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STABILIZATION OF CAUCHY PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATIONS

СТАБІЛІЗАЦІЯ ЗАДАЧІ КОШІ ДЛЯ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

In the present paper, we obtain the criterion of stabilization of Cauchy problem for an integro-differential equation in the class of functions of polynomial growth $\gamma \geq 0$.

Одержано критерій стабілізації задачі Коші для інтегро-диференціального рівняння у класі функцій з поліноміальним зростанням $\gamma \geq 0$.

1. Introduction. In the present paper, we consider the integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} = P\left(\frac{\partial}{\partial x}\right)u(x,t) + Q\left(\frac{\partial}{\partial x}\right)\int_0^t u(x,\tau)d\tau, \quad (x,t) \in \Pi_\infty = \mathbb{R}^n \times [0,+\infty), \tag{1.1}$$

under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $P(\sigma)$ and $Q(\sigma)$ are arbitrary polynomials with complex constant coefficients $(\sigma \in \mathbb{R}^n)$; here $u : \Pi_\infty \to \mathbb{C}$ is the unknown function; $u_0 : \mathbb{R}^n \to \mathbb{C}$ is a given function; $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$. We study problem (1.1), (1.2) under the condition $Q(\sigma) \neq 0$ $(\forall\, \sigma \in \mathbb{R}^n)$. Here $\int_0^t u(x,\tau)d\tau$ is a control (the system input).

Introduce the following Banach space of functions of some polynomial growth $\gamma \geq 0$:

$$H_{m,\gamma} = \left\{ f \in C^m(\mathbb{R}^n) : \| f \|_{m,\gamma} = \max_{|\alpha| \leq m} \sup_{\mathbb{R}^n} \left| \frac{\partial^{|\alpha|} f(x)}{\partial x^\alpha} \right| (1+|x|)^{-\gamma} < +\infty \right\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multiindex $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and $\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}, \ldots, \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\right)$.

Definition 1.1. *We say that problem* (1.1), (1.2) *is stable in the class of functions of polynomial growth* $\gamma \geq 0$ *if for every nonnegative integer* m *there exists a nonnegative integer* l*, so that for every initial function* $u_0(x)$ *of space* $H_{l,\gamma}$*, each solution* $u(x,t)$ *of problem* (1.1), (1.2) *belongs to the space* $H_{m,\gamma}$ *for each* $t \in [0,T]$*, and*

$$\left\| \frac{\partial^j u(\cdot,t)}{\partial t^j} \right\|_{m,\gamma} \to 0, \quad t \to +\infty, \quad j = 0, 1. \tag{1.3}$$

If we consider problem (1.1), (1.2) in the space S (where S is the Schwartz space and S' is the dual space of tempered distribution [1]) and apply the Fourier transform, we obtain

$$\frac{\partial v(\sigma,t)}{\partial t} = P(i\sigma)v(\sigma,t) + Q(i\sigma)\int_0^t v(\sigma,\tau)\,d\tau \quad (\text{in } S'), \tag{1.4}$$

$$v(\sigma,0) = v_0(\sigma) \quad (\text{in } S'), \tag{1.5}$$

where $v(\sigma, t)$ and $v_0(\sigma)$ are the Fourier transforms of $u(x, t)$ and $u_0(x)$ respectively:

$$v(\cdot, t) = F_x\{u(\cdot, t)\}, \qquad v_0 = F_x\{u_0\}$$

(F_x is the operator of Fourier transform with respect to x).

If we introduce the vector function

$$\mathbf{v}(\sigma, t) = \left(v, \frac{dv}{dt}\right)^T,$$

it is easily seen from (1.4) and (1.5) that $\mathbf{v}(\sigma, t)$ is a solution of the following Cauchy problem:

$$\frac{d\mathbf{v}(\sigma,t)}{dt} = \mathbf{A}(\sigma)\mathbf{v}, \qquad \mathbf{v}(\sigma,0) = \mathbf{v}_0(\sigma) \quad (\text{in } S'), \tag{1.6}$$

where

$$A(\sigma) = \begin{pmatrix} 0 & 1 \\ Q(i\sigma) & P(i\sigma) \end{pmatrix} \quad \text{and} \quad \mathbf{v}_0(\sigma) = v_0(\sigma)(1, P(i\sigma))^T.$$

In Section 2, we prove some auxiliary lemmas. The criterion of stabilization of problem (1.1), (1.2) in the class of functions of polynomial growth is established in Section 3.

2. Preliminaries. Let $\lambda_1(\sigma)$ and $\lambda_2(\sigma)$ be the eigenvalues of matrix $\mathbf{A}(\sigma)$ and let

$$\Lambda(\sigma) = \max\{\operatorname{Re}\lambda_1(\sigma), \operatorname{Re}\lambda_2(\sigma)\};$$

here $\operatorname{Re} z$ is the real part of the complex z. Because $Q(i\sigma) \neq 0$ for every $\sigma \in \mathbb{R}^n$, we conclude that $\lambda_1(\sigma)\lambda_2(\sigma) \neq 0$ $(\forall\, \sigma \in \mathbb{R}^n)$.

Lemma 2.1. *Let the function* $\Lambda(\sigma)$ *satisfy the condition*

$$\Lambda(\sigma) < 0 \quad (\forall\, \sigma \in \mathbb{R}^n). \tag{2.1}$$

Then there exist constants $\beta < 0$ *and* $q \in \mathbf{Q}$ *such that*

$$\Lambda(\sigma) < \beta\sqrt{\left(1+|\sigma|^2\right)^q} \quad (\forall\, \sigma \in \mathbb{R}^n). \tag{2.2}$$

Proof. Let $\delta(r)$ be a real function defined as

$$\delta(r) = \sup_{\sigma\in\mathbb{R}^n : |\sigma| = r} \{\Lambda(\sigma)\}.$$

It is obvious that $\delta(r)$ is defined on $[0, +\infty)$. It follows from (2.1) that $\delta(r) < 0$ for all $r \geq 0$. By applying the results of [2] (Appendix A) to $\delta(r)$, we find that $\delta(r)$ is piecewise continuous on $[0, +\infty)$ and for some constants $M < 0$ and $q \in \mathbf{Q}$,

$$\delta(r) = M r^q (1 + o(1)) \quad (r \to +\infty);$$

therefore there exists $\beta < 0$ such that $\delta(r) \leq \beta\sqrt{\left(1+r^2\right)^q}$ for all $r \geq 0$, which implies the estimate (2.2) and Lemma 2.1 is proved.

Lemma 2.2. *Let the function* $\Lambda(\sigma)$ *satisfy condition* (2.1). *Then*

$$\mathbb{R}(\sigma, t) = \begin{cases} \dfrac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} P(e^{t\lambda_1} - e^{t\lambda_2}) & \lambda_1 e^{t\lambda_2} - \lambda_2 e^{t\lambda_1} \\ P(\lambda_1 e^{t\lambda_1} - \lambda_2 e^{t\lambda_2}) & \lambda_1\lambda_2(e^{t\lambda_2} - e^{t\lambda_1}) \end{pmatrix}, & \text{if } P^2 + 4Q \neq 0, \\ \begin{pmatrix} 1 & \lambda_1(1+2t) \\ \lambda_1 & 2\lambda_1 + \lambda_1^2(1+2t) \end{pmatrix} e^{t\lambda_1}, & \text{if } P^2 + 4Q = 0 \end{cases} \tag{2.3}$$

is a multiplicator in S (*here* $\lambda_j = \lambda_j(\sigma)$, $P = P(i\sigma)$ *and* $Q = Q(i\sigma)$).

Proof. By using the estimate of a matrix exponential in [3] (Chap. 1, Sect. 6) (see also [4]) and estimate (2.2), we obtain

$$\|\mathbb{R}(\sigma, t)\| \leq C(1+|\sigma|)^d e^{t\beta(1+|\sigma|)^q} \qquad (\forall\, \sigma \in \mathbb{R}^n,\ \forall\, t \geq 0),$$

where $C > 0$, and $d = \max(\deg P, \deg Q)$. Therefore

$$\left\| \frac{\partial^{|\alpha|}\mathbb{R}(\sigma, t)}{\partial\sigma^\alpha} \right\| \leq C_\alpha (1+|\sigma|)^{(|\alpha|+1)d - |\alpha|} e^{t\beta(1+|\sigma|)^q} \qquad (\forall\, \sigma \in \mathbb{R}^n,\ \forall\, t \geq 0) \tag{2.4}$$

for any miltiindex α and some $C_\alpha > 0$. Hence, $\mathbb{R}(\sigma, t)$ is a multiplicator in S.

***Corollary* 2.1.** *If conditions* (2.1) *is satisfied, then the solution of Cauchy problem* (1.6) *in* S' *reads*

$$\mathbf{v}(\sigma, t) = \mathbb{R}(\sigma, t)(1,1)^T v_0(\sigma) \qquad (in\ S') \quad (t \geq 0). \tag{2.5}$$

In fact, if conditions (2.1) is valid, then function $\mathbb{R}(\sigma, t)$ given by (2.3) will be a multiplicator in S, and (2.5) follows from estimate (2.4).

3. Criterion of the stabilization of problem (1.1), (1.2).

Theorem 3.1. *In order that the Cauchy problem* (1.1), (1.2) *should be stable in the space of functions of polynomial growth* $\gamma \geq 0$, *it is necessary and sufficient that condition* (2.1) *should be valid.*

Proof. *Necessity.* Let problem (1.1), (1.2) be stable in the space of functions of polynomial growth $\gamma \geq 0$. Assume on contrary that condition (2.1) is violated. Then for some $\sigma_0 \in \mathbb{R}^n$ we have $\Lambda(\sigma_0) > 0$. Without loss of generality, suppose that $\operatorname{Re}\lambda_1(\sigma_0) = \Lambda(\sigma_0) \geq 0$ and $\operatorname{Re}\lambda_1(\sigma_0) \geq \operatorname{Re}\lambda_2(\sigma_0)$. Further we find the solution of the Cauchy problem for equation (1.1) with the initial condition

$$u(x, 0) = \begin{cases} \dfrac{\lambda_1(\sigma_0) - \lambda_2(\sigma_0)}{\lambda_1(\sigma_0)} e^{ix.\sigma_0}, & \text{if } \lambda_1(\sigma_0) \neq \lambda_2(\sigma_0), \\ e^{ix.\sigma_0}, & \text{if } \lambda_1(\sigma_0) = \lambda_2(\sigma_0); \end{cases}$$

here

$$x.\sigma_0 = \sum_{i=1}^n x_i \sigma_{0i},$$

if $x = (x_1, \ldots, x_n)$, $\sigma_0 = (\sigma_{01}, \ldots, \sigma_{0n})$. Obviously, the solution of this problem reads

$$u(x, t) = \begin{cases} e^{t\lambda_1(\sigma_0) + ix.\sigma_0} - \dfrac{\lambda_2(\sigma_0)}{\lambda_1(\sigma_0)} e^{t\lambda_2(\sigma_0) + ix.\sigma_0}, & \text{if } \lambda_1(\sigma_0) \neq \lambda_2(\sigma_0), \\ (1 + \lambda_1(\sigma_0)t) e^{t\lambda_1(\sigma_0) + ix.\sigma_0}, & \text{if } \lambda_1(\sigma_0) = \lambda_2(\sigma_0). \end{cases}$$

If $\lambda_1(\sigma_0) = \lambda_2(\sigma_0)$ then

$$\left| u(x,t) \right| = |1+\lambda_1(\sigma_0)t|e^{t\Delta(\sigma_0)}$$

and we have

$$\overline{\lim_{t\to+\infty}} |u(x,t)| > 0,$$

which contradicts the hypothesis that problem (1.1), (1.2) is stable in the space of functions of polynomial growth $\gamma \ge 0$.

If $\lambda_1(\sigma_0) \ne \lambda_2(\sigma_0)$ then

$$\left| u(x,t) \right| = e^{t\Lambda(\sigma_0)} \left| 1 - \frac{\lambda_2(\sigma_0)}{\lambda_1(\sigma_0)} e^{t(\lambda_2(\sigma_0)-\lambda_1(\sigma_0))} \right| \ge$$

$$\ge e^{t\Lambda(\sigma_0)} \left| 1 - \left|\frac{\lambda_2(\sigma_0)}{\lambda_1(\sigma_0)}\right| e^{t(\operatorname{Re}\lambda_2(\sigma_0)-\operatorname{Re}\lambda_1(\sigma_0))} \right| > 0,$$

and we have

$$\overline{\lim_{t\to+\infty}} |u(x,t)| > 0,$$

which contradicts the hypothesis that problem (1.1), (1.2) is stable in the space of functions of polynomial growth $\gamma \ge 0$.

Sufficiency. Consider for equation (1.1) the Cauchy problem with the initial condition

$$u(x,0) = u^0(x), \quad x \in \mathbb{R}^n. \tag{3.1}$$

Because of the fulfillment of condition (2.1), the solution of Cauchy problem (1.6) associated to the Cauchy problem (1.1), (3.1) is given by (2.5) and the first component of vector $\mathbf{v}(\sigma,t)$ is the solution (in S') of the Cauchy problem for equation (1.4) with the initial condition $v(\sigma,0) = v^0(\sigma) = F_x\{u^0\}$:

$$v(\sigma,t) = \begin{cases} \dfrac{1}{\lambda_1(\sigma)-\lambda_2(\sigma)}\left[(P(i\sigma)-\lambda_2(\sigma))e^{t\lambda_1(\sigma)} + (\lambda_1(\sigma)-P(i\sigma))^{t\lambda_2(\sigma)}\right]v^0(\sigma), \\ \text{if } \lambda_1(\sigma)\ne\lambda_2(\sigma), \\ \\ [1+\lambda_1(\sigma)(1+2t)]e^{t\lambda_1(\sigma)}v^0(\sigma), \\ \text{if } \lambda_1(\sigma)\ne\lambda_2(\sigma). \end{cases}$$

Therefore the function

$$u(x,t) = F_\sigma^{-1}\{v(\sigma,t)\} \quad (\sigma \in \mathbb{R}^n)$$

is the unique solution of Cauchy problem (1.1), (3.1) in S' (see [3, 5–6]). Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and show that for some large $l \in \mathbb{N}$ the function $u(x,t)$ belongs (with respect to x) to the class $H_{m,\gamma}$ (for every $t \ge 0$) and satisfies condition (1.3) as soon as $u^0 \in H_{l,\gamma}$. Let $e(x)$ be a compactly supported infinitely differentiable function on $\mathbb{R}^n$ satisfying the condition

$$\sum_{j\in\mathbb{Z}^n} e(x-j) \equiv 1$$

and whose support lies in $\left\{x\in\mathbb{R}^n : |x| \le 1\right\}$ (see [7–11]).

Let $u_j^0(x) = e(x)u^0(x+j)$ and $v_j^0(\sigma) = F_x\{u_j^0\}$. Then the function

$$
v_j(\sigma, t) = \begin{cases} \dfrac{1}{\lambda_1(\sigma)-\lambda_2(\sigma)}\Big[\big(P(i\sigma)-\lambda_2(\sigma)\big)e^{t\lambda_1(\sigma)} + \big(\lambda_1(\sigma)-P(i\sigma)\big)^{t\lambda_2(\sigma)}\Big]v_j^0(\sigma), \\ \text{if } \lambda_1(\sigma) \neq \lambda_2(\sigma), \\ \\ [1+\lambda_1(\sigma)(1+2t)]e^{t\lambda_1(\sigma)}v_j^0(\sigma), \\ \text{if } \lambda_1(\sigma) \neq \lambda_2(\sigma), \end{cases}
$$

is the solution of the Cauchy problem (1.4), (1.5) in which the initial function $v_0(\sigma)$ is replaced by $v_j^0(\sigma)$. Therefore $u_j(x, t) = F_\sigma^{-1}\{v_j(\cdot, t)\}$ is the solution of problem (1.1), (3.1) with $u^0(x)$ replaced by $u_j^0(x)$; here $j \in \mathbb{Z}^n$. Because $u_j^0(x) \equiv e(x)u^0(x+j)$, it is evident that for some $M > 0$ that does not depend on $j \in \mathbb{Z}^n$, we have

$$
\|u_j^0\|_{l,\gamma} \leq M\|u^0\|_{l,\gamma}(1+|j|)^\gamma .
$$

From estimate (2.4) and estimate

$$
\left|\sigma^\lambda \frac{\partial^{|\alpha|}}{\partial\sigma^\alpha}\big(\sigma^\nu v_j^0(\sigma)\big)\right| \leq C_{\alpha,\nu,\lambda}\|u^0\|_{l,\gamma}(1+|j|)^\gamma ,
$$

$\sigma \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ an arbitrary multiindex, $|\nu| + |\lambda| \leq l$, it follows that

$$
\left|\frac{\partial^{|\alpha|}}{\partial\sigma^\alpha}\big(\sigma^\nu v_j(\sigma, t)\big)\right| \leq M_{1(\alpha,\nu,\lambda)}\|u^0\|_{l,\gamma}(1+|\sigma|)^{(|\alpha|+1)d-|\alpha|-|\lambda|}e^{t\beta(1+|\sigma|)^q}(1+|j|)^\gamma ,
$$

where $|\nu| + |\lambda| \leq l$ and α is arbitrary. If we choose λ from the condition

$$
|\lambda| = (|\alpha|+1)d - |\alpha| + n + 1,
$$

then we obtain

$$
\left|x^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}u_j(x, t)\right| \leq M_{2(\alpha,\nu)}\rho(t)\|u^0\|_{l,\gamma}(1+|j|)^\gamma ,
$$

where α is arbitrary,

$$
|\nu| < l - (|\alpha|+1)d + |\alpha| - n - 1,
$$

and

$$
\rho(t) = \begin{cases} (1+t)^{1/q} & \text{for } q < 0, \\ \exp(\beta t) & \text{for } q \geq 0. \end{cases}
$$

Because $(1+|j|)^\gamma \leq (1+|x+j|)^\gamma(1+|x|)^\gamma$, if we choose an α from the condition $|\alpha| = n - E(-\gamma) + 1$ (here $E(-\gamma)$ stands for the integer part of $-\gamma$), we obtain

$$
\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}u_j(x, t)\right| \leq M_\nu \rho(t)\|u^0\|_{l,\gamma}(1+|x+j|)^\gamma(1+|x|)^{-n-1}, \tag{3.2}
$$

where $\nu \leq m$, $l \geq m + (n - E(-\gamma) + 2)d + E(-\gamma)$; consequently

$$
u(x, t) = \sum_{j \in \mathbf{Z}} u_j(x - j, t) \tag{3.3}
$$

is solution of the Cauchy problem (1.1), (3.1), belongs to $H_{m,\gamma}$ for all $t \geq 0$ and satisfies the condition

$$\|u(\cdot, t)\|_{m,\gamma} \le M_m \rho(t) \left\| u^0 \right\|_{l,\gamma} \quad (\forall\, t \ge 0). \tag{3.4}$$

Because $\rho(t) \to 0$ as $t \to +\infty$, we conclude from (3.2)–(3.4) that $u(\cdot, t) \in H_{m,\gamma}$ $(t \ge 0)$.

By analogy, we prove that

$$\left\| \frac{\partial u}{\partial t}(\cdot, t) \right\| \le M'_m \rho(t) \left\| u^0 \right\|_{l,\gamma} \quad (\forall\, t \ge 0). \tag{3.5}$$

It is sufficient to notice that the Cauchy problem (1.6) is equivalent to the Cauchy problem

$$\frac{d^2 v(\sigma, t)}{dt^2} = P(i\sigma) \frac{dv(\sigma, t)}{dt} + Q(i\sigma) v(\sigma, t),$$

$$v(\sigma, 0) = v_0(\sigma), \qquad v'_t(\sigma, 0) = P(i\sigma) v_0(\sigma).$$

It follows from (3.4) and (3.5) that $u(x, t)$ satisfies the condition (1.3). Hence Cauchy problem (1.1), (1.2) is stable in the class of functions of some polynomial growth $\gamma \ge 0$ and Theorem 3.1 is proved.

***Example* 3.1.** Consider the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} - 4\int_0^t u(x, \tau)\, d\tau, \qquad x \in \mathbb{R}, \quad t \ge 0.$$

For this equation, $P(i\sigma) = -\sigma^2$, $Q(i\sigma) = -4$, and

$$\Lambda(\sigma) = \begin{cases} -\sigma^2 + \sqrt{\sigma^4 - 16} & \text{for } \sigma \in (-\infty, -2] \cup [2, +\infty), \\ -\sigma^2 & \text{for } \sigma \in (-2, 2). \end{cases}$$

Therefore $\Lambda(\sigma) < 0$ for every $\sigma \in \mathbb{R}$, and by Theorem 3.1, the Cauchy problem for this equation is stable in the classes of polynomial growth.

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