K(z-t_2)\, dt_2\right] \times$$

$$\times W\left(a_n(u_1-\ell_0)\right)W\left(a_n\left(u_1+\frac{z}{\lambda_n}-\ell_0\right)\right)du_1\,dz\leq$$

$$\leq c_7a_n^2\int\left[\int|t|K(t)\,dt\int W\left(a_n(u_1-\ell_0)\right)du_1\right]\leq c_8a_n.$$

Thus

$$(\lambda_na_n)^{-1}\sigma_n^2(f)=2\iint f^2\left(\ell_0+\frac{v}{a_n}\right)K_0^2(z)W^2(v)\,dv\,dz+$$

$$+A_{n3}+O\left(\frac{1}{\lambda_n}\right)+O\left(\frac{1}{\lambda_na_n}\right), \tag{3}$$

where

$$A_{n3}=2\iint f^2\left(\ell_0+\frac{v}{a_n}\right)K_0^2(z)W^2(v)\left[W\left(v+\frac{a_n}{\lambda_n}\,z\right)-W(v)\right]dv\,dz,$$

and also

$$|A_{n3}|\leq 2\iint f^2\left(\ell_0+\frac{v}{a_n}\right)K_0^2(z)W(v)\left|W\left(v+\frac{a_n}{\lambda_n}\,z\right)-W(v)\right|dz\,dv\leq$$

$$\leq c_9\int K_0^2(z)\omega_1\left(\frac{a_n}{\lambda_n}\,z\right)dz. \tag{4}$$

The expression $\omega_1(h)=\int\big|W(v+h)-W(v)\big|\,dv$ is the L_1-modulus of continuity of the function $W(x)$. It is evidently bounded as a function of h since $\omega_1(h)\leq 2\|W\|_{L_1}$. Moreover, $\omega_1(h)\to 0$ as $h\to 0$. Therefore, by the Lebesque theorem on majorized convergence, the integral in the right-hand part of (4) converges to zero as $n\to\infty$.

So, using this fact and (3) we obtain

$$(\lambda_na_n)^{-1}\sigma_n^2(f)\longrightarrow\sigma^2(f).$$

The lemma is proved.

Theorem 1. *Let $K(x)\in H$, $f(x)\in F$, $W(x)$ be bounded and $W(x)\in L_1(-\infty,\infty)$. If $\lambda_n\to\infty$, $a_n\to\infty$, $a_n/\lambda_n\to 0$ and $\dfrac{\lambda_na_n^2}{n}\to 0$ as $n\to\infty$, then*

$$\frac{U_n^{(1)}-\Delta_n(f)}{\sigma_n(f)}\xrightarrow{d}N(0,1),$$

where d denoted convergence in distribution, and $N(a,\sigma)$ a random value having a normal distribution with mean a and dispersion σ^2.

Proof. We have

$$\sigma_n^{-1}(U_n^{(1)}-\Delta_n)=\sqrt{\frac{n-1}{n}}\,H_n^{(1)}+H_n^{(2)},$$

where

$$H_n^{(1)}=\sum_{j=1}^n\xi_j,$$

$$H_n^{(2)} = \frac{1}{n\sigma_n} \sum_{j=1}^{n} \Bigg(\int \alpha_n^2(x, X_j) a_n W\left(a_n(x - \ell_0)\right) dx -$$

$$-E \int \alpha_n^2(x - X_j) a_n W\left(a_n(x - \ell_0)\right) dx \Bigg),$$

$$\sigma_n^2 \equiv \sigma_n^2(f).$$

We will first establish the convergence of $H_n^{(2)}$ to zero in probability.

Indeed,

$$\operatorname{Var} H_n^{(2)} \le c_{10} n^{-1} \sigma_n^{-2} \lambda_n^4 E \Bigg[\int K^2\left(\lambda_n(x - X_1)\right) a_n W\left(a_n(x - \ell_0)\right) dx +$$

$$+ \int \left(EK\left(\lambda_n(x - X_1)\right)\right)^2 a_n W\left(a_n(x - \ell_0)\right) dx \Bigg]^2 \le$$

$$\le c_{11} n^{-1} \sigma_n^{-2} \lambda_n^4 \Bigg\{ E \Bigg[\int K^2\left(\lambda_n(x - X_1)\right) a_n W\left(a_n(x - \ell_0)\right) dx \Bigg]^2 +$$

$$+ \Bigg[\int \left(EK\left(\lambda_n(x - X_1)\right)\right)^2 a_n W\left(a_n(x - \ell_0)\right) dx \Bigg]^2 \Bigg\} =$$

$$= I_n^{(1)} + I_n^{(2)},$$

and also

$$I_n^{(1)} \le c_{12} \lambda_n^4 n^{-1} \sigma_n^{-2} \lambda_n^{-2} a_n^2 = c_{12} \left(\frac{\lambda_n a_n}{\sigma_n^2} \right) \frac{\lambda_n a_n}{n} \longrightarrow 0,$$

$$I_n^{(2)} \le c_{13} n^{-1} \sigma_n^{-2} \longrightarrow 0.$$

Therefore,

$$\operatorname{Var} H_n^{(2)} = O\left(\frac{\lambda_n a_n}{n} \right) + O\left(\frac{1}{n\sigma_n^2} \right).$$

Hence $H_n^{(2)} \xrightarrow{P} 0$ as $n \to \infty$ (here and in the sequel the letter p above the arrow denote convergence in probability).

To prove the assertion of Theorem 1 we need to show that $H_n^{(1)} \xrightarrow{d} N(0, 1)$. To this end, we use Theorem 4 from [6, p. 580] which contains the conditions of the central limit theorem for sequences that form a difference-martingale. Let us show that our sequence $\{\xi_k, \mathcal{F}_k\}$ satisfies these conditions. Note that $\sum_{j=1}^{n} E\xi_j^2 = 1$ since, as can be easily verified, $E\xi_j^2 = 2(j - 1)\big[n(n - 1)\big]^{-1}$. Asymptotic normality takes place if for every $\varepsilon \in (0, 1]$ and $n \to \infty$

$$\sum_{k=1}^{n} E\Big[\xi_k^2 I\left(|\xi_k| \ge \varepsilon\right) \Big| \mathcal{F}_{k-1}\Big] \xrightarrow{P} 0$$

(the Lindeberg condition) and

$$V_n^2 = \sum_{k=1}^{n} E\left(\xi_k^2 \mid \mathcal{F}_{k-1}\right) \xrightarrow{P} 1,$$

i.e., then

$$H_n^{(1)} = \sum_{k=1}^{n} \xi_k \xrightarrow{d} N(0,1).$$

In the first place we verify that

$$V_n^2 = \sum_{k=1}^{n} E\left(\xi_k^2 \mid \mathcal{F}_{k-1}\right) \xrightarrow{P} 1 \quad \text{as} \quad n \to \infty. \tag{5}$$

Taking the definition of ξ_j into account, we represent V_n^2 in the form

$$V_n^2 = \sum_{j=2}^{n} E\left(\xi_j^2 \mid X_1, \ldots, X_{j-1}\right) =$$

$$= \sum_{j=2}^{n} E\left[\left(\sum_{i=1}^{j-1} \eta_{ij}\right)^2 \middle| X_1, \ldots, X_{j-1}\right] =$$

$$= \sum_{j=2}^{n} E\left(\sum_{i=1}^{j-1} \eta_{ij}^2 \middle| X_1, \ldots, X_{j-1}\right) + 2\sum_{i<\ell} E\left(\eta_{ij}\eta_{\ell j} \middle| X_1, \ldots, X_{j-1}\right) =$$

$$= \sum_{j=2}^{n} E\left(\sum_{i=1}^{j-1} \eta_{ij}^2 \middle| X_1, \ldots, X_{j-1}\right) +$$

$$+2\sum_{j=3}^{n} E\left(\sum_{i=1}^{j-2} \sum_{\ell=i+1}^{j-1} \eta_{ij}\eta_{\ell j} \middle| X_1, \ldots, X_{j-1}\right) =$$

$$= V_{n1} + V_{n2}.$$

Let us show that $V_{n1} \xrightarrow{P} 1$ and $V_{n2} \xrightarrow{P} 0$ as $n \to \infty$.

Denote

$$\Phi_n(x,y) = EK\left(\lambda_n(x - X_1)\right) K\left(\lambda_n(y - X_1)\right) -$$

$$-EK\left(\lambda_n(x - X_1)\right) EK\left(\lambda_n(y - X_1)\right),$$

$$\varepsilon_i = \lambda_n^{-2} \iint \Big[\alpha_n(x, X_i)\alpha_n(y, X_i)\Phi_n(x,y)W_n(x)W_n(y)\Big]\, dx\, dy,$$

$$\overline{\sigma}_n = \iint \Phi_n^2(x,y)W_n(x)W_n(y)\, dx\, dy,$$

$$Z_j = \sum_{i=1}^{j-1} (\varepsilon_i - \overline{\sigma}_n).$$

Further, the definition of η_{ij} implies that

$$V_{n1} = \frac{4}{n^2\sigma_n^2} \sum_{j=2}^{n} E\Bigg(\sum_{i=1}^{j-1} \iint \alpha_n(x, X_i)\alpha_n(y, X_i)\alpha_n(x, X_j)\alpha_n(y, X_j)\times$$

$$\times W_n(x)W_n(y)\,dx\,dy \,\Bigg|\, X_1, \ldots, X_{j-1} \Bigg) =$$

$$= \frac{4}{n^2\sigma_n^2} \sum_{j=2}^{n}\sum_{i=1}^{j-1} \iint \alpha_n(x, X_i)\alpha_n(y, X_i)E\alpha_n(x, X_j)\alpha_n(y, X_j)W_n(x)W_n(y)\,dx\,dy$$

and since

$$E\alpha_n(x, X_i)\alpha_n(y, X_i) = \lambda_n^2 \Big(E\left(K\left(\lambda_n(x - X_1)\right) K\left(\lambda_n(y - X_1)\right)\right) -$$

$$-EK\left(\lambda_n(x - X_1)\right) EK\left(\lambda_n(y - X_1)\right)\Big)$$

we have

$$V_{n1} = \frac{4\lambda_n^2}{n^2\sigma_n^2} \sum_{j=2}^{n}\sum_{i=1}^{j-1} \iint \alpha_n(x, X_i)\alpha_n(y, X_i)\Phi_n(x, y)W_n(x)W_n(y)\,dx\,dy.$$

Therefore,

$$\operatorname{Var} V_{n1} = E(V_{n1} - EV_{n1})^2 = \frac{16\lambda_n^8}{n^4\sigma_n^4}\, E\left[\sum_{j=2}^{n}\sum_{i=1}^{j-1} (\varepsilon_i - \overline{\sigma}_n) \right]^2 =$$

$$= D_n \sum_{j=2}^{n} E\left(\sum_{i=1}^{j-1} (\varepsilon_i - \overline{\sigma}_n) \right)^2 + 2D_n \sum_{i=2}^{n-1} E\left(Z_i(Z_{i+1} + \ldots + Z_n)\right),$$

where

$$D_n = \frac{16\lambda_n^8}{n^4\sigma_n^4}.$$

It is obvious that

$$Z_{i+1} = Z_i + (\varepsilon_i - \overline{\sigma}_n),$$

$$Z_{i+2} = Z_i + (\varepsilon_i - \overline{\sigma}_n) + (\varepsilon_{i+1} - \overline{\sigma}_n),$$

....

$$Z_n = Z_i + (\varepsilon_i - \overline{\sigma}_n) + \ldots + (\varepsilon_{n-1} - \overline{\sigma}_n).$$

Hence

$$E(V_{n1} - EV_{n1})^2 =$$

$$= D_n \sum_{j=2}^{n} E\left(\sum_{i=1}^{j-1} (\varepsilon_i - \overline{\sigma}_n) \right)^2 + 2D_n \sum_{i=2}^{n-1} EZ_i^2(n - i) =$$

$$= D_n \sum_{j=2}^{n} (j-1) E(\varepsilon_1 - \overline{\sigma}_n)^2 + 2D_n \sum_{i=2}^{n-1} E Z_i^2 (n-i) = B_{n1} + B_{n2}.$$

Let us estimate B_{n1} and B_{n2}.

We have

$$B_{n1} \leq c_{14} \frac{\lambda_n^8}{n^2 \sigma_n^4} E(\varepsilon_1 - \overline{\sigma}_n)^2. \tag{6}$$

But

$$E(\varepsilon_1 - \overline{\sigma}_n)^2 = \int \Big\{ \iint \Big[K(\lambda_n(x-t)) - EK(\lambda_n(x - X_1)) \Big] \times$$
$$\times \Big[K(\lambda_n(y-t)) - EK(\lambda_n(y - X_1)) \Big] \times$$
$$\times \Phi_n(x,y) W_n(x) W_n(y)\, dx\, dy \Big\}^2 f(t)\, dt =$$
$$= c_{15} \frac{\lambda_n^8}{n^2 \sigma_n^4} \left(A_{n1} + A_{n2} + A_{n3} + A_{n4} \right), \tag{7}$$

where

$$A_{n1} = \int \Big[\iint K(\lambda_n(x-t))\, K(\lambda_n(y-t))\, \Phi_n(x,y) \times$$
$$\times W_n(x) W_n(y)\, dx\, dy \Big]^2 f(t)\, dt,$$

$$A_{n2} = \int \Big[\iint K(\lambda_n(x-t))\, EK(\lambda_n(y-X_1))\, \Phi_n(x,y) \times$$
$$\times W_n(x) W_n(y)\, dx\, dy \Big]^2 f(t)\, dt,$$

$$A_{n3} = \int \Big[\iint K(\lambda_n(y-t))\, EK(\lambda_n(x-X_1))\, \Phi_n(x,y) \times$$
$$\times W_n(x) W_n(y)\, dx\, dy \Big]^2 f(t)\, dt,$$

$$A_{n4} = \int \Big[\iint EK(\lambda_n(x-X_1))\, EK(\lambda_n(y-X_1))\, \Phi_n(x,y) \times$$
$$\times W_n(x) W_n(y)\, dx\, dy \Big]^2 f(t)\, dt.$$

Since $|\Phi_n(x,y)| \leq c_{16} \lambda_n^{-1}$, it can be easily established that

$$A_{n1} \leq c_{17} \frac{a_n^4}{\lambda_n^6}, \qquad A_{n2} \leq c_{18} \frac{a_n^2}{\lambda_n^6},$$
$$A_{n3} \leq c_{18} \frac{a_n^2}{\lambda_n^6}, \qquad A_{n4} \leq c_{19} \lambda_n^{-6}. \tag{8}$$

Using (7) and (8) we obtain

$$E(\varepsilon_1 - \overline{\sigma}_n)^2 \leq c_{20} \left(\frac{a_n^4}{\lambda_n^6} + \frac{a_n^2}{\lambda_n^6} + \frac{1}{\lambda_n^6} \right). \tag{9}$$

This and (6) imply

$$B_{n1} \leq c_{21} \left(\frac{\lambda_n a_n}{\sigma_n^2} \right)^2 \frac{a_n^2}{n^2} \longrightarrow 0. \tag{10}$$

Further, it is not difficult to see that

$$B_{n2} = \frac{32\lambda_n^8}{n^4\sigma_n^4} \sum_{i=2}^{n-1} EZ_i^2(n-i) = \frac{32\lambda_n^8}{n^4\sigma_n^4} \sum_{i=2}^{n-1} (n-i)(i-1)E(\varepsilon_1 - \overline{\sigma}_n)^2.$$

This and (9) imply that

$$B_{n2} \leq c_{22} \frac{\lambda_n^2 a_n^4}{n\sigma_n^4} = c_{22} \left(\frac{\lambda_n a_n}{\sigma_n^2} \right)^2 \frac{a_n^2}{n} \longrightarrow 0.$$

Therefore,

$$\operatorname{Var} V_{n1} = E(V_{n1} - EV_{n1})^2 \longrightarrow 0.$$

On the other hand,

$$EV_{n1} = \frac{4\lambda_n^2}{n^2\sigma_n^2} \sum_{j=2}^{n} \sum_{i=1}^{j-1} \iint E\alpha_n(x, X_i)\alpha_n(y, X_i)\Phi_n(x, y)W_n(x)W_n(y)\, dx\, dy =$$

$$= \frac{4}{n^2\sigma_n^2} \sum_{j=2}^{n} \sum_{i=1}^{j-1} \iint \left(E\alpha_n(x, X_1)\alpha_n(y, X_1)\right)^2 W_n(x)W_n(y)\, dx\, dy =$$

$$= \frac{n^2 - n}{n^2} = 1 - \frac{1}{n} \longrightarrow 1.$$

Therefore,

$$V_{n1} \xrightarrow{P} 1.$$

Now let consider V_{n2} and show that $V_{n2} \xrightarrow{P} 0$. Taking the inequality

$$E\left(\sum_{i=1}^{m} Z_i \right)^2 \leq \left(\sum_{i=1}^{n} (EZ_i^2)^{1/2} \right)^2,$$

into account, we obtain

$$EV_{n2}^2 = D_n E\Bigg(\sum_{j=3}^{n} \sum_{i=1}^{j-2} \sum_{\ell=i+1}^{j-1} \iint \overline{\alpha}_n(x, X_i)\, \overline{\alpha}_n(y, X_\ell)\Phi_n(x, y) \times$$

$$\times W_n(x)W_n(y)\, dx\, dy \Bigg)^2 =$$

$$= D_n E\left(\sum_{j=3}^{n} \iint \sum_{i=1}^{j-2} \overline{\alpha}_n(x, X_i) g_i(y)\Phi_n(x, y)W_n(x)W_n(y)\, dx\, dy \right)^2 \leq$$

$$\leq D_n\Bigg[\sum_{j=3}^{n}\Bigg\{E\Bigg(\iint\sum_{i=1}^{j-1}\overline{\alpha}_n(x,X_i)g_i(y)\Phi_n(x,y)\times$$

$$\times W_n(x)W_n(y)\,dx\,dy\Bigg)^2\Bigg\}^{1/2}\Bigg]^2, \tag{11}$$

where

$$D_n=\frac{16\lambda_n^8}{n^4\sigma_n^4},$$

$$\overline{\alpha}_n(x,X_i)=K\left(\lambda_n(x-X_i)\right)-EK\left(\lambda_n(x-X_1)\right),$$

$$g_i(y)=\sum_{\ell=i+1}^{j-1}\overline{\alpha}_n(y,X_\ell).$$

Since

$$E\,\overline{\alpha}_n(x,X_i)g_i(y)\,\overline{\alpha}_n(y,X_r)g_r(y)=0\quad\text{as}\quad i<r,$$

(11) takes the form

$$EV_{n2}^2=$$

$$=D_n\Bigg[\sum_{j=3}^{n}\Bigg\{\sum_{i=1}^{j-2}E\Bigg(\iint\overline{\alpha}_n(x,X_i)g_i(y)\Phi_n(x,y)W_n(x)W_n(y)\,dx\,dy\Bigg)^2\Bigg\}^{1/2}\Bigg]^2. \tag{12}$$

Next, elementary calculations show that

$$E\sum_{i=1}^{j-2}\Bigg(\iint\overline{\alpha}_n(x,X_i)g_i(y)\Phi_n(x,y)W_n(x)W_n(y)\,dx\,dy\Bigg)^2=$$

$$=\sum_{i=1}^{j-2}E\iiiint\overline{\alpha}_n(x_1,X_i)\,\overline{\alpha}_n(x_2,X_i)\times$$

$$\times\sum_{\ell_1=i+1}^{j-1}\sum_{\ell_2=i+1}^{j-1}\overline{\alpha}_n(y_1,X_{\ell_1})\,\overline{\alpha}_n(y_2,X_{\ell_2})\Phi_n(x_1,y_1)\Phi_n(x_2,y_2)\times$$

$$\times W_n(x_1)W_n(x_2)W_n(y_1)W_n(y_2)\,dx_1\,dx_2\,dy_1\,dy_2=$$

$$=\sum_{i=1}^{j-2}\iiiint E\,\overline{\alpha}_n(x_1,X_i)\,\overline{\alpha}_n(x_2,X_i)\times$$

$$\times\sum_{\ell_1=i+1}^{j-1}E\,\overline{\alpha}_n(y_1,X_\ell)\,\overline{\alpha}_n(y_2,X_\ell)\Phi_n(x_1,y_1)\Phi_n(x_2,y_2)\times$$

$$\times W_n(x_1)W_n(x_2)W_n(y_1)W_n(y_2)\,dx_1\,dx_2\,dy_1\,dy_2=$$

$$= O\biggl(j^2 \iiiint \Bigl| E\,\overline{\alpha}_n(x_1, X_1)\,\overline{\alpha}_n(x_2, X_1) E\,\overline{\alpha}_n(y_1, X_1)\,\overline{\alpha}_n(y_2, X_1) \Bigr| \times$$

$$\times W_n(x_1) W_n(x_2) W_n(y_1) W_n(y_2)\, dx_1\, dx_2\, dy_1\, dy_2 \biggr). \tag{13}$$

Recalling the definition of $\overline{\alpha}_n(x, y)$ and performing the change of variables, we obtain

$$\iiiint \Bigl| E\,\overline{\alpha}_n(x_1, X_1)\,\overline{\alpha}_n(x_2, X_1) E\,\overline{\alpha}_n(y_1, X_1)\,\overline{\alpha}_n(y_2, X_1) \Bigr| \times$$

$$\times W_n(x_1) W_n(x_2) W_n(y_1) W_n(y_2)\, dx_1\, dx_2\, dy_1\, dy_2 \leq$$

$$\leq c_{23} \lambda_n^{-7} \iiiint \Biggl[\int K(w_1) K(z_1 - w_1) f\left(x_1 - \frac{w_1}{\lambda_n} \right) dw_1 +$$

$$+ \lambda_n E K\left(\lambda_n (x_1 - X_1) \right) E K\left(\lambda_n (x_1 - X_1) + z_1 \right) \Biggr] \times$$

$$\times \Biggl[\int K(w_2) K(z_2 - z_3 - w_2) f\left(x_1 + \frac{z_2}{\lambda_n} - \frac{w_2}{\lambda_n} \right) dw_2 +$$

$$+ \lambda_n E K\left(\lambda_n (x_1 - X_1) + z_2 \right) E K\left(\lambda_n (x_1 - X_1) + z_3 \right) \Biggr] \times$$

$$\times \Biggl[\int K(w_3) K(z_2 - w_3) f\left(x_1 - \frac{w_3}{\lambda_n} \right) dw_3 +$$

$$+ \lambda_n E K\left(\lambda_n (x_1 - X_1) \right) E K\left(\lambda_n (x_1 - X_1) + z_2 \right) \Biggr] \times$$

$$\times \Biggl[\int K(w_4) K(z_3 - z_1 - w_4) f\left(x_1 + \frac{z_1}{\lambda_n} - \frac{w_4}{\lambda_n} \right) dw_4 +$$

$$+ \lambda_n E K\left(\lambda_n (x_1 - X_1) + z_1 \right) E K\left(\lambda_n (x_1 - X_1) + z_3 \right) \Biggr] \times$$

$$\times a_n^4 W\left(a_n (x_1 - \ell_0) \right) W\left(\left(x_1 + \frac{z_1}{\lambda_n} - \ell_0 \right) a_n \right) \times$$

$$\times W\left(a_n \left(x_1 + \frac{z_2}{\lambda_n} - \ell_0 \right) \right) W\left(a_n \left(x_1 + \frac{z_3}{\lambda_n} - \ell_0 \right) \right) dx_1\, dz_1\, dz_2\, dz_3 \leq$$

$$\leq c_{24} \lambda_n^{-7} a_n^3, \tag{14}$$

since the integrals contained in (12) are majorised by the value a_n^3.

Thus, by virtue of (13) and (14), from (12) it follows that

$$E V_{n2}^2 \leq c_{25} \left(\frac{\lambda_n a_n}{\sigma_n^2} \right)^2 \frac{a_n}{\lambda_n} \longrightarrow 0.$$

Therefore,

$$\sum_{j=1}^{n} E\left(\xi_j^2 \mid \mathcal{F}_{k-1} \right) = V_{n1} + V_{n2} \xrightarrow{P} 0. \tag{15}$$

Let us now proceed to establishing the validity of the Lindeberg condition

$$\sum_{k=1}^{n} E\Big[\xi_k^2 I\left(|\xi_k| \geq \varepsilon\right) \mid \mathcal{F}_{k-1}\Big] \xrightarrow{P} 0. \tag{16}$$

For (16) to be valid it suffices to show that

$$\sum_{j=1}^{n} E\xi_j^4 \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

Indeed,

$$P\left\{\sum_{k=1}^{n} E\Big[\xi_k^2 I\left(|\xi_k| \geq \varepsilon\right) \Big| \mathcal{F}_{k-1}\Big] \geq \delta\right\} \leq$$

$$\leq \delta^{-1} \sum_{j=1}^{n} E\left(E\Big[\xi_j^2 \left(I(\xi_j \geq \varepsilon)\right) \Big| \mathcal{F}_{k-1}\Big]\right) = \delta^{-1} \sum_{j=1}^{n} E\Big[\xi_j^2 \left(I(\xi_j \geq \varepsilon)\right)\Big] \leq$$

$$\leq \delta^{-1}\varepsilon^{-2} \sum_{j=1}^{n} E\left(\xi_j^4 I(\xi_j \geq \varepsilon)\right) \leq \delta^{-1}\varepsilon^{-2} \sum_{j=1}^{4} E\xi_j^4.$$

We will prove (17). By the definitions of η_{ik} and ξ_k, we obtain

$$\sum_{k=1}^{n} E\xi_k^4 = \frac{16}{n^4\sigma_n^4} \left(M_n^{(1)} + M_n^{(2)}\right), \tag{18}$$

where

$$M_n^{(1)} = \sum_{k=1}^{n} (k-1) \int \left[E \prod_{i=1}^{4} \alpha_n(x_i, X_1)\right]^4 \prod_{i=1}^{4} W_n(x_i)\, dx,$$

$$M_n^{(2)} = 3\sum_{k=1}^{n} (k-1)(k-2) \int \Bigg[E \prod_{i=0}^{1} \alpha_n(x_{i+1}, X_1)\alpha_n(x_{i+3}, X_2) \times$$

$$\times E \prod_{i=1}^{4} \alpha_n(x_i, X_1) W_n(x_i)\Bigg]\, dx, \quad dx = dx_1 \ldots dx_4.$$

Let us estimate $M_n^{(1)}$ and $M_n^{(2)}$. We have

$$M_n^{(1)} =$$

$$= \sum_{k=1}^{n} (k-1) \iint \left(\int \alpha_n(x_1, u)\alpha_n(x_1, v) W_n(x_1)\, dx_1\right)^4 f(u) f(v)\, du\, dv \leq$$

$$\leq c_{26} n^2 \iint \Bigg[\delta^4(u, v) + \left(\int f(t)\delta(u, t)\, dt\right)^4 +$$

$$+ \left(\int f(t)\delta(v, t)\, dt\right)^4 + \left(\iint \delta(x, y) f(x) f(y)\, dx\, dy\right)^4\Bigg] f(u) f(v)\, du\, dv, \tag{19}$$

where

$$\delta(x,y)=\lambda_n^2\int K\left(\lambda_n(x-u)\right)K\left(\lambda_n(x-v)\right)W_n(x)\,dx.$$

Since

$$\sup_v\int\delta^s(u,v)f(u)\,du\le c_{27}a_n^s\lambda_n^s\sup_v\int K_0^s\left(\lambda_n(u-v)\right)du\le c_{28}a_n^s\lambda_n^{s-1},$$

$$s=2,4,$$

formula (19) implies

$$M_n^{(1)}\le c_{29}n^2a_n^4\lambda_n^3, \tag{20}$$

and also

$$M_n^{(2)}=3\sum_{k=1}^n(k-1)(k-2)\iiiint\left(\int\alpha_n(x_1,u)\alpha_n(x_1,t)W_n(x_1)\,dx_1\right)^2\times$$

$$\times\left(\int\alpha_n(x_2,v)\alpha_n(x_2,t)W_n(x_2)\,dx_2\right)^2 f(u)f(v)f(t)\,du\,dv\,dt\le$$

$$\le 3\sum_{k=1}^n(k-1)(k-2)\iiint\left(\delta_n^2(u,t)+O(a_n^4\lambda_n)\right)\times$$

$$\times\left(\delta_n^2(v,t)+O(\lambda_na_n^4)\right)f(u)f(v)f(t)\,du\,dv\,dt\le$$

$$\le c_{30}n^3a_n^4\lambda_n^3. \tag{21}$$

From (18), (20) and (21) it follows that

$$\sum_{k=1}^n E\xi_k^4\le c_{31}\,\frac{\lambda_n^3a_n^4}{n\sigma_n^4}=c_{31}\left(\frac{a_n\lambda_n}{\sigma_n^2}\right)^2\frac{a_n^2\lambda_n}{n}.$$

Therefore,

$$\lim_{n\to\infty}\sum_{k=1}^n E\xi_k^4=0.$$

The theorem is proved.

Theorem 2. *Let $K(x)\in H$, $f(x)\in F$, $W(x)$ be bounded and $W(x)\in L_1(-\infty,\infty)$. If $\lambda_n\to\infty$, $a_n\to\infty$, $\dfrac{a_n}{\lambda_n}\to 0$ and $\dfrac{\lambda_na_n^2}{n}\to 0$ as $n\to\infty$, then*

$$(\lambda_na_n)^{-1/2}\sigma^{-1}(f)\left(U_n^{(1)}-\Delta_n(f)\right)\xrightarrow{d}N(0,1),$$

where

$$\sigma^2(f)=2f^2(\ell_0)\int K_0^2(z)\,dz\int W^2(v)\,dv,\quad f(\ell_0)\ne 0.$$

Proof. Follows from Lemma 2 and Theorem 1.

Theorem 3. *Let $K(x)\in H$, $f(x)\in F$, $W(x)$ be bounded, $W(-x)=W(x)$, $x\in R$ and $x^2W(x)\in L_1(R)$. If $\lambda_n\to\infty$, $a_n\to\infty$, $\dfrac{a_n}{\lambda_n}\to 0$, $\dfrac{\lambda_na_n^2}{n}\to 0$ and $\lambda_na_n^{-5}\to 0$, then*

$$\left(\frac{\lambda_n}{a_n}\right)^{1/2}\sigma^{-1}(f)\left(U_n^{(2)}-\Delta(f)\right)\xrightarrow{d}N(0,1),$$

where

$$\Delta(f)=f(\ell_0)\int K^2(u)\,du\int W(x)\,dx,\quad U_n^{(2)}=\lambda_n^{-1}U_n^{(1)}.$$

Proof. We have

$$\Delta_n(f)=n\int E\left(f_n(x)-Ef_n(x)\right)^2W_n(x)\,dx=$$

$$=\lambda_n\iint K^2(u)f\left(x-\frac{u}{\lambda_n}\right)a_nW\left(a_n(x-\ell_0)\right)du\,dx-$$

$$-\int\left(\int K(v)f\left(x-\frac{v}{\lambda_n}\right)dv\right)^2a_nW\left(a_n(x-\ell_0)\right)dx. \tag{22}$$

Since

$$\lambda_n\iint K^2(u)f\left(x-\frac{u}{\lambda_n}\right)a_nW\left(a_n(x-\ell_0)\right)du\,dx=$$

$$=\lambda_n\int K^2(u)\,du\int f\left(\ell_0+\frac{v}{a_n}\right)W(v)\,dv+O(1)=$$

$$=f(\ell_0)\lambda_n\int K^2(u)\,du\int W(u)\,du+O\left(\frac{\lambda_n}{a_n^2}\right)+O(1)$$

and

$$\int\left(\int K(v)f\left(x-\frac{v}{\lambda_n}\right)dv\right)^2a_nW\left(a_n(x-\ell_0)\right)dx=O(1),$$

from these formulas and (22) we obtain

$$\Delta_n(f)=\lambda_n\left[f(\ell_0)\int K^2(u)\,du\int W(v)\,dv+O\left(\frac{1}{a_n^2}\right)+O\left(\frac{1}{\lambda_n}\right)\right]=$$

$$=\lambda_n\Big[\Delta(f)+O(a_n^{-2})+O(\lambda_n^{-1})\Big].$$

Therefore,

$$(\lambda_na_n)^{-1/2}\sigma^{-1}\left(U_n^{(1)}-\Delta_n\right)-\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}\left(U_n^{(2)}-\Delta\right)=$$

$$=O\left(\frac{\sqrt{\lambda_n}}{\sqrt{a_n}\,a_n^2}\right)+O\left(\left(\frac{1}{\lambda_na_n}\right)^{1/2}\right).$$

Since the right-hand part of the latter equality tends to zero by virtue of the condition $\lambda_n/a_n^{-5}\to 0$.

The theorem is proved.

The case where in $U_n^{(2)}$ $Ef_n(x)$ is replaced by $f(x)$ is more natural for applications.

Theorem 4. *Let $K(x)$, $f(x)$ and $W(x)$ satisfy the conditions of Theorem 3. If*

$$\lambda_n \to \infty, \quad a_n \to \infty, \quad \frac{a_n}{\lambda_n} \to 0, \quad \frac{\lambda_n a_n^2}{n} \to 0,$$

$$\frac{\lambda_n}{a_n^5} \to 0, \quad \sqrt{n a_n}\, \lambda_n^{-5/2} \to 0 \quad \textit{and} \quad \frac{n}{\sqrt{a_n}}\, \lambda_n^{-9/2} \to 0,$$

then

$$\left(\frac{\lambda_n}{a_n}\right)^{1/2} \sigma^{-1}(f)\, (U_n - \Delta(f)) \xrightarrow{d} N(0,1),$$

$$U_n = \frac{n}{\lambda_n} \int (f_n(x) - f(x))^2\, W_n(x)\, dx.$$

Proof. We have

$$\sqrt{\frac{\lambda_n}{a_n}} \left(U_n - U_n^{(2)}\right) = \sqrt{\frac{\lambda_n}{a_n}}\, \Theta_n + \sqrt{\frac{\lambda_n}{a_n}}\, R_n,$$

where

$$\Theta_n = \frac{n}{\lambda_n} \int (Ef_n(x) - f(x))^2\, W_n(x)\, dx,$$

$$R_n = 2\, \frac{n}{\lambda_n} \int (f_n(x) - Ef_n(x))\, (Ef_n(x) - f(x))\, W_n(x)\, dx.$$

Let us estimate $\sqrt{\lambda_n / a_n}\, E|R_n|$. Since

$$\operatorname{cov}(f_n(x), f_n(y)) = n^{-1}\lambda_n^2 \Big\{ \int K(\lambda_n(x-u))\, K(\lambda_n(y-u))\, f(u)\, du -$$

$$- \int K(\lambda_n(x-u))\, f(u)\, du \int K(\lambda_n(y-u))\, f(u)\, du \Big\},$$

we obtain

$$\sqrt{\frac{\lambda_n}{a_n}}\, E|R_n| \le$$

$$\le 2\, \frac{n}{\sqrt{a_n \lambda_n}} \left\{ \frac{\lambda_n^2}{n}\, E\left[\int K(\lambda_n(x - X_1))\, (Ef_n(x) - f(x))\, W_n(x)\, dx \right]^2 - \right.$$

$$\left. - \frac{\lambda_n^2}{n} \left[\int EK(\lambda_n(x - X_1))\, (Ef_n(x) - f(x))\, W_n(x)\, dx \right]^2 \right\}^{1/2} \le$$

$$\le 2\, \frac{n}{\sqrt{a_n \lambda_n}} \left\{ \frac{\lambda_n^2}{n} \int f(u)\, du \left[\int K(\lambda_n(x-u))\, (Ef_n(x) - f(x))\, W_n(x)\, dx \right]^2 \right\}^{1/2}. \tag{23}$$

Further, since

$$Ef_n(x) - f(x) = O(\lambda_n^{-2}),$$

uniformly with respect to $x \in R = (-\infty, \infty)$, from (23) we obtain

$$\sqrt{\frac{\lambda_n}{a_n}}\, E|R_n| \le c_{32}\sqrt{na_n}\,\lambda_n^{-5/2} \longrightarrow 0,$$

and also

$$\sqrt{\frac{\lambda_n}{a_n}}\,\Theta_n \le c_{33}\,\frac{n}{\sqrt{a_n}}\,\lambda_n^{-9/2} \longrightarrow 0.$$

The theorem is proved.

The conditions of Theorem 4 for λ_n and a_n are fulfilled, for instance, if it is assumed that $\lambda_n = n^{1/2+\varepsilon}$ and $a_n = n^{\varepsilon}$, $1/8 < \varepsilon < 1/6$.

2. Asymptotic power of the goodness-of-fit test based on U_n. The assertion of Theorem 4 enables us to construct tests of an asymptotic level α for verifying the hypothesis H_0 by which $f(x) = f_0(x)$ and $f_0(\ell_0) \ne 0$. To this end, we should calculate U_n and discard H_0 if

$$U_n \ge d_n(\alpha) = \Delta(f_0) + \left(\frac{\lambda_n}{a_n}\right)^{-1/2} \varepsilon_\alpha \sigma(f_0), \tag{24}$$

where

$$\Delta(f_0) = f_0(\ell_0)\int K^2(u)\,du \int W(x)\,dx,$$

$$\sigma^2(f_0) = 2f_0^2(\ell_0)\int K_0^2(z)\,dz \int W^2(v)\,dv,$$

$$K_0 = K * K,$$

ε_α is the quantile of the level α of the standard normal distribution $\Phi(x)$.

Now we will investigate the asymptotic behavior of test (24) or, more exactly, the behavior of the power function for $n \to \infty$.

Let us consider the question whether test (24) is consistent.

The following statement is true.

Theorem 5. *Let the conditions of Theorem* 4 *be fulfilled. Then for* $n \to \infty$

$$\Pi_n(f_1) = P_{H_1}\big\{U_n \ge d_n(\alpha)\big\} \longrightarrow 1.$$

Therefore the test defined in (24) *is consistent against any alternative* $H_1 \colon f(x) = f_1(x)$, $f_1(x) \ne f_0(x)$ *on the set of a positive Lebesgue measure and* $f_1(\ell_0) \ne f_0(\ell_0)$.

Proof. It is easy to see that

$$\Pi_n(f_1) =$$

$$= P_{H_1}\left\{\frac{n}{\lambda_n}\int (f_n(x) - f_0(x))^2\, W_n(x)\,dx \ge \Delta(f_0) + \left(\frac{\lambda_n}{a_n}\right)^{-1/2} \sigma(f_0)\varepsilon_\alpha\right\} =$$

$$= P_{H_1}\left\{\frac{n}{\lambda_n}\int (f_n(x) - f_1(x))^2\, W_n(x)\,dx \ge\right.$$

$$\ge \Delta(f_0) + \left(\frac{\lambda_n}{a_n}\right)^{-1/2} \sigma(f_0)\varepsilon_\alpha - \frac{n}{\lambda_n}\,R_n -$$

$$-2\int (f_n(x)-f_1(x))\,W_n(x)\psi_n(x)\,dx\,\frac{n}{\lambda_n}\Bigg\} =$$

$$= P_{H_1}\Bigg\{\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)\,(U_n^*-\Delta(f_1)) \geq$$

$$\geq \sqrt{\frac{\lambda_n}{a_n}}\,\big[\Delta(f_0)-\Delta(f_1)\big]\sigma^{-1}(f_1)+\frac{\sigma(f_0)}{\sigma(f_1)}\,\varepsilon_\alpha-\frac{n}{\sqrt{\lambda_n a_n}}\,\sigma^{-1}(f_1)R_n-$$

$$-2\,\frac{n}{\sqrt{\lambda_n a_n}}\int (f_n(x)-f_1(x))\,W_n(x)\psi_n(x)\,dx\,\sigma^{-1}(f_1)\Bigg\} =$$

$$= P_{H_1}\Bigg\{\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)\,(U_n^*-\Delta(f_1)) \geq$$

$$\geq -\frac{n}{\sqrt{\lambda_n a_n}}\Bigg[\sigma^{-1}(f_1)R_n+\frac{\lambda_n}{n}\,(\Delta(f_1)-\Delta(f_0))\,\sigma^{-1}(f_1)+$$

$$+2\sigma^{-1}(f_1)\int (f_n(x)-f_1(x))\,\psi_n(x)W_n(x)\,dx+\frac{\sqrt{\lambda_n a_n}}{n}\,\sigma(f_0)\sigma^{-1}(f_1)\varepsilon_\alpha\Bigg]\Bigg\}, \quad (25)$$

where

$$U_n^* = \frac{n}{\lambda_n}\int (f_n(x)-f_1(x))^2\,W_n(x)\,dx,$$

$$\psi_n(x) = (f_1(x)-f_0(x))\,W_n(x), \qquad W_n(x) = a_n W\left(a_n(x-\ell_0)\right),$$

$$R_n = \int (f_1(x)-f_0(x))^2\,W_n(x)\,dx.$$

Furthermore, using the inequalities

$$E\bigg(\int (f_n(x)-f_1(x))^2\left(f_1^2(x)+f_0^2(x)\right)dx\bigg) \leq c_{34}\,\frac{\lambda_n}{n}+c_{35}\,\lambda_n^{-4}$$

and

$$\int W_n^2(x)\,dx \leq c_{36}a_n,$$

we can state that

$$\int (f_n(x)-f_1(x))\,\psi_n(x)\,dx \xrightarrow{P} 0.$$

Therefore, (25) implies

$$\Pi_n(f_1) = P_{H_1}\Bigg\{\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)\,(U_n^*-\Delta(f_1)) \geq$$

$$\geq -\frac{n}{\sqrt{\lambda_n a_n}}\,\left(\sigma^{-1}(f_1)R_n+o_p(1)\right)\Bigg\}. \quad (26)$$

Since $\sqrt{\dfrac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)(U_n^* - \Delta(f_1))$ is distributes assimptotically normally to $(0,1)$ in the case of the hypothesis H_1,

$$\frac{n}{\sqrt{\lambda_n a_n}} \longrightarrow \infty$$

and

$$R_n \longrightarrow (f_1(\ell_0) - f_0(\ell_0))^2 \int W(u)\,du > 0 \text{ as } n \to \infty,$$

from (26) it follows that $\Pi_n(f_1) \to 1$.

The theorem is proved.

Thus the power of test (24) for any fixed alternative tends to 1 as $n \to \infty$. Nevertheless more profound properties of the test are revealed when investigating the question how the test reacts to “small” deviations from the verified hypothesis, i.e., when instead of the fixed alternative we consider the sequence of alternatives $\{H_{1n}\}$ approaching with the basic hypothesis H_0 as $n \to \infty$. Let us consider the sequence of alternatives of the form [2, 3]

$$H_1\colon\ f_1(x) = f_0(x) + \alpha_n \varphi\left(\frac{x - \ell_n}{\gamma_n}\right) + o(\alpha_n\gamma_n), \quad \ell_n = \ell_0 + o(\gamma_n),$$

where $\alpha_n \downarrow 0$, $\gamma_n \downarrow 0$, the function $\varphi(x) \in F$ and $\int \varphi(x)\,dx = 0$.

Theorem 6. *Let $K(x)$, $f_1(x)$, $W(x)$, λ_n and a_n satisfy the conditions of Theorem* 4. *If, in addition to this,*

$$W(0) \neq 0, \qquad \alpha_n\gamma_n = o(n^{-1/2}),$$

$$n\lambda_n^{-1/2} a_n^{1/2} \gamma_n \alpha_n^2 \longrightarrow \gamma_0, \qquad \lambda_n a_n^{-1} \alpha_n^2 \longrightarrow 0, \quad a_n\gamma_n \longrightarrow 0,$$

$$\sqrt{n}\,\alpha_n \sqrt{a_n}\,\lambda_n^{-5/2} \gamma_n^{-2} \longrightarrow 0, \qquad \lambda_n^4 \gamma_n^5 a_n \longrightarrow \infty,$$

and

$$a_n \lambda_n \alpha_n \longrightarrow \infty \quad \textit{as} \quad n \to \infty,$$

then

$$P_{H_1}\{U_n \geq d_n(\alpha)\} \longrightarrow 1 - \Phi\bigg(\varepsilon_\alpha - \frac{\gamma_0 W(0)}{\sigma(f_0)} \int \varphi^2(x)\,dx\bigg).$$

Proof. We write U_n as a sum

$$U_n = U_n^{(2)} + A_{n1} + A_{n2},$$

$$U_n^{(2)} = \frac{n}{\lambda_n} \int (f_n(x) - Ef_n(x))^2\, W_n(x)\,dx,$$

$$A_{n1} = \frac{n}{\lambda_n} \int (Ef_n(x) - f_0(x))^2\, W_n(x)\,dx,$$

$$A_{n2} = 2\,\frac{n}{\lambda_n} \int (f_n(x) - Ef_n(x))\,(Ef_n(x) - f_0(x))\, W_n(x)\,dx,$$

where $E(\cdot)$ is the mathematical expectation under the hypothesis H_1. Therefore, we obtain

$$P_{H_1}\big\{U_n \geq d_n(\alpha)\big\} = P_{H_1}\bigg\{\frac{1}{\sqrt{a_n\lambda_n}}\,\sigma^{-1}(f_1)\big[U_n^{(1)} - EU_n^{(1)}\big] \geq$$

$$\geq \frac{\sigma(f_0)}{\sigma(f_1)}\,\varepsilon_\alpha + \sqrt{\frac{\lambda}{a_n}}\,\sigma^{-1}(f_1)\big[\lambda_n^{-1}\Delta_n(f_1) - \Delta(f_0)\big]-$$

$$-\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)A_{n1} + \sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)A_{n2}\bigg\}. \tag{27}$$

Tracing the proofs of Theorems 1 and 2, it is not difficult to make sure that

$$\frac{1}{\sqrt{a_n\lambda_n}}\,\sigma^{-1}(f_1)\Big[U_n^{(1)} - EU_n^{(1)}\Big] \xrightarrow{d} N(0,1).$$

Let us show that

$$\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)A_{n2} \xrightarrow{P} 0.$$

Indeed,

$$\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)E|A_{n2}| \leq$$

$$\leq c_1\frac{1}{\sqrt{a_n\lambda_n}}\bigg\{\frac{\lambda_n^2}{n}\int f_1(u)\,du\bigg[\int K\left(\lambda_n(x-u)\right)\left(Ef_n(x) - f_1(x)\right)W_n(x)\,dx\bigg]^2\bigg\}^{1/2}$$

and also

$$Ef_n(x) = f_1(x) + O(\lambda_n^{-2}) + O\left(\frac{\alpha_n}{\lambda_n^2\gamma_n^2}\right).$$

Hence

$$\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)E|A_{n2}| = O\left(\left(\frac{na_n}{\lambda_n^5}\right)^{1/2}\right) + O\left(\frac{\sqrt{n}\,\alpha_n\sqrt{a_n}}{\lambda_n^{5/2}\gamma_n^2}\right).$$

Therefore,

$$\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)A_{n2} \xrightarrow{P} 0. \tag{28}$$

Next, by using the condition $n\lambda_n^{-1/2}a_n^{1/2}\gamma_n\alpha_n^2 \longrightarrow \gamma_0$ it is not difficult to establish that

$$\sqrt{\frac{\lambda_n}{a_n}}\,A_{n1} = \frac{n\alpha_n^2}{\sqrt{\lambda_n a_n}}\int\varphi^2\left(\frac{x-\ell_n}{\gamma_n}\right)W_n(x)\,dx + O(na_n^{-1/2}\lambda_n^{-9/2})+$$

$$+O(\lambda_n^{-4}\gamma_n^{-5}a_n^{-1}) + O\left(\frac{1}{\alpha_n\lambda_n^2}\right) + O\left(\frac{1}{\lambda_n\gamma_n}\right) + O(a_n^{-1}\alpha_n^{-1}\lambda_n^{-1}).$$

From this and the Lebesgue theorem on majorized convergence it follows that

$$\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)A_{n1} =$$

$$= \sigma^{-1}(f_1)n\lambda_n^{-1/2}a_n^{1/2}\alpha_n^2\gamma_n\frac{1}{a_n\gamma_n}\int\varphi^2\left(\frac{t}{a_n\gamma_n}\right)W(t)\,dt+$$

$$+O\big(na_n^{-1/2}\lambda_n^{-9/2}\big) + O\big(\lambda_n^{-4}\gamma_n^{-5}a_n^{-1}\big) + O\left(\frac{1}{\alpha_n\lambda_n^2}\right) + O\left(\frac{1}{\lambda_n\gamma_n}\right)+$$

$$+O\big(a_n^{-1}\alpha_n^{-1}\lambda_n^{-1}\big) \longrightarrow \frac{\gamma_0 W(0)}{\sigma(f_0)} \int \varphi^2(u)\,du. \tag{29}$$

Finally, we can easily show that

$$\sqrt{\frac{\lambda_n}{a_n}}\,\sigma^{-1}(f_1)\big[\lambda_n^{-1}\Delta_n(f_1) - \Delta(f_0)\big] = O\left(\left(\frac{\lambda_n}{a_n^5}\right)^{1/2}\right) + O\left(\frac{\lambda_n \alpha_n^2}{a_n}\right). \tag{30}$$

Thus (27) – (30) imply

$$P_{H_1}\big\{U_n \geq d_n(\alpha)\big\} \longrightarrow 1 - \Phi\bigg(\varepsilon_\alpha - \frac{\gamma_0(W_0)}{\sigma(f_0)} \int \varphi^2(u)\,du\bigg).$$

The theorem is proved.

It is well known that the limit power of the Rosenblatt – Bickel test [1 – 3]

$$T_n \geq \int f_0(x) W(x)\,dx \int K^2(u)\,du + \lambda_n^{-1/2}\varepsilon_\alpha \sigma_0,$$

$$T_n = \frac{n}{\lambda_n} \int \left(f_n(x) - f_0(x)\right)^2 W(x)\,dx, \tag{31}$$

$$\sigma_0^2 = 2 \int f_0^2(x) W^2(x)\,dx \int K_0^2(u)\,du.$$

For verifying the hypothesis $H_0\colon f(x) = f_0(x)$ against the alternative

$$H_1\colon\ f_1(x) = f_0(x) + \alpha_n \varphi\left(\frac{x - \ell_n}{\gamma_n}\right) + o(\alpha_n \gamma_n), \quad \ell_n = \ell_0 + o(\gamma_n),$$

where $\lambda_n = n^\delta$, $\alpha_n = n^{-\alpha}$, $\gamma_n = n^{-\beta}$ for some α, β and δ, for which $\alpha + \beta > 1/2$, $1 - 2\alpha - \beta = \delta/2$ (for example, $\alpha = 9/35$, $\beta = 2/7$, $\delta = 2/5$ or $\alpha = 1/6$, $\beta = 5/12$, $\delta = 1/2$), is equal to,

$$\gamma(T) = 1 - \Phi\bigg(\varepsilon_\alpha - \frac{W(\ell_0)}{\sigma_0} \int \varphi^2(u)\,du\bigg),$$

while the limit power $\gamma(U)$ of test (24) is equal to 1 (see (29)) for $a_n = n^\varepsilon$, $\varepsilon < \delta$.

However, for some α, β, δ and ε, for which $\alpha + \beta > 1/2$, $1 - 2\alpha - \beta + \varepsilon/2 = \delta/2$ (for example, $\alpha = 9/35$, $\beta = 2/7$, $\varepsilon = 1/6$, $\delta = 17/30$), the limit power of test (24), by Theorem 6, is equal to

$$\gamma(U) = 1 - \Phi\bigg(\varepsilon_\alpha - \frac{W(0)}{\sigma(f_0)} \int \varphi^2(u)\,du\bigg),$$

while the limit power $\gamma(T)$ of test (31) is equal to $1 - \Phi(\varepsilon_\alpha)$. Therefore, choosing between (31) and (24), we will prefer the test based on U_n. Moreover, for weight functions $W(x)$, for which $W(0) > 0$, the goodness-of-fit test of the hypothesis H_0 against alternatives of form H_1 are asymptotically strictly unbiased since the mathematical expectation $\gamma_0 W(0) \int \varphi^2(u)\,du > 0$ is equal to zero if and only if $\varphi(x) = 0$, $x \in (-\infty, \infty)$.

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