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NEW SECOND BRANCH OF SPECTRA OF THE BCS HAMILTONIAN AND “PSEUDOGAP”*

НОВА ДРУГА ВІТКА СПЕКТРА ГАМІЛЬТОНІАНА БКШ ТА “ПСЕВДОЩІЛИНА”

The BCS Hamiltonian of superconductivity has the second branch of eigenvalues and eigenvectors. It consists from wave functions of pairs of electrons in ground and excited states. The continuous spectra of excited pairs is divided by different from zero gap from the point of discrete spectra corresponding to the pair in ground state. The corresponding grand partition function and free energy is exactly calculated. It follows from it that for low temperatures system is in condensate of pairs in ground state. The sequence of correlation functions is exactly calculated in the thermodynamic limit and it coincides with corresponding sequence of system with approximating Hamiltonian. The gap in spectra of excitations depends continuously on temperature and is different from zero above the critical temperature corresponding to the first branch of spectra. It seems to us that this fact explains the phenomena of “pseudogap”.

Гамільтоніан БКШ теорії надпровідності має другу вітку власних значень та власних векторів. Ця вітка складається з хвильових функцій пар електронів в основному та збуджених станах. Неперервний спектр збуджених пар відділений відмінною від нуля щільною від точки дискретного спектра, що відповідає парі в основному стані. Відповідна велика статистична сума та вільна енергія вираховані точно. Звідси випливає, що при низьких температурах система є в конденсаті пар в основному стані. Послідовність кореляційних функцій вирахована точно у термодинамічній границі і збігається з відповідною послідовністю системи з апроксимуючим гамільтоніаном. Щільність в спектрі збуджень залежить неперервно від температури і є відмінною від нуля і на відрізьку вище критичної температури, що відповідає першій вітці спектра. На думку автора, цей факт пояснює феномен „псевдощільнини”.

Introduction. In the series of papers [1–7] we investigated the eigenvalues and eigenvectors of the BCS Hamiltonian for system of electrons in finite cube Λ with periodic boundary condition and in the entire space \mathbf{R}^3 . It was shown that the BCS Hamiltonian has two branches of eigenvalues and eigenvectors — the first is well known ground state and its excitations with corresponding eigenvalues discovered by Bardeen, Cooper and Schrieffer [8]. Bogolyubov [9] showed that the mean energies per volume of ground states of the BCS Hamiltonian and of the approximating Hamiltonian coincide in the thermodynamic limit as $\Lambda \rightarrow \mathbf{R}^3$ in some sense.

Recently we showed that the mean energies per volume of all the excited states of the both Hamiltonians coincide in the thermodynamic limit, and the BCS Hamiltonian and the approximating Hamiltonian coincide as the quadratic form in the thermodynamic limit [5].

The second branch of eigenvalues and eigenvectors has been discovered by author first directly for infinite system [6] and recently for finite system in cube Λ [1–4]. It was shown that the eigenvalues determined in the cube Λ tend to the corresponding eigenvalues determined in the entire \mathbf{R}^3 in the thermodynamic limit. For the second branch of the eigenvectors mean energies per volume of the ground and the excited states of the BCS and approximating Hamiltonians also coincide in the thermodynamic limit. On this second branch of the eigenvectors the BCS and the approximating Hamiltonians also coincide as the quadratic forms, in the thermodynamic limit.

Describe shortly the second branch of eigenvectors and eigenvalues directly for infinite system in \mathbf{R}^3 . Consider the Hamiltonian H_2 for wave function of one pair of

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electrons with opposite momenta and spin

$$H_2 f(k) = \left(\frac{2k^2}{2m} - 2\mu \right) f(k) + gv(k) \int v(p) f(p) dp, \tag{1}$$

where m is mass of electron, μ — chemical potential, $v(k)v(p)$ separated potential, g — coupling constant. Denote by $f_0(k)$ the eigenvector with lowest eigenvalue E_0 . $f_0(k)$ and E_0 satisfy equations

$$\begin{aligned} \left(\frac{2k^2}{2m} - 2\mu \right) f_0(k) + c_0 v(k) &= E_0 f_0(k), \quad c_0 = g \int v(p) f_0(p) dp, \tag{2} \\ 1 &= g \int \frac{v^2(p) dp}{E_0 - \frac{2p^2}{2m} + 2\mu}. \end{aligned}$$

For certain potential $v(k) = v(|k|)$ with support in layer $\left| \frac{k^2}{2m} - \mu \right| < \omega$, $\omega > 0$, equation (2) for eigenvalue E_0 has unique solution $E_0 < -2\omega$ with gap $|E_0 + 2\omega| = \Delta \neq 0$ and corresponding eigenvector (normalized to unity)

$$f_0(k) = \frac{v(k)}{E_0 - \frac{2p^2}{2m} + 2\mu} \left(\int \frac{v(p)^2 dp}{\left(E_0 - \frac{2p^2}{2m} + 2\mu \right)^2} \right)^{-\frac{1}{2}}. \tag{3}$$

E_0 is discrete eigenvalue. These results has been obtained by Cooper [10] and Yamaguchi [11].

We consider the operator H_2 on functions with support in layer $\left| \frac{k^2}{2m} - \mu \right| < \omega$, $\omega > 0$. The Hamiltonian H_2 has also eigenvectors corresponding to the continuous spectra $-2\omega < E - 2\mu < 2\omega$. Namely, some of these eigenvectors $f_E(k)$ with eigenvalue $E - 2\mu$ are orthogonal to $v(k)$, i.e., $\int v(p) f_E(p) dp = 0$, and satisfy equations

$$\left(\frac{2k^2}{2m} - 2\mu \right) f_E(k) = (E - 2\mu) f_E(k), \quad E = \frac{p^2}{m}. \tag{4}$$

Eigenvectors $f_E(k)$ can be represented as superposition of the following eigenfunctions:

$$f_{E,nl}(k) = \left(\frac{m|k|}{2} \right)^{\frac{1}{2}} \frac{\delta(|k| - |p|)}{|k|^2} Y_{nl}(\theta, \varphi), \quad |n| + l \geq 1, \tag{5}$$

where $Y_{nl}(\theta, \varphi)$ is spherical function.

For $l = 0$ we have

$$\begin{aligned} f_{E,00}(k) &= \left(\frac{m|k|}{2} \right)^{\frac{1}{2}} \frac{\delta(|k| - |p|)}{|k|^2} + \frac{gv(|k|)c_1}{\frac{k^2}{m} - E - i\varepsilon}, \\ c_1 &= \left(\frac{m|p|}{2} \right)^{\frac{1}{2}} 4\pi v(|p|) \left(1 - 4\pi g \int \frac{v^2(|p'|)|p'|^2 d|p'|}{\frac{p'^2}{m} - E - i\varepsilon} \right)^{-1}. \end{aligned} \tag{6}$$

One supposes $\lim_{\varepsilon \rightarrow 0}$ in (6). Eigenvalue $E - 2\mu$ is degenerated $2l + 1$ times because for fixed l number n takes value $n = 0, \pm 1, \dots, \pm l$. Eigenvalue $E - 2\mu$ corresponds to excited pair with eigenvectors $f_{E,nl}(k)$.

The $2n$ -particle Hamiltonian or the Hamiltonian H_n for n pairs is defined as follows:

$$H_n = H_2 \otimes I \otimes \dots \otimes I + \dots + I \otimes \dots \otimes I \otimes H_2, \quad n > 2, \quad (7)$$

and it has eigenfunctions proportional to

$$f_0(k_1) \dots f_0(k_s) f_{E_1}(k_{s+1}) \dots f_{E_{n-s}}(k_n) \quad (8)$$

with eigenvalues

$$sE_0 + E_1 - 2\mu + \dots + E_{n-s} - 2\mu, \quad -2\omega + 2\mu \leq E_i \leq 2\omega + 2\mu, \quad 1 \leq i \leq n - s.$$

Note that formulae (7) is crucial point in our paper. It defines the BCS Hamiltonian in subspace of n pairs and has been discovered in our paper [6]. It means that BCS Hamiltonian is identical to the Hamiltonians $H_n(7)$ in subspace of n pairs and general Hilbert space of translation invariant functions [6]. It follows from (7) that the Hamiltonian $H_2(1)$ introduced by Cooper [10] in connection with the theory of superconductivity for wave function of one pair defines the BCS Hamiltonian for arbitrary numbers of n pairs by formulae (7).

Note that we obtained wave functions of excited pairs (5), (6) with arbitrary angular momenta $l \geq 0$.

The Hamiltonians H_n , $n \geq 2$, coincide with the BCS Hamiltonian [8]

$$H = \int \left(\frac{p^2}{2m} - \mu \right) a^+(\bar{p}) a(\bar{p}) d\bar{p} + \frac{g(2\pi)^3}{V} \int v(p)v(p') a^+(p) a^+(-p) a(-p') a(p') dp dp' \quad (9)$$

on eigenfunctions (8) (description of notation used in (9) will be given in Section 1). These facts have been discovered in our papers [1–6]. This means that n wave functions of pairs of electrons with opposite momenta and spins in ground or excited states are eigenvectors of the BCS Hamiltonian. The interaction of the BCS Hamiltonian is only cause to create bound state of pairs of electrons, but pairs do not interact between themselves.

Define the ground state ϕ_0 as coherent state of pairs with wave functions $f_0(k)$ (3)

$$\phi_0 = e^{\int f_0(k) a^+(k) a^+(-k) dk} |0\rangle$$

and the BCS ground state

$$\phi_0^a = e^{\int f_0^a(k) a^+(k) a^+(-k) dk} |0\rangle,$$

where

$$f_0^a(k) = - \left((\varepsilon^2(k) + c^2 v^2(k))^{\frac{1}{2}} - \varepsilon(k) \right)^{\frac{1}{2}} \left((\varepsilon^2(k) + c^2 v^2(k))^{\frac{1}{2}} + \varepsilon(k) \right)^{-\frac{1}{2}},$$

$$\varepsilon(k) = \frac{k^2}{2m} - \mu.$$

The constant c is defined from condition of minimum of $(\phi_0^a, H \phi_0^a)$.

The main difference between the ground state discovered by Bardeen, Cooper and Schrieffer ϕ_0^a and the second ground state ϕ_0 consists in the following. The BCS ground state ϕ_0^a is determined from condition of minimum of mean energy $(\phi_0^a, H \phi_0^a)$ for the all coherent state of pairs ϕ_0^a , but the second ground state ϕ_0 is determined from condition of minimum of mean energy $(\phi_0^n, H \phi_0^n)$ for the all states ϕ_0^n of n pairs of the coherent state ϕ_0 of pairs.

For these second branch of spectra we calculated exactly grand partition function

$$\Xi = e^{(2\pi)^{-3} V e^{-\beta E_0}} \exp \left\{ (2\pi)^{-3} V \int_{-2\omega}^{2\omega} e^{-\beta(\frac{k^2}{2m} - 2\mu)} \left[\alpha \left(\frac{k^2}{m} \right) + \sum_{l=1}^{\infty} (2l + 1) e^{-\beta_1(l+1)l} \right] d \left(\frac{k^2}{m} - 2\mu \right) \right\}, \quad (10)$$

where β is inverse temperature, $\beta_1 = \frac{\beta}{I}$ and I is inertia momenta and $\alpha \left(\frac{k^2}{m} \right)$ is some function which will be defined later. Free energy per volume is equal to

$$-\frac{1}{\beta} \lim_{V \rightarrow \infty} \frac{1}{V} \ln \Xi = -\frac{(2\pi)^{-3}}{\beta} \left(e^{-\beta E_0} + \int_{-2\omega}^{2\omega} e^{-\beta(\frac{k^2}{2m} - 2\mu)} \left[\alpha \left(\frac{k^2}{m} \right) + \sum_{l=0}^{\infty} (2l + 1) e^{-\beta_1(l+1)l} \right] d \left(\frac{k^2}{m} - 2\mu \right) \right). \quad (11)$$

It follows from (11) that for low temperature ($\beta \rightarrow \infty$) and due to the gap in spectra of the Hamiltonian H_2 system exhibits condensation of pairs in ground state

$$-\frac{1}{\beta} \lim_{V \rightarrow \infty} \frac{1}{V} \ln \Xi \approx -\frac{(2\pi)^{-3}}{\beta} e^{-\beta E_0}.$$

We also proved that the correlation functions associated with the second branch of eigenvalues and eigenvectors (8) of the BCS Hamiltonian coincide with the correlation functions associated with the following approximating Hamiltonian [9]:

$$H_{\text{appr}} = \int a^+(\bar{k}) \left(\frac{k^2}{2m} - \mu \right) a(\bar{k}) d\bar{k} + c \int v(k) a^+(k) a^+(-k) dk + c \int v(k) a(-k) a(k) dk + C(c)V, \quad (12)$$

where constant c is defined as follows:

$$c = \int v(p) \left[f_0(p) e^{-\beta E_0} + \int_{-2\omega+2\mu}^{2\omega+2\mu} f_{E,00}(p) e^{-\beta(E-2\mu)} \alpha(E) dE \right] dp \quad (13)$$

and constant $C(c)$ is determined from condition of coincidence of grand partition functions (10) of the BCS (9) and the approximating (12) Hamiltonians.

Stress that constant c is defined directly by formulae (13), it is different from zero for arbitrary $0 \leq \beta < \infty$ and depends on β continuously. Recall that in the approximating Hamiltonian that corresponds to the first branch of spectra constant c is defined from condition of minimum of the free energy with respect to c . The condition of minimum is reduced to certain nonlinear equation that has nontrivial solution for the temperature T less than some critical T_c . In the case of the second branch of eigenvalues and eigenvectors constant c is different from zero for all the temperatures.

If seems to us that this fact explains the phenomena of “pseudogap”. Indeed, the eigenvalues of the approximating Hamiltonian (12) that correspond to n -particle excitation with momenta p_1, \dots, p_n are defined through formulae

$$E(k_1) + \dots + E(k_n), E(k_1) = \sqrt{\left(\frac{k^2}{2m} - \mu \right)^2 + c^2 v^2(k)},$$

where $c^2v^2(k)$ characterizes the gap in spectra and according to (13) the gap is different from zero for all the temperatures.

1. The model BCS Hamiltonian. 1.1. Equation for ground state. Consider the model BCS Hamiltonian [8] for infinite cube $\Lambda = \mathcal{R}^3$

$$H = \int \left(\frac{p^2}{2m} - \mu \right) a^+(\bar{p})a(\bar{p})d\bar{p} + \frac{g(2\pi)^3}{V} \int v(p)v(p')a^+(p)a^+(-p)a(-p')a(p')dpdp' = H_0 + H_I, \quad (1.1)$$

where $V = V(\mathcal{R}^3)$ is the volume of the three-dimensional space \mathbf{R}^3 , g is coupling constant, \bar{p} denote momenta p and spin $\sigma = \pm 1$, $d\bar{p}$ means integration with respect to p and summation with respect to $\sigma = \pm 1$, $p = (p, 1)$, $-p = (-p, -1)$ $a^+(\bar{p}), a(\bar{p})$ are the operators of creation and annihilation of electrons with momenta p and spin σ .

The model Hamiltonian (1.1) has a rigorous meaning in the Hilbert space of translation-invariant functions and its spectra has been investigated in detail [4, 6]. We present a short review of these results. We consider the Hamiltonian H (1.1) on functions with support in layer $\left| \frac{k^2}{2m} - \mu \right| < \omega, \omega > 0$.

Let us consider the following coherent state:

$$\begin{aligned} \Phi_0 &= e^{\int f_0(k)a^+(k)a^+(-k)dk} |0\rangle = \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \int f_0(k_1) \dots f_0(k_r) \times \\ &\times a^+(k_1)a^+(-k_1) \dots a^+(k_r)a^+(-k_r)dk_1 \dots dk_r |0\rangle = \sum_{r=0}^{\infty} \frac{1}{r!} \Phi_0^r \end{aligned} \quad (1.2)$$

and determine the normalized to unity function $f_0(k)$ from condition that each Φ_0^r is an eigenvector of H with the lowest eigenvalue. From these conditions we obtain

$$\begin{aligned} &\sum_{i=1}^r \left(\frac{2k_i^2}{2m} - 2\mu \right) f_0(k_1) \dots f_0(k_r) + \\ &+ \sum_{i=1}^n g \int v(k)f_0(k)dk f_0(k_1) \dots \frac{i}{v(k_i)} \dots f_0(k_r) = \\ &= rE_0 f_0(k_1) \dots f_0(k_r), \quad H\Phi_0^r = rE_0\Phi_0^r. \end{aligned} \quad (1.3)$$

Obtaining (1.3) we used the identity $\frac{(2\pi)^3}{V}\delta(0) = 1$, and the fact that, according to the Fermi statistics, in Φ_0 pairs with the same momenta are absent. By using the method of separation of variables one concludes that $f_0(k)$ is solution of the equation

$$\begin{aligned} \left(\frac{2k^2}{2m} - 2\mu \right) f_0(k) + c_0v(k) &= E_0f_0(k), \\ c_0 &= g \int v(k)f_0(k)dk, \end{aligned} \quad (1.4)$$

and eigenvalue E_0 is solution of the equation

$$1 = g \int \frac{v^2(p)}{E_0 - \frac{2p^2}{2m} + 2\mu} dp. \quad (1.5)$$

From (1.4) one gets normalized to unity $f_0(k)$

$$f_0(k) = \frac{v(k)}{E_0 - \frac{2k^2}{2m} + 2\mu} \left(\int \frac{v^2(p)dp}{\left(E_0 - \frac{2p^2}{2m} + 2\mu\right)^2} \right)^{-\frac{1}{2}}.$$

Let proceed to investigation of equation (1.5).

1.2. Equation for discrete eigenvalue. We have the following equation for eigenvalue (1.5):

$$1 = g \int \frac{v^2(p)}{E - \left(\frac{2p^2}{2m} - 2\mu\right)} dp,$$

where $v(p)$ is different from zero in layer $-\omega + \mu \leq \frac{p^2}{2m} \leq \omega + \mu$ and for the sake of simplicity we put $v^2(p) = \frac{\alpha}{|p|}$, $\alpha > 0$. By using spherical system of coordinate equation is performed to the following form:

$$\begin{aligned} 1 &= 2\pi g\alpha m \int_{-\omega + \mu < \frac{p^2}{2m} < \omega + \mu} \frac{\frac{2|p|}{m} d|p|}{E - \left(\frac{2p^2}{2m} - 2\mu\right)} = \\ &= ga \int_{-2\omega}^{2\omega} \frac{dx}{E - x} = |g|a \int_{-2\omega}^{2\omega} \frac{dx}{x - E}, \end{aligned} \tag{1.6}$$

where $a = 2\pi\alpha m > 0$.

We calculate the last integral in (1.6) supposing that $E < -2\omega$, and obtain

$$1 = |g|a \ln \frac{2\omega - E}{-2\omega - E}. \tag{1.7}$$

This equation has the unique solution

$$E_0 = 2\omega \frac{1 + e^{\frac{1}{a|g|}}}{1 - e^{\frac{1}{a|g|}}} < -2\omega < 0 \tag{1.8}$$

that is eigenvalue corresponding to the following normalized to unity eigenfunction

$$f_0(k) = \frac{v(k)}{E_0 - \frac{2k^2}{2m} + 2\mu} \left(\int \frac{v^2(p)}{\left(E_0 - \frac{2p^2}{2m} + 2\mu\right)^2} dp \right)^{-\frac{1}{2}}. \tag{1.9}$$

Consider equation (1.6) for $-2\omega < E < 2\omega$, i.e., $2\omega - E > 0$, $-2\omega - E < 0$. The function $\ln \frac{2\omega - E}{-2\omega - E}$ is holomorphic function and for its negative argument $\frac{2\omega - E}{-2\omega - E} < 0$ equation is defined as follows:

$$1 = \ln \frac{2\omega - E}{2\omega + E} + i\pi, \quad \frac{2\omega - E}{2\omega + E} > 0.$$

If means that equation (1.7) has not solution with $-2\omega < E < 2\omega$.

Show that equation (1.7) has also not solution with $E > 2\omega$. Indeed in this case

$$\frac{2\omega - E}{-2\omega - E} = \frac{E - 2\omega}{E + 2\omega} > 0$$

and equation (1.7)

$$1 = |g|a \ln \frac{E - 2\omega}{E + 2\omega}$$

has solution

$$E = 2\omega \frac{1 + e^{\frac{1}{a}|g|}}{1 - e^{\frac{1}{a}|g|}} < -2\omega < 0.$$

But this contradicts our assumption that $E > 2\omega$.

Now show that in general case equation

$$1 = g \int \frac{v^2(p)}{E - \frac{2p^2}{2m} + 2\mu} dp = \varphi(E)$$

can have only finite number of real solutions $-2\omega < E < 2\omega$. Indeed, the right-hand side of equation $\varphi(E)$ is holomorphic function with respect to E in complex plane outside the interval $\text{Im}E = 0, -2\omega < \text{Re}E < 2\omega$. For potential $v(p)$ that is holomorphic function in a neighborhood of this interval the function $\varphi(E)$ has boundary value on this interval and is here a holomorphic function. (This easy follows from method of holomorphic continuation by using deformation of the contour of integration.) Therefore $\varphi(E)$ can take the value 1 only in finite number of points on the interval $\text{Im}E = 0, -2\omega < \text{Re}E < 2\omega$.

In what follows we will consider only the case $v^2(p) = \frac{\alpha}{|p|}$, and in this case equation (1.6) has unique solution (1.8) $E_0 < -2\omega$.

1.3. Eigenfunction of continuous spectra. Consider equation for eigenfunctions corresponding to continuous spectra

$$H_2 f_E(k) = \left(\frac{2k^2}{2m} - 2\mu \right) f(k) + gv(k) \int v(p')f(p')d\bar{p}' = (E - 2\mu)f(k), \quad (1.10)$$

$$-2\omega + 2\mu < E < 2\omega + 2\mu.$$

This equation is equivalent to the following integral equation [11]:

$$f_E(k) = \left(\frac{m|k|}{2} \right)^{\frac{1}{2}} \delta(k - p) + \frac{gv(k)}{\frac{2k^2}{2m} - E - i\varepsilon} \int v(p')f(p')dp', \quad \frac{2p^2}{2m} = E, \quad \varepsilon > 0. \quad (1.11)$$

Represent $f_E(k)$ and $\delta(k - p)$ as follows:

$$\begin{aligned} f_E(k) &= \sum_{l=0}^{\infty} \sum_{n=-l}^l \Psi_{E,l}(|k|) Y_{ln}(\hat{k}) Y_{ln}^*(\hat{p}) = \\ &= \sum_{l=0}^{\infty} \Psi_{E,l}(|k|) 4\pi(2l+1) P_l(\hat{k} \cdot \hat{p}), \quad |\hat{k}| = 1, \quad |\hat{p}| = 1, \\ \delta(k - p) &= \sum_{ln} \frac{1}{k^2} \delta(|k| - |p|) Y_{ln}(\hat{k}) Y_{ln}^*(\hat{p}) = \\ &= \sum_{l=0}^{\infty} \frac{1}{k^2} \delta(|k| - |p|) 4\pi(2l+1) P_l(\hat{k} \cdot \hat{p}). \end{aligned} \quad (1.12)$$

There $Y_{ln}(\hat{k})$ is normalized spherical function, $\hat{k} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$

$$Y_{ln}(\hat{k}) = e^{in\varphi} P_l^{|n|}(\cos \theta) N_{ln},$$

$$N_{ln} = \left(\frac{(l - |n|)!(2l + 1)}{(l + |n|)!4\pi} \right)^{\frac{1}{2}},$$

$$P_l^{|n|}(\xi) = (1 - \xi^2)^{\frac{|n|}{2}} \frac{d^{|n|}}{d\xi^{|n|}} P_l(\xi),$$

$P_l(\xi)$ is Legendre polinom.

Note that $\sum_{l=0}^{\infty} \sum_{n=-l}^l Y_{ln}(\hat{k}) Y_{ln}^*(\hat{p})$ is δ -function $\delta(\hat{k}, \hat{p})$ on sphere and if $\varphi(\hat{k})$ is arbitrary smooth function then

$$\int \sum_{l=0}^{\infty} \sum_{n=-l}^l Y_{ln}(\hat{k}) Y_{ln}^*(\hat{p}) \varphi(\hat{p}) d\hat{p} = \varphi(\hat{k}),$$

where $d\hat{p}$ is the element of unit sphere $|\hat{p}| = 1$. Substituting expressions (1.12) into (1.11) and using orthogonality $Y_{ln}^*(\hat{p}')$ to $v(p')$, $l + |n| \geq 1$ one obtains

$$\Psi_{E,0}(|k|) = \left(\frac{m|k|}{2} \right)^{\frac{1}{2}} \frac{\delta(|k| - |p|)}{k^2} + \frac{gv(k)c_1}{\frac{2k^2}{2m} - E - i\varepsilon},$$

$$c_1 = \left(\frac{m|p|}{2} \right)^{\frac{1}{2}} v(p) 4\pi \left(1 - 4\pi g \int \frac{v^2(p') p'^2 dp'}{\frac{2p'^2}{2m} - E - i\varepsilon} \right)^{-1},$$

$$\Psi_{E,l}(k) = \left(\frac{m|k|}{2} \right)^{\frac{1}{2}} \frac{\delta(|k| - |p|)}{k^2}.$$

Denote as above by $E_k = \frac{k^2}{m}$, $E_p = \frac{p^2}{m}$ and use the following obvious formula:

$$\delta(E_k - E_p) = m\delta(k^2 - p^2) = \frac{m}{2|k|} \delta(|k| - |p|). \tag{1.13}$$

It follows from (1.12) and (1.13) that

$$\Psi_{E,l}(|k|) = \frac{\left(\frac{|m|k|}{2} \right)^{\frac{1}{2}}}{k^2} \delta(|k| - |p|) = \left(\frac{4}{E_k m^3} \right)^{\frac{1}{4}} \delta(E_k - E_p), \quad l \geq 1. \tag{1.14}$$

The eigenfunction of the operator H_2 that corresponds to continuous eigenvalue $-2\omega \leq E - 2\mu \leq 2\omega$ and orbital momenta $l \geq 1$ can be represented follows:

$$f_{E,l}(k, p) = \Psi_{E,l}(|k|) \sum_{n=-l}^l Y_{ln}(\hat{k}) Y_{ln}^*(\hat{p}) =$$

$$= \left(\frac{4}{E_k m^3} \right)^{\frac{1}{4}} \delta(E_k - E_p) \sum_{n=-l}^l Y_{ln}(\hat{k}) Y_{ln}^*(\hat{p}) =$$

$$= \left(\frac{4}{E_k m^3} \right)^{\frac{1}{4}} \delta(E_k - E_p) \frac{1}{4\pi} (2l + 1) P_l(\hat{k} \cdot \hat{p}). \tag{1.15}$$

1.4. Some formulae. Now calculate the following expression by using commutation relations:

$$\begin{aligned}
& \langle 0 | \int d\hat{p} \int dk_1 \int_{E_1}^{E_2} dE_{p_1} \left(\frac{4}{E_{k_1} m^3} \right)^{\frac{1}{4}} \delta(E_{k_1} - E_{p_1}) \times \\
& \quad \times \sum_{n_1=-l}^l Y_{ln_1}^*(\hat{k}_1) Y_{ln_1}(\hat{p}) a(-k_1) a(k_1) \times \\
& \quad \times \int dk_2 \int_{E_1}^{E_2} dE_{p_2} \left(\frac{4}{E_{k_2} m^3} \right)^{\frac{1}{4}} \delta(E_{k_2} - E_{p_2}) \times \\
& \quad \times \sum_{n_2=-l}^l Y_{ln_2}(\hat{k}_2) Y_{ln_2}^*(\hat{p}) a^+(k_2) a(-k_2) |0\rangle = \\
& \quad = (2\pi)^{-3} V(2l+1)(E_2 - E_1), \tag{1.16} \\
& \quad l \geq 1, \quad E_1 = -2\omega + 2\mu, \quad E_2 = 2\omega + 2\mu.
\end{aligned}$$

If one first performs integration with respect to \hat{p} using orthogonality of $Y_{ln}(\hat{p})$ then one obtains the following equivalent representation of (1.16)

$$\begin{aligned}
& \langle 0 | \int dk_1 \int_{E_1}^{E_2} dE_{p_1} \left(\frac{4}{E_{k_1} m^3} \right)^{\frac{1}{4}} \delta(E_{k_1} - E_{p_1}) a(-k_1) a(k_1) \times \\
& \quad \times \int dk_2 \int_{E_1}^{E_2} dE_{p_2} \left(\frac{4}{E_{k_2} m^3} \right)^{\frac{1}{4}} \delta(E_{k_2} - E_{p_2}) \times \\
& \quad \times \sum_{n=-l}^l Y_{ln}^*(\hat{k}_1) Y_{ln}(k_2) a^+(k_2) a^+(-k_2) |0\rangle = \\
& \quad = (2\pi)^{-3} V(2l+1)(E_2 - E_1).
\end{aligned}$$

The same result will be obtained if one considers more general expression

$$\begin{aligned}
& \langle 0 | \int dk_1 \int_{E_1}^{E_2} dE_{p_1} \left(\frac{4}{E_{k_1} m^3} \right)^{\frac{1}{4}} \delta(E_{k_1} - E_{p_1}) \sum_{n_1=-l}^l Y_{ln_1}^*(\hat{k}_1) a(-k_1) a(k_1) \times \\
& \quad \times \int dk_2 \int_{E_1}^{E_2} dE_{p_2} \left(\frac{4}{E_{k_2} m^3} \right)^{\frac{1}{4}} \delta(E_{k_2} - E_{p_2}) \times \\
& \quad \times \sum_{n_2=-l}^l Y_{ln_2}(\hat{k}_2) a^+(k_2) a^+(-k_2) |0\rangle = \\
& \quad = (2\pi)^{-3} V(2l+1)(E_2 - E_1). \tag{1.17}
\end{aligned}$$

We omit the same calculation as in (1.16).

It is obvious that functions

$$\begin{aligned}
f_{E,l}(k) &= \sum_{n=-l}^l f_{E,ln}(|k|) = \left(\frac{4}{E_k m^3} \right)^{\frac{1}{4}} \delta(E_k - E_p) \sum_{n=-l}^l Y_{ln}(\hat{k}), \quad l \geq 1, \\
f_{E,ln}(k) &= \left(\frac{4}{E_k m^3} \right)^{\frac{1}{4}} \delta(E_k - E_p) Y_{ln}(\hat{k}), \quad n = 0, \pm 1, \dots, \pm l,
\end{aligned} \tag{1.18}$$

are also eigenfunctions of the operator H_2 with continuous eigenvalue $-2\omega \leq E \leq 2\omega$, because

$$\int v(p') \sum_{n=-l}^l Y_{nl}(\hat{p}') dp' = 0, \quad l \geq 1,$$

and

$$\left(\frac{2k^2}{2m} - 2\mu\right) f_{E,l}(k) = (E_p - 2\mu) f_{E,l}(k), \quad E_p = E_k.$$

In what follows for the sake of simplicity we will use expression (1.17), (1.18) calculating different averages.

Remark that we use the factor $\left(\frac{m|k|}{2}\right)^{\frac{1}{2}}$ in (1.14) in order to have in formulae (1.16)–(1.18) $(2\pi)^{-3}V(2l+1) \int_{E_1}^{E_2} dE$ without any factor depending on E [12].

Note that $f_0(k)$ has been found by Cooper [10], but he did not calculate eigenfunction corresponding to continuous spectra. We also used papers of Yamaguchi [11].

1.5. Superposition of eigenfunctions corresponding to continuous spectra. Denote by $f_{ln}(k)$ and $f_l(k)$ functions

$$f_{ln}(k) = \int_{E_1}^{E_2} \left(\frac{4}{E_k m^3}\right)^{\frac{1}{4}} \delta(E_k - E) Y_{ln}(\hat{k}) dE, \quad n = 0, \pm 1, \dots, \pm l,$$

$$f_l(k) = \int_{E_1}^{E_2} f_{E,l}(k) dE = \int_{E_1}^{E_2} \left(\frac{4}{E_k m^3}\right)^{\frac{1}{4}} \delta(E_k - E) \sum_{n=-l}^l Y_{ln}(\hat{k}) dE =$$

$$= \sum_{n=-l}^l f_{ln}(k), \quad l \geq 1,$$

$$f^0(k) = \int_{E_1}^{E_2} \left[\left(\frac{4}{E_k m^3}\right)^{\frac{1}{4}} \delta(E_k - E) + \frac{gv(k)c_1}{\frac{2k^2}{2m} - E - i\varepsilon} \right] dE$$

that is superposition of functions $f_{E,l}(k)$ with respect to E with fixed unit vector \hat{p} . Note that $f_0(k) \neq f^0(k)$ because $f_0(k)$ corresponds to discrete eigenvalue E_0 and $f^0(k)$ corresponds to superposition of eigenvalues with continuous eigenvalues $-2\omega \leq E - 2\mu \leq 2\omega$ and $l = 0$. The constant c_1 was defined in Subsection 1.3.

Denote by $f(k)$ function

$$f(k) = \sum_{l=0}^{\infty} f_l(k). \tag{1.19}$$

Note that function $f(k)$ is given by formal series (1.19) and it contains the wave functions of excited pairs with arbitrary angular momenta.

2. Ground and excited states and grand partition function. 2.1. Ground and excited states of infinite system. The ground state Φ_0 of infinite system can be represented as follows:

$$\Phi_0 = \left(1, 0, \int f_0(k_1) a^+(k_1) a^+(-k_1) dk_1 |0\rangle, 0, \dots\right)$$

$$\begin{aligned} & \dots, \frac{1}{r!} \int f_0(k_1) a^+(k_1) a^+(-k_1) dk_1 \dots \\ & \dots \int f_0(k_r) a^+(k_r) a^+(-k_r) dk_r |0\rangle, 0 \dots \end{aligned} \quad (2.1)$$

The state Φ_0 is the coherent state of pair of electrons with opposite momenta and spin with wave function $f_0(k)$ that corresponds to the lowest eigenvalue $E_0 < 0$ of H_2 . It can be represented as follows:

$$\Phi_0 = \exp \left(\int f(k) a^+(k) a^+(-k) dk \right) |0\rangle. \quad (2.2)$$

Φ_0 is the ground state of system with the Hamiltonian H , because

$$\begin{aligned} & H \int f_0(k_1) a^+(k_1) a^+(-k_1) dk_1 \dots \int f_0(k_r) a^+(k_r) a^+(-k_r) dk_r |0\rangle = \\ & = rE_0 \int f_0(k_1) a^+(k_1) a^+(-k_1) dk_1 \dots \int f_0(k_r) a^+(k_r) a^+(-k_r) dk_r |0\rangle. \end{aligned}$$

Define the following state:

$$\Phi_s = \frac{1}{s!} \int f(k_1) a^+(k_1) a^+(-k_1) dk_1 \dots \int f(k_s) a^+(k_s) a^+(-k_s) dk_s \Phi_0. \quad (2.3)$$

The state $f_{E_1}(k_1) a^+(k_1) a^+(-k_1) \dots f_{E_s}(k_s) a^+(k_s) a^+(-k_s) \Phi_0^r$ is the eigenvector of H with eigenvalue $(E_1 - 2\mu) + \dots + (E_s - 2\mu) + rE_0$. The state $f(k_1) a^+(k_1) \times \dots \times a^+(-k_1) \dots f(k_s) a^+(k_s) a^+(-k_s) \Phi_0^r$ is superposition of states $f_{E_1}(k_1) a^+(k_1) \times \dots \times a^+(-k_1) \dots f_{E_s}(k_s) \Phi_0^r$. Φ_s is excitation of the ground state Φ_0 by s excited pairs with wave functions $f(k)$. Vectors Φ_s are orthogonal to Φ_0 and themselves and their linear combinations forms the Hilbert space of states. In this Hilbert space all the averages will be calculated.

Define the state

$$\begin{aligned} \Phi &= \sum_{s=0}^{\infty} \frac{1}{s!} \int f(k_1) a^+(k_1) a^+(-k_1) \dots \int f(k_l) a^+(k_s) a^+(-k_s) \Phi_0 = \\ &= e^{\int f(k) a^+(k) a^+(-k) dk} e^{\int f_0(k) a^+(k) a^+(-k) dk} |0\rangle. \end{aligned} \quad (2.4)$$

Φ is excitation of the ground state Φ_0 by arbitrary number of pairs with wave functions $f(k)$. Note that $f(k)$ is orthogonal to $f_0(k)$ and therefore excitations are orthogonal to Φ_0 . It is easy to construct the operator of creation and annihilation for which Φ_0 is the vacuum (see [2]).

Remark 2.1. Now explain how to obtain Φ_0 , Φ_s and Φ for infinite system in \mathbf{R}^3 from those for finite system in cube Λ with center at the origin of coordinate. For system situated in Λ the ground state is defined as follows by analogy with the BCS ground state [8]

$$\begin{aligned} \Phi_0^\Lambda &= \prod_k (1 + f_0(k) a_k^+ a_{-k}^+) |0\rangle = \left(1, 0, \sum_k f_0(k_1) a_{k_1}^+ a_{-k_1}^+ |0\rangle, 0, \dots \right. \\ & \quad \left. \dots, \frac{1}{r!} \sum_{k_1 \neq \dots \neq k_r} f_0(k_1) a_{k_1}^+ a_{-k_1}^+ \dots f_0(k_r) a_{k_r}^+ a_{-k_r}^+ |0\rangle, \dots \right), \\ \Phi_s^\Lambda &= \frac{1}{s!} \sum_{k_1 \neq \dots \neq k_s} f(k_1) a_{k_1}^+ a_{-k_1}^+ \dots f(k_s) a_{k_s}^+ a_{-k_s}^+ \Phi_0^\Lambda, \\ \Phi^\Lambda &= \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{k_1 \neq \dots \neq k_s} f(k_1) a_{k_1}^+ a_{-k_1}^+ \dots f(k_s) a_{k_s}^+ a_{-k_s}^+ \Phi_0^\Lambda, \end{aligned} \quad (2.5)$$

where k is quasimomenta $k = \frac{2\pi}{L}n$, $n = (n_1, n_2, n_3)$, $n_i \in Z$, and L is length of the edge of the cube Λ . Note that in (2.5) summation is carried out over all $k_1 \neq \dots \neq k_r$ and $k_1 \neq \dots \neq k_s$. We have taken into account that pairs of electrons cannot have the same momenta, but they can have the same wave functions because the operators $a_{k_i}^+ a_{-k_i}^+, a_{k_j}^+ a_{-k_j}^+$ commute for $k_i \neq k_j$.

In order to obtain Φ_0, Φ_s, Φ for infinite system in \mathbf{R}^3 one formally replaces the operators a_k^+, a_{-k}^+ by $a^+(k), a^+(-k)$ and sums \sum_k by integrals $\int dk$ according to the formulas

$$a^+(\pm k) = \lim_{V \rightarrow \infty} \left(\frac{V}{(2\pi)^3} \right)^{\frac{1}{2}} a_{\pm k}^+, \quad \lim_{V \rightarrow \infty} \frac{(2\pi)^3}{V} \sum_k f(k) = \int f(k) dk.$$

It is easy to see that $\Phi_0^\Lambda, \Phi_s^\Lambda, \Phi^\Lambda$ (2.5) become Φ_0, Φ_s, Φ (2.2)–(2.4) in this limit.

2.2. Grand partition function. Eigenvalues $E_1 \leq E \leq E_2$ are degenerated by angular momenta $l = 0, 1, 2, \dots$. Each l is again degenerated $(2l + 1)$ time. The eigenfunction $f_{E,l}(k)$ corresponds to energy E and above described angular momenta.

According to law of quantum statistical mechanics it is necessary to take into account the angular momenta together with energy. Denote by M the operator of angular momentum. In spherical system of coordinate

$$M^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}, \quad M^2 Y_{ln}(\hat{k}) = (l + 1)l Y_{ln}(\hat{k}),$$

$$\hat{k} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta).$$

Denote by

$$M_s^2 = M^2 \otimes I \otimes \dots \otimes I + \dots + I \otimes \dots \otimes M^2 \tag{2.6}$$

and by

$$\hat{M}^2 = \sum_{s=1}^{\infty} M_s^2. \tag{2.7}$$

Recall that

$$M_s^2(f(k_1) \dots f(k_s)) = (M^2 f(k_1)) \dots f(k_s) + \dots + f(k_1) \dots (M^2 f(k_s)). \tag{2.8}$$

Denote by β the inverse temperature, and by $\beta_1 = \frac{\beta}{2I_l}$, where I_l is inertia momenta of pair with angular momenta l .

In statistical averages of considered system at the inverse temperature β one should use the operator $e^{-\beta(H + \frac{1}{2I} \hat{M}^2)} = e^{-\beta H - \beta_1 \hat{M}^2}$ instead of the operator $e^{-\beta H}$. (Recall that $I, \beta_1 = \frac{\beta}{2I}$ depend on l when the operator \hat{M}^2 acts on function with angular momentum l , for the sake of simplicity one omits sign l in β_1 and often in I .)

Consider expression

$$(\Phi, e^{-\beta H - \beta_1 \hat{M}^2} \Phi) = \langle 0 | \sum_{s_1=0}^{\infty} \frac{1}{s_1!} \int f(k_1) a(-k_1) a(k_1) dk_1 \dots$$

$$\dots \int f(k_{s_1}) a(-k_{s_1}) a(k_{s_1}) dk_{s_1} \times$$

$$\begin{aligned}
 & \times \sum_{r_1=0}^{\infty} \frac{1}{r_1!} \int f_0(p_1) a(-p_1) a(p_1) dp_1 \dots \\
 & \dots \int f_0(p_{r_1}) a(-p_{r_1}) a(p_{r_1}) dp_{r_1} \times \\
 & \times e^{-\beta H - \beta_1 \hat{M}^2} \sum_{s=0}^{\infty} \frac{1}{s!} \int f(k'_1) a^+(k'_1) a^+(-k'_1) dk'_1 \dots \\
 & \dots \int f(k'_s) a^+(k'_s) a^+(-k'_s) dk'_s \times \\
 & \times \sum_{r=0}^{\infty} \frac{1}{r!} \int f_0(p'_1) a^+(p'_1) a^+(-p'_1) dp'_1 \dots \\
 & \dots \int f_0(p'_r) a^+(p'_r) a^+(-p'_r) dp'_r |0\rangle. \tag{2.9}
 \end{aligned}$$

It is obvious that expression (2.9) is grand partition function Ξ because $(s + n)$ -particle states are invariant with respect to action of the Hamiltonian H and operator \hat{M}^2 . Therefore in $(\Phi, e^{-\beta H - \beta_1 \hat{M}^2} \Phi)$ different from zero contributions belong to equal $s = s_1, r = r_1, l = l_1$ and $E = E'$, but it is the grand partition function. Thus we have taken into account pairs in ground state and all excited states of pairs with all energies and angular momenta and therefore

$$\Xi = (\Phi, e^{-\beta H - \beta_1 \hat{M}^2} \Phi) = \text{Tr} e^{-\beta H - \beta_1 \hat{M}^2}. \tag{2.10}$$

Now calculate Ξ . We begin with calculation of some integrals

$$\begin{aligned}
 \langle 0 | \int f_0(k_1) a(-k_1) a(k_1) dk_1 \int f_0(k_2) a^+(k_2) a^+(-k_2) dk_2 |0\rangle &= \\
 = V \int f_0(k)^2 dk = V, \quad \int f_0(k)^2 dk = 1, & \tag{2.11} \\
 \langle 0 | \int \left\{ \int_{E_1}^{E_2} dE \left(\frac{4}{E_k m^3} \right)^{\frac{1}{4}} \delta(E_k - E) Y_{ln}^*(\hat{k}_1) a(-k_1) a(k_1) \right\} dk_1 \times & \\
 \times \int f_0(k_2) a^+(k_2) a^+(-k_2) dk_2 |0\rangle = 0, \quad l + |n| \geq 1. &
 \end{aligned}$$

In these equalities we used orthogonality $f_0(k)$ to $Y_{ln}(\hat{k})$, $l + |n| \geq 1$. We will also use equality (1.16) or equivalent equality (1.17) with $f_{E,l}(k)$ (1.18), and orthogonality $f_0(k)$ and $f^0(k)$ that corresponds to different eigenvalues of H_2 .

It is obvious that

$$\begin{aligned}
 & e^{-\beta H - \beta_1 \hat{M}^2} \frac{1}{s!} \left(\int f_{ln}(k) a^+(k) a^+(-k) dk \right)^s \times \\
 & \times \frac{1}{r!} \left(\int f_0(k) a^+(k) a^+(-k) dk \right)^r |0\rangle = \\
 & = \frac{1}{s!} \left(\int f_{ln}(k_1) a^+(k_1) a^+(-k_1) dk_1 \dots \right. \\
 & \dots \left. \int f_{ln}(k_s) a^+(k_s) a^+(-k_s) e^{\sum_{i=1}^s [-\beta(\frac{2k_i^2}{2m} - 2\mu) - \beta_1(l+1)l]} dk_s \right) \times
 \end{aligned}$$

$$\begin{aligned}
 & \times \frac{1}{r!} \left(\int f_0(k_{s+1}) a^+(k_{s+1}) a^+(-k_{s+1}) dk_{s+1} \dots \right. \\
 & \dots \left. \int f_0(k_{s+r}) a^+(k_{s+r}) a^+(-k_{s+r}) dk_{s+r} e^{-\beta E_0 r} \right) |0\rangle = \\
 & = \frac{1}{s!} \left(\int f_{ln}(k) e^{-\beta(\frac{2k^2}{2m} - 2\mu)} e^{-\beta_1(l+1)l} a^+(k) a^+(-k) dk \right)^s \times \\
 & \times \frac{1}{r!} \left(\int f_0(k) e^{-\beta E_0} a^+(k) a^+(-k) dk \right)^r |0\rangle. \tag{2.12}
 \end{aligned}$$

We used formulae

$$e^{-\beta_1 \hat{M}^2} Y_{ln}(\hat{k}) = e^{-\beta_1(l+1)l} Y_{ln}(\hat{k}), \quad n = 0, \pm 1, \dots, \pm l.$$

If one considers $f_{l_i n_i}(k_i)$ with different $l_i n_i, i = 1, \dots, s$, then one will have in (2.12)

$$\begin{aligned}
 & \frac{1}{s!} \left(\int f_{l_1 n_1}(k_1) e^{-\beta(\frac{2k_1^2}{2m} - 2\mu)} e^{-\beta_1(l_1+1)l_1} a^+(k_1) a^+(-k_1) dk_1 \dots \right. \\
 & \dots \left. \int f_{l_s n_s}(k_s) e^{-\beta(\frac{2k_s^2}{2m} - 2\mu)} e^{-\beta_1(l_s+1)l_s} a^+(k_s) a^+(-k_s) dk_s \right).
 \end{aligned}$$

Recall that spectra of the operator M^2 with eigenvalue $(l + 1)l$ is degenerated $2l + 1$ times. Using formulas (1.16), (2.11), (2.12) one obtains

$$\begin{aligned}
 \Xi & = \langle \Phi, e^{-\beta H - \beta_1 \hat{M}^2} \Phi \rangle = \langle 0 | \sum_{s_1=0}^{\infty} \frac{1}{s_1!} \left(\int f(k) a(-k) a(k) dk \right)^{s_1} \times \\
 & \times \sum_{r_1=0}^{\infty} \frac{1}{r_1!} \left(\int f_0(k) a(-k) a(k) dk \right)^{r_1} \times \\
 & \times \sum_{s=0}^{\infty} \frac{1}{s!} \left(\int dk \int_{E_1}^{E_2} dE \sum_{l+|n|\geq 0} f_{E,ln}(k) e^{-\beta(E-2\mu) - \beta_1(l+1)l} a^+(k) a^+(-k) dk \right)^s \times \\
 & \times \sum_{r=0}^{\infty} \frac{1}{r!} \left(\int f_0(k) e^{-\beta E_0} a^+(k) a^+(-k) dk \right)^r = \\
 & = \sum_{s=0}^{\infty} \frac{1}{s!} \left[(2\pi)^{-3} V \int_{E_1}^{E_2} e^{-\beta(E-2\mu)} \left(\alpha(E) + \sum_{l=1}^{\infty} (2l+1) e^{-\beta_1(l+1)l} \right) dE \right]^s \times \\
 & \times \sum_{n=0}^{\infty} \frac{1}{n!} ((2\pi)^{-3} V e^{-\beta E_0})^n = \\
 & = \exp \left[(2\pi)^{-3} V \int_{E_1}^{E_2} e^{-\beta(E-2\mu)} \left(\alpha(E) + \sum_{l=1}^{\infty} (2l+1) e^{-\beta_1(l+1)l} \right) dE \right] \times \\
 & \times \exp((2\pi)^{-3} V e^{-\beta E_0}) = \\
 & = \exp \left[(2\pi)^{-3} V \int_{-2\omega}^{2\omega} e^{-\beta(\frac{k^2}{m} - 2\mu)} \left(\alpha\left(\frac{k^2}{m}\right) + \sum_{l=1}^{\infty} (2l+1) e^{-\beta_1(l+1)l} \right) \times \right.
 \end{aligned}$$

$$\times d \left(\frac{k^2}{m} - 2\mu \right) \Big] \exp((2\pi)^{-3} V e^{-\beta E_0}), \quad (2.13)$$

$$E_1 = -2\omega + 2\mu, \quad E_2 = 2\omega + 2\mu.$$

Note that factor $2l + 1$ is connected with degeneracy of eigenvalue $(l + 1)l$ by $2l + 1$ time ($n = 0, \pm 1, \dots, \pm l$).

Remark 2.2. We calculated exactly factor $e^{-\beta(E - 2\mu)}$ only for $l \geq 1$ (see (1.16), (1.17)). Show that for $l = 0$ we have the factor $e^{-\beta(E - 2\mu)} \alpha(E)$ where function $\alpha(E)$ will be defined below. It follows from formulae

$$\begin{aligned} & \int \left\{ \int_{E_1}^{E_2} \bar{\Psi}_{E',0}(|k|) dE' \int \Psi_{E,0}(|k|) e^{-\beta(E-2\mu)} dE \right\} d|k| = \\ & = \int_{E_1}^{E_2} dE' \int_{E_1}^{E_2} dE e^{-\beta(E-2\mu)} \left\{ \int \bar{\Psi}_{E',0}(|k|) \Psi_{E,0}(|k|) d|k| \right\} = \\ & = \int_{E_1}^{E_2} dE' \int_{E_1}^{E_2} dE e^{-\beta(E-2\mu)} \delta(E' - E) \alpha(E') = \int_{E_1}^{E_2} dE e^{-\beta(E-2\mu)} \alpha(E). \end{aligned}$$

We used condition of orthogonality of eigenvectors $\Psi_{E,0}(|k|)$

$$\int \bar{\Psi}_{E'}(|k|) \Psi_E(|k|) d|k| = \delta(E' - E) \alpha(E').$$

We can not prove directly that $\alpha(E) = 1$. Thus we calculated exactly the grand partition function for system of pairs with one ground state with eigenvalue E_0 and excited states with continuous spectra in interval $-2\omega < E - 2\mu < 2\omega$. This system of pairs can be considered as system of unpenetrated bosons in momentum space, because according to the Fermi statistics pairs can not occupy the same momenta. We used extensively that state Φ is the coherent state of pairs in ground and excited states.

Note that obtained grand partition function does not coincide with those for Bose–Einstein or Fermi–Dirac statistics. It is the grand partition function of system of noninteracting pairs of electrons with opposite momenta and spins. We have taken into account that pairs of electrons cannot have the same momenta, but they can have the same wave functions because the operators $a_{k_i}^+ a_{-k_i}^+, a_{k_j}^+ a_{-k_j}^+$ commute for $k_i \neq k_j$.

2.3. Definition of energy connected with orbital momenta. Now we proceed to calculate inertia momenta. Use the following formula [12] (see formulae (2.57))

$$\begin{aligned} e^{i\vec{k}r} &= \frac{4\pi}{kr} \left(\frac{mk}{2} \right)^{\frac{1}{2}} \sum_{ln} i^l u_l(kr) Y_{nl}(\hat{r}) Y_{ln}^*(\hat{k}) = \\ &= \frac{1}{kr} \left(\frac{mk}{2} \right)^{\frac{1}{2}} \sum_{l=0} i^l u_l(kr) (2l + 1) P_l(\hat{r} \cdot \hat{k}), \quad |\hat{k}| = 1, |\hat{r}| = 1, \quad (2.14) \end{aligned}$$

where

$$u_l(kr) = \left(\frac{1}{2} \pi kr \right)^{\frac{1}{2}} j_{l+\frac{1}{2}}(kr)$$

and $j_{l+\frac{1}{2}}(kr)$ is the Bessel function, $P_l(\hat{r} \cdot \hat{k})$ is the Legendre polinom.

Function $u_l(kr)$ has the following asymptotic expression:

$$u_l(z) \underset{z \rightarrow \infty}{\approx} \sin(z - \frac{1}{2}\pi l), \quad u_l(z) \underset{z \rightarrow 0}{\approx} \frac{z^{l+1}}{(2l+1)!!}. \tag{2.15}$$

Recall that $\left| \frac{4\pi}{kr} \left(\frac{mk}{2}\right)^{\frac{1}{2}} u_l(kr) \right|^2 r^2$ is the density of probability with respect to r for given l, n . Now define energy connected with orbital momenta l :

$$\begin{aligned} \frac{(l+1)l}{I_l} &= (l+1)l \int_0^\infty \frac{mk}{2r^2} \left(\frac{4\pi}{kr}\right)^2 |u_l(kr)|^2 r^2 dr = \\ &= (l+1)l \int_0^\infty \left(\frac{4\pi}{kr}\right)^2 \frac{mk}{2} |u_l(kr)|^2 dr, \quad l \geq 1. \end{aligned} \tag{2.16}$$

It follows from asymptotic behavior of function $u_l(kr)$ (2.14) that integral (2.15) is convergent and inertia momenta I_l depend on l . Moreover, I_l can be calculated exactly.

We have (see [13, p. 443], §13.42, formulae (1))

$$\begin{aligned} \frac{(l+1)l}{I_l} &= (l+1)l \int_0^\infty \left(\frac{4\pi}{kr}\right)^2 \frac{mk}{2} \frac{1}{2} \pi kr |j_{l+\frac{1}{2}}(kr)|^2 dr = \\ &= \frac{(l+1)l(2\pi)^3 m}{2} \int_0^\infty \frac{|j_{l+\frac{1}{2}}(kr)|^2}{r} dr = \frac{(l+1)l(2\pi)^3 m}{2(2l+1)}. \end{aligned} \tag{2.17}$$

It follows from (2.17) that

$$\frac{1}{I_l} = \frac{(2\pi)^3 m}{2(2l+1)}. \tag{2.18}$$

This means that inertia momenta

$$I_l = \frac{2(2l+1)}{(2\pi)^3 m}$$

is proportional to $2l+1$ and it grows together with l . The series

$$\sum_{l=1}^\infty (2l+1) e^{-\frac{\beta(l+1)l}{I_l}} = \sum_{l=1}^\infty (2l+1) e^{-\frac{\beta(l+1)l(2\pi)^3 m}{2(2l+1)}}$$

is absolutely convergent.

2.4. Free energy. As known the free energy per volume is defined as follows:

$$\begin{aligned} &-\frac{1}{\beta} \lim_{V \rightarrow \infty} \frac{1}{V} \ln \Xi = \\ &= -\frac{(2\pi)^{-3}}{\beta} \left(e^{-\beta E_0} + \int_{E_1}^{E_2} e^{-\beta(E-2\mu)} \left(\alpha(E) + \sum_{l=1}^\infty (2l+1) e^{-\beta_1(l+1)l} \right) dE \right). \end{aligned} \tag{2.19}$$

Consider asymptotic of the free energy in limit of low temperature $\beta \rightarrow \infty$. It is obvious that due to the fact that $E_0 < 0$ we have for $\beta \rightarrow \infty$

$$\begin{aligned}
 -\frac{1}{\beta} \frac{1}{V} \ln \Xi &\cong -\frac{(2\pi)^{-3}}{\beta} \left(e^{-\beta E_0} + \int_{-2\omega}^0 e^{-\beta(\frac{k^2}{m} - 2\mu)} \times \right. \\
 &\times \left. \left(\alpha \left(\frac{k^2}{m} \right) + \sum_{l=1}^{\infty} (2l+1) e^{-\beta_1(l+1)l} \right) d \left(\frac{k^2}{m} - 2\mu \right) \right). \tag{2.20}
 \end{aligned}$$

It follows from (2.20) that for $\beta \rightarrow \infty$ system exists in layer $-2\omega \leq \frac{2k^2}{2m} - 2\mu \leq 0$ below Fermi sphere. If one takes into account that E_0 is divided from the continuous spectra by gap $\Delta = \min_E (E_0 - E) < 0$ then

$$\begin{aligned}
 -\frac{1}{\beta} \frac{1}{V} \ln \Xi &\cong -\frac{(2\pi)^{-3}}{\beta} e^{-\beta E_0} \left(1 + \int_{-2\omega}^0 e^{\beta(E_0 - \frac{k^2}{m} + 2\mu)} \left(\alpha \left(\frac{k^2}{m} \right) + \right. \right. \\
 &\left. \left. + \sum_{l=1}^{\infty} (2l+1) e^{-\beta_1(l+1)l} \right) d \left(\frac{k^2}{m} - 2\mu \right) \right) \approx e^{-\beta E_0}. \tag{2.21}
 \end{aligned}$$

This means that for low temperature $\frac{1}{V} \ln \Xi \cong e^{-\beta E_0}$ and system is in state of pairs of ground state, i.e., we have condensation of pairs in ground state. May be this phenomena of condensation explains hypothesis of Schafroth, Butler and Blatt [14]. Note that this condensation of pairs in ground state is different from Bose–Einstein condensation of free boson system. There does not exists any critical temperature. Note that we are able to calculate the grand partition function and free energy per volume directly for infinite volume. We obtained complete and detailed description of eigenvectors and eigenvalues of the BCS Hamiltonian and it permits us to prove that the grand partition function is exponent which depends on volume multiplied by finite expression proportional to the free energy.

3. Equation for correlation functions and “pseudogap”. 3.1. Equation for correlation functions. Consider the operator $a(\vec{p}), a^+(\vec{p})$ in the Heisenberg representation

$$a(t, \vec{p}) = \bar{e}^{iHt} a(\vec{p}) e^{iHt}, \quad a^+(t, \vec{p}) = e^{-iHt} a^+(\vec{p}) e^{iHt}. \tag{3.1}$$

They satisfy the Heisenberg equations

$$\begin{aligned}
 &i \frac{\partial a(t, p)}{\partial t} = \\
 &= \left(\frac{p^2}{2m} - \mu \right) a(t, p) + v(p) a^+(t, -p) \frac{(2\pi)^3}{V} \int v(p') a(t, -p') a(t, p') dp', \\
 &-i \frac{\partial a^+(t, p)}{\partial t} = \\
 &= \left(\frac{p^2}{2m} - \mu \right) a^+(t, p) + \frac{(2\pi)^3}{V} \int v(p') a^+(t, p') a^+(t, -p') dp' v(p) a(t, -p).
 \end{aligned} \tag{3.2}$$

It is easy to derive equations for $a(t, -p), a^+(t, -p)$.

Consider the following correlation functions:

$$\begin{aligned}
 \langle a^+(t_1, p_1) a(t_2, p_2) \rangle &= \frac{1}{\Xi} \text{Tr} \left(a^+(t_1, p_1) a(t_2, p_2) e^{-\beta H - \beta_1 \hat{M}^2} \right), \\
 \langle a^+(t_1, p_1) a^+(t_2, p_2) \rangle &= \frac{1}{\Xi} \text{Tr} \left(a^+(t_1, p_1) a^+(t_2, p_2) e^{-\beta H - \beta_1 \hat{M}^2} \right), \\
 \langle a(t_1, p_1) a(t_2, p_2) \rangle &= \frac{1}{\Xi} \text{Tr} \left(a(t_1, p_1) a(t_2, p_2) e^{-\beta H - \beta_1 \hat{M}^2} \right),
 \end{aligned} \tag{3.3}$$

and derive for them equations using the Heisenberg equations (3.1). Begin with the first correlation function (3.3). One obtains

$$i \frac{\partial}{\partial t^2} \langle a^+(t_1, p_1) a(t_2, p_2) \rangle = \left(\frac{p_2^2}{2m} - \mu \right) \langle a^+(t_1, p_1) a(t_2, p_2) \rangle + v(p_2) \frac{(2\pi)^3}{V} \int v(p') \langle a^+(t_1, p_1) a^+(t_2, -p_2) a(t_2, -p') a(t_2, p') \rangle dp'. \tag{3.4}$$

Note that average $\langle a^+(t_1, p_1) a^+(t_2, -p_2) a(t_2, -p') a(t_2, p') \rangle$ is invariant with respect to transformation $t_1 \rightarrow t_1 + t_0, t_2 \rightarrow t_2 + t_0$ with arbitrary t_0 . For example, substitute in the average operators $e^{-iH(-t_2)} a^+(t_1, p_1) e^{iH(-t_2)}, e^{-iH(-t_2)} \times a^+(t_2, -p_2) e^{iH(-t_2)}, e^{-iH(-t_2)} a(t_2, -p') e^{iH(-t_2)}, e^{-iH(-t_2)} a(t_2, p') e^{iH(-t_2)}$ instead of the operators $a^+(t_1, p_1), a^+(t_2, -p_2), a(t_2, -p'), a(t_2, p')$ and use that average is invariant with respect to cyclic permutation of operators. Then one obtains the equality

$$\begin{aligned} &\langle a^+(t_1, p_1) a^+(t_2, -p_2) a(t_2, -p') a(t_2, p') \rangle = \\ &= \langle a^+(t_1 - t_2, p_1) a^+(0, p_2) a(0, -p') a(0, p') \rangle, \end{aligned} \tag{3.5}$$

$a^+(0, p_2) = a^+(p_2), a(0, -p') = a(-p'), a(0, p') = a(p')$ and substitutes it in (3.4).

Proceed to investigate the second term in the right-hand side of (3.4).

Consider expression

$$\begin{aligned} &\frac{(2\pi)^3}{V} \int v(p') a(-p') a(p') dp' e^{-\beta H - \beta_1 \hat{M}^2} \Phi = \\ &= \frac{(2\pi)^3}{V} \int v(p') a(-p') a(p') dp' e^{-\beta H - \beta_1 \hat{M}^2} \times \\ &\quad \times \sum_{s=0}^{\infty} \frac{1}{s!} \left(\int f(k) a^+(k) a^+(-k) dk \right)^s \times \\ &\quad \times \sum_{r=0}^{\infty} \frac{1}{r!} \left(\int f_0(k) a^+(k) a^+(-k) dk \right)^r |0\rangle = \\ &= \frac{(2\pi)^3}{V} \int v(p') a(-p') a(p') dp' \times \\ &\quad \times \sum_{s=0}^{\infty} \frac{1}{s!} \left(\int dk \int_{E_1}^{E_2} dE \sum_{l+|n|\geq 0} f_{E,ln}(k) \times \right. \\ &\quad \left. \times e^{-\beta(E-2\mu) - \beta_1(l+1)l} a^+(k) a^+(-k) \right)^s \times \\ &\quad \times \sum_{r=0}^{\infty} \frac{1}{r!} \left(\int f_0(k) e^{-\beta E_0} a^+(k) a^+(-k) dk \right)^r |0\rangle = \\ &= \int v(p') \left[f_0(p') e^{-\beta E_0} + \int_{E_1}^{E_2} f_{E,0}(p') e^{-\beta(E-2\mu)} \alpha(E) dE \right] dp' \times \\ &\quad \times \sum_{s=0}^{\infty} \frac{1}{s!} \left(\int dk \int_{E_1}^{E_2} dE \sum_{l+|n|\geq 0} f_{E,ln}(k) \times \right. \end{aligned}$$

$$\begin{aligned}
& \times e^{-\beta(E-2\mu)-\beta_1(l+1)l} a^+(k) a^+(-k) \Big)^s \times \\
& \times \sum_{r=0}^{\infty} \frac{1}{r!} \left(\int f_0(k) e^{-\beta E_0} a^+(k) a^+(-k) dk \right)^r |0\rangle = \\
& = c e^{-\beta H - \beta_1 \hat{M}^2} \sum_{s=0}^{\infty} \frac{1}{s!} \left(\int dk \int_{-2\omega}^{2\omega} dE \sum_{l+|n|\geq 1} f_{E,ln}(k) a^+(k) a^+(-k) \right)^s \times \\
& \times \sum_{r=0}^{\infty} \frac{1}{r!} \left(\int f_0(k) a^+(k) a^+(-k) dk \right)^r |0\rangle = c e^{-\beta H - \beta_1 \hat{M}^2} \Phi, \\
& c = \int v(p) \left[f_0(p) e^{-\beta E_0} + \int_{E_1}^{E_2} f_{E,0}(p) e^{-\beta(E-2\mu)} \alpha(E) dE \right] dp.
\end{aligned} \tag{3.6}$$

We used orthogonality of $v(k)$ to $f_{E,ln}(k)$ with $l \geq 1$.

Now substitute obtained expression in the second term of the right-hand side of (3.4).

One obtains

$$\begin{aligned}
& \frac{v(p_2)}{\Xi} \frac{(2\pi)^3}{V} \int v(p') \langle \Phi, a^+(t_1, p_1) a^+(t_2, -p_2) a(t_2, -p') a(t_2, p') e^{-\beta H - \beta_1 \hat{M}^2} \Phi \rangle = \\
& = c \frac{v(p_2)}{\Xi} \langle \Phi, a^+(t_1 - t_2, p_1) a^+(0, -p_2) e^{-\beta H - \beta_1 \hat{M}^2} \Phi \rangle = \\
& = cv(p_2) \langle a^+(t_1, p_1), a^+(t_2, p_2) \rangle.
\end{aligned}$$

We used again that averages are invariant with respect to cyclic permutation of operators and therefore $\langle a^+(t_1 - t_2, p_1) a^+(0, p_2) \rangle = \langle a^+(t_1, p_1) a^+(t_2, p_2) \rangle$.

Taking the last formula into account one obtains equation (3.4) in the final form

$$\begin{aligned}
& i \frac{\partial}{\partial t_2} \langle a^+(t_1, p_1) a(t_2, p_2) \rangle = \\
& = \left(\frac{p_2^2}{2m} - \mu \right) \langle a^+(t_1, p_1) a(t_2, p_2) \rangle + cv(p_2) \langle a^+(t_1, p_1) a^+(t_2, -p_2) \rangle.
\end{aligned} \tag{3.7}$$

Derive equation for the third correlation function (3.3). One obtains

$$\begin{aligned}
& i \frac{\partial}{\partial t_2} \langle a(t_1, p_1) a(t_2, p_2) \rangle = \\
& = \left(\frac{p_2^2}{2m} - \mu \right) \langle a(t_1, p_1) a(t_2, p_2) \rangle + \\
& + v(p_2) \frac{(2\pi)^3}{V} \int v(p') \langle a(t_1, p_1) a^+(t_2, -p_2) a(t_2, -p') a(t_2, p') \rangle dp'.
\end{aligned} \tag{3.8}$$

By using formulae (3.6) in the second term of the right-hand side of (3.8) one represent equation (3.8) in the following form:

$$\begin{aligned}
& i \frac{\partial}{\partial t_2} \langle a(t_1, p_1) a(t_2, p_2) \rangle = \\
& = \left(\frac{p_2^2}{2m} - \mu \right) \langle a(t_1, p_1) a(t_2, p_2) \rangle + cv(p_2) \langle a(t_1, p_1) a^+(t_2, -p_2) \rangle.
\end{aligned} \tag{3.9}$$

Equation for the second correlation function one obtains by conjugation of the equation, for $\langle a(t_1, p_1)a(t_2, p_2) \rangle$

$$\begin{aligned}
 & -i \frac{\partial}{\partial t_2} \langle a^+(t_2, p_2)a^+(t_1, p_1) \rangle = \\
 & = \left(\frac{p_2^2}{2m} - \mu \right) \langle a^+(t_2, p_2)a^+(t_1, p_1) \rangle + \bar{c}v(p_2) \langle a(t_2, -p_2)a^+(t_1, p_1) \rangle. \tag{3.10}
 \end{aligned}$$

Recall that $f_{E,0}(k)$ is complex function and therefore constant c is complex number.

3.2. Cluster property and equation for Green functions. In the second term of the right-hand side of (3.4) one is faced with the problem of giving a meaning to the integral with inverse volume V . The analogical problem has been already solved for equations for correlation function for models of superconductivity, superfluidity, Huang–Yang–Luttinger model and other model [15]. We will follow the method used for above mentioned models.

Namely we suppose that correlation functions satisfy the cluster property. For example

$$\begin{aligned}
 & \langle a^+(t_1, p_1)a^+(t_2, p_2)a(t_2, -p')a^+(t_2, p') \rangle = \\
 & = \langle a^+(t_1, p_1)a^+(t_2, p_2) \rangle \langle a(t_2, -p')a(t_2, p') \rangle + \dots, \tag{3.11}
 \end{aligned}$$

where the rest terms consist from sum of all possible two particle correlation functions.

If one substitutes the right-hand side of (3.11) in (3.4) then one obtains

$$\begin{aligned}
 & i \frac{\partial}{\partial t_2} \langle a^+(t_1, p_1)a(t_2, p_2) \rangle = \\
 & = \left(\frac{p_2^2}{2m} - \mu \right) \langle a^+(t_1, p_1)a(t_2, p_2) \rangle + cv(p_2) \langle a^+(t_1, p_1)a^+(t_2, -p_2) \rangle,
 \end{aligned}$$

where

$$c = \frac{(2\pi)^3}{V} \int v(p') \langle a(0, -p')a(0, p') \rangle dp'. \tag{3.12}$$

It was used that correlation function $\langle a(t_2, -p')a(t_2, p') \rangle$ does not depend on time (it depends on difference of times of operators $a(t_2, -p')$ and $a(t_2, p')$).

One can calculate $\frac{(2\pi)^3}{V} \int v(p') \langle a(0, -p')a(0, p') \rangle dp'$ exactly as in Subsection 3.1

$$\frac{(\Phi, \frac{(2\pi)^3}{V} \int v(p')a(0, -p')a(0, p')dp' e^{-\beta H - \beta_1 \hat{M}^2} \Phi)}{(\Phi, e^{-\beta H - \beta_1 \hat{M}^2} \Phi)} = c.$$

Thus we obtained the same result as before in Subsection 3.1.

4. Model and approximating Hamiltonian. 4.1. Model Hamiltonian on state Φ .

Consider model BCS Hamiltonian on the state Φ

$$\begin{aligned}
 H\Phi & = \left[\int a^+(\bar{k}) \left(\frac{k^2}{2m} - \mu \right) a(\bar{k}) d\bar{k} + \right. \\
 & \left. + \frac{(2\pi)^3}{V} \int \int v(k)a^+(k)a^+(-k)v(k')a(-k')a(k')dkdk' \right] \times \\
 & \quad \times e^{\int f(k)a^+(k)a^+(-k)dk} e^{\int f_0(k)a^+(k)a^+(-k)dk} |0\rangle = \\
 & = \left[\int a^+(\bar{k}) \left(\frac{k^2}{2m} - \mu \right) a(\bar{k}) d\bar{k} + c \int v(k)a^+(k)a^+(-k)dk \right] \Phi, \tag{4.1}
 \end{aligned}$$

where

$$c = \int v(p) \left[f_0(p) + \int_{E_1}^{E_2} dE f_{E,0}(p) \right] dp. \quad (4.2)$$

Calculating $H\Phi$ one uses that

$$\begin{aligned} & \frac{(2\pi)^3}{V} \int v(k') a(-k') a(k') dk' \int f(k) a^+(k) a^+(-k) dk |0\rangle = \\ & = \frac{(2\pi)^3}{V} \int \delta(k-k) v(k) f(k) dk |0\rangle = \frac{V}{V} \int v(k) \left(\int_{E_1}^{E_2} f_{E,0}(k) dE \right) dk \end{aligned}$$

because $\int f_{E,l}(k) v(k) dk = 0$ for $l \geq 1$. It was also used that

$$\begin{aligned} & \frac{(2\pi)^3}{V} \int v(k') a(-k') a(k') dk' \int f_0(k) a^+(k) a^+(-k) dk |0\rangle = \\ & = \frac{(2\pi)^3}{V} \int \delta(k-k) v(k) f_0(k) dk |0\rangle = \frac{V}{V} \int v(k) f_0(k) dk |0\rangle. \end{aligned}$$

Taking into account that

$$\Phi = e^{\int f(k) a^+(k) a^+(-k) dk} e^{\int f_0(k) a^+(k) a^+(-k) dk} |0\rangle$$

one obtains that

$$\frac{(2\pi)^3}{V} \int v(k') a(-k') a(k') dk' \Phi = c\Phi. \quad (4.3)$$

4.2. Approximating Hamiltonian on state Φ . Define the approximating Hamiltonian

$$\begin{aligned} H_a &= \int a^+(\bar{k}) \left(\frac{k^2}{2m} - \mu \right) a(\bar{k}) d\bar{k} + \\ &+ c \int v(k) a^+(k) a^+(-k) dk + \bar{c} \int v(k) a(-k) a(k) dk - |c|^2 V, \end{aligned} \quad (4.4)$$

where c is defined according to (4.2). Show that

$$H\Phi = H_a\Phi. \quad (4.5)$$

The action $H\Phi$ was already calculated (4.1), (4.2). To prove (4.5) it is sufficient to show that

$$\left(\bar{c} \int v(k) a(-k) a(k) dk - |c|^2 V \right) \Phi = 0, \quad (4.6)$$

but it is simple consequence of (4.3).

Remark 4.1. It is obvious that

$$\Phi = e^{\int f(k) a^+(k) a^+(-k) dk} e^{\int f_0(k) a^+(k) a^+(-k) dk} |0\rangle = e^{\int [f(k)+f_0(k)] a^+(k) a^+(-k) dk} |0\rangle.$$

Include $f^0(k)$ in $f_0(k)$ and introduce the functions

$$\tilde{f}_0(k) = f_0(k) + f^0(k), \quad \tilde{f}(k) = \sum_{l=1}^{\infty} f^l(k),$$

and define the following vectors:

$$\tilde{\Phi}_0 = e^{\int \tilde{f}_0(k)a^+(k)a^+(-k)dk} |0\rangle,$$

$$\tilde{\Phi}_s = \frac{1}{s!} \int \tilde{f}(k_1)a^+(k_1)a^+(-k_1)dk_1 \dots \int \tilde{f}(k_s)a^+(k_s)a^+(-k_s)dk_s \tilde{\Phi}_0, \quad s \geq 0.$$

The vector Φ can be represented as follows:

$$\begin{aligned} \Phi &= \sum_{s=0}^{\infty} \tilde{\Phi}_s = e^{\int \tilde{f}(k)a^+(k)a^+(-k)dk} e^{\int \tilde{f}_0(k)a^+(k)a^+(-k)dk} |0\rangle = \\ &= e^{\int [\tilde{f}(k)+\tilde{f}_0(k)]a^+(k)a^+(-k)dk} |0\rangle = e^{\int [f(k)+f_0(k)]a^+(k)a^+(-k)dk} |0\rangle = \\ &= e^{\int f(k)a^+(k)a^+(-k)dk} e^{\int f_0(k)a^+(k)a^+(-k)dk} |0\rangle. \end{aligned}$$

We have proved that $H\Phi = H_a\Phi$, but we also have $H\tilde{\Phi}_s = H_a\tilde{\Phi}_s, s \geq 0$, because

$$\frac{(2\pi)^3}{V} \int v(p)a(-p)a(p)dp \tilde{\Phi}_s = c\tilde{\Phi}_s, \quad \int v(p)\tilde{f}(p)dp = 0,$$

where as above $c = \int v(p)[f_0(p) + f^0(p)]dp$, and it was used that $\int v(p)\tilde{f}(p)dp = 0$.

This means that H coincides with H_a not only on vectors Φ and $\tilde{\Phi}_0$ but also on all the excitations $\tilde{\Phi}_s, s \geq 1$, of the state $\tilde{\Phi}_0$. Excitations $\tilde{\Phi}_s$ are orthogonal to $\tilde{\Phi}_0$ and to themselves and linear combinations of $\tilde{\Phi}_0, \tilde{\Phi}_s, s \geq 1$, form the Hilbert space of states. On this Hilbert space the BCS and approximating Hamiltonians coincide.

The approximating Hamiltonian has its own branch of eigenvalues and eigenvectors. It is operator — quadratic form of operators of creations and annihilations and therefore it can be diagonalized

$$H_a = \int E(k)\alpha^+(\bar{k})\alpha(\bar{k})d\bar{k} + C(c)V, \tag{4.7}$$

where

$$E(k) = \sqrt{\varepsilon(k)^2 + c^2v^2(k)}, \quad C(c) = \int [\varepsilon(k) - \sqrt{\varepsilon(k)^2 + c^2v^2(k)}] dk - c^2, \tag{4.8}$$

$$\varepsilon(k) = \frac{k^2}{2m} - \mu.$$

The operators $\alpha^+(\bar{k}), \alpha(\bar{k})$ satisfy the same canonical anticommutation relations as the operators $a^+(\bar{k}), a(k)$ and are expressed through the operators $a^+(\bar{k}), a(k)$ by the following formulae [9, 16]:

$$\begin{aligned} \alpha^+(k) &= u(k)a^+(k) + w(k)a(-k), \quad \alpha^+(-k) = u(k)a^+(-k) - w(k)a(k), \\ \alpha(k) &= u(k)a(k) + w(k)a^+(-k), \quad \alpha(-k) = u(k)a(-k) - w(k)a(k), \end{aligned} \tag{4.9}$$

$$u(k) = \frac{1}{\sqrt{2}}(1 + \varepsilon(k)(\varepsilon^2(k) + c^2v^2(k))^{-\frac{1}{2}})^{\frac{1}{2}},$$

$$w(k) = \frac{1}{\sqrt{2}}(1 - \varepsilon(k)(\varepsilon^2(k) + c^2v^2(k))^{-\frac{1}{2}})^{\frac{1}{2}}.$$

The approximating Hamiltonian H_a (4.4) has the following eigenvectors:

$$\Phi_0^a = e^{\int f_0^a(k)a^+(k)a^+(-k)dk} |0\rangle, \tag{4.10}$$

where

$$f_0^a(k) = -\left((\varepsilon^2(k) + c^2 v^2(k))^{\frac{1}{2}} - \varepsilon(k)\right)^{\frac{1}{2}} \left((\varepsilon^2(k) + c^2 v^2(k))^{\frac{1}{2}} + \varepsilon(k)\right)^{-\frac{1}{2}}. \quad (4.11)$$

Eigenvector Φ_0^a is the vacuum for operators $\alpha^+(\bar{k}), \alpha(\bar{k})$ because $\alpha(\bar{k})\Phi_0^a = 0$, and eigenvector for H_a

$$H_a \Phi_0^a = C(c) V \Phi_0^a. \quad (4.12)$$

The operator H_a has also eigenvectors $\alpha^+(\bar{k}_1) \alpha \dots \alpha^+(\bar{k}_n) \Phi_0^a$

$$H_a \alpha^+(\bar{k}_1) \dots \alpha^+(\bar{k}_n) \Phi_0^a = (E(k_1) + \dots + E(k_n)) \alpha^+(\bar{k}_1) \dots \alpha^+(\bar{k}_n) \Phi_0^a. \quad (4.13)$$

(Note that in this case the constant c is defined from condition of minimum of the function $C(c)$ and it is different from that defined according to (4.2).)

These eigenvectors are n -particle excitations of Φ_0^a . The eigenvectors (4.11), (4.14) with $\bar{k}_i \neq -\bar{k}_j$, $(i, j) \subset (1, \dots, n)$ are also eigenvectors of the operator H with the same eigenvalues (see [8, 9]).

To eigenvectors (4.14) corresponds the following grand partition function

$$\Xi_a = e^{V \int \frac{dp}{e^{\beta E(p)} + 1} + V C(c)},$$

i.e., the grand partition function of free system of fermions with energy $E(p)$. Recall again that in this subsection the constant c is defined according to (4.2).

Remark 4.2. Note that the wave function of one pair of electrons with opposite momenta and spin $f_0^a(k)$ is not the eigenfunction of the two particle Hamiltonian H_2 . The wave function of $n \geq 2$ pairs $f_0^a(k_1) \dots f_0^a(k_n)$ also are not the eigenfunction of the BCS Hamiltonian. Only the coherent state of pairs Φ_0^a is eigenfunction of the BCS Hamiltonian in the following sense [5, 9]

$$\lim_{V \rightarrow \infty} \frac{1}{V} (\Phi_{0,\Lambda}^a, H_\Lambda \Phi_{0,\Lambda}^a) = \lim_{V \rightarrow \infty} \frac{1}{V} (\Phi_{0,\Lambda}^a, H_{a,\Lambda} \Phi_{0,\Lambda}^a),$$

where $\Phi_{0,\Lambda}^a, H_{a,\Lambda}, H_\Lambda$ are the restriction of Φ_0^a, H_a, H in a cube Λ . This formula is also true if one puts excitations (4.14) instead of Φ_0^a .

Usually in presentation of the theory of superconductivity authors begin with equation (2) for the Cooper pair in the ground state $f_0(k)$, but later continue with coherent state (4.11), from condition of minimum of $(\Phi_{0,\Lambda}^a, H_\Lambda \Phi_{0,\Lambda}^a)$ define $f_0^a(k)$ (4.12) that is different from the wave function of the original Cooper pair $f_0(k)$ and is not eigenfunction of H_2 . They also consider excitation of the ground state Φ_0^a (see for example [17]).

It is surprise that the eigenvalue problem has not been considered for H in $2n$ -particle space in spite of promise made by Cooper in his pioneering paper [10]. This problem has solved in our papers [1–6].

4.3. Approximating Hamiltonian and Green functions. Consider the approximating Hamiltonian (4.4) but with parameter c defined as in equations for Green functions

$$c = \int v(p) \left[f_0(p) e^{-\beta E_0} + \int_{E_1}^{E_2} f_{E,0}(p) e^{-\beta(E-2\mu)} \alpha(E) dE \right] dp. \quad (4.14)$$

It is easy to check that equations (3.7)–(3.10) for correlations functions of the BSC model Hamiltonian H completely coincide with equation for the same correlation function of the approximating Hamiltonian H_a (4.4) but with the constant (4.15).

We omit almost obvious calculation for arbitrary correlation functions and simple proof that the equations for them for the BCS model Hamiltonian H coincide with the corresponding equations for the approximating Hamiltonian H_a .

Remark 4.3. The sequence of correlation functions do not depend on the last fourth term in (4.4). In our case of the second branch of eigenvalues and eigenvectors the fourth term should be determined from condition of coincidence of the free energy of the Hamiltonian (4.4) (with unknown fourth term) with the free energy (2.19).

4.4. Pseudogap. We have proved that equations for correlation functions of the model with the BCS Hamiltonian H and those equations of the model with approximating Hamiltonian H_a with constant c (4.15) coincide. This means that their solutions, i.e., correlation functions also coincide if initial data coincide. In this sense model with the BCS Hamiltonian H and model with the approximating Hamiltonian H_a are thermodynamically equivalent, i.e., their states, described by correlation functions, coincide.

At first sight we obtained well known result, first established by Bogolyubov [9] for zero temperature and by Bogolyubov [16] for arbitrary temperatures concerning thermodynamic equivalence of the BCS and approximating Hamiltonians.

In fact there is one fundamental difference. It consists in the following. In our case for the second branch of eigenvalues and eigenvectors the constant

$$c = \int v(p) \left[f_0(p)e^{-\beta E_0} + \int_{E_1}^{E_2} f_{E,0}(p)e^{-\beta(E-2\mu)}\alpha(E)dE \right] dp$$

in the approximating Hamiltonian H_a is calculated exactly and it does depend on temperature, and it is different from zero for arbitrary $\beta > 0$.

In Bogolyubov’s cases [9, 16] constant c in his approximating Hamiltonian is solution of nonlinear equation that defines minimum of free energy and it also depends on temperature. It is different from zero below certain critical temperature T_c and equal to zero for temperature greater than T_c .

The constant c defines the gap in spectrum of excitation, namely one particle excitation with momentum p has the following energy:

$$E(p) = \sqrt{\left(\frac{p^2}{2m} - \mu\right) + |c|^2v^2(p)}$$

with the gap $\Delta(p) = |c|^2v^2(p)$.

In our case for the second branch of eigenvalues and eigenvectors the constant c also depends on temperature, the one particle excitation with momenta p has also the following energy

$$E(p) = \sqrt{\left(\frac{p^2}{2m} - \mu\right) + |c|^2v^2(p)}$$

with the gap $\Delta(p) = |c|^2v^2(p)$, but the gap in our case does not vanish for temperature greater than T_c , i.e., in the BCS model exists “pseudogap” that depends also on temperature and exists for all the temperatures:

$$c = \int v(p) \left[f_0(p)e^{-\beta E_0} + \int_{E_1}^{E_2} f_{E,0}(p)e^{-\beta(E-2\mu)}dE \right] dp.$$

Note that for the second branch of eigenvalues and eigenvectors the approximating Hamiltonian appears in two different cases. In the first one it coincides with BCS Hamiltonian H on Φ , $(\tilde{\Phi}_0, \tilde{\Phi}_s)$ and with the constant c (4.2). In the second one it appears

in the equations for the correlation functions, i.e., equations for correlation functions defined for model with BCS Hamiltonian H coincide with those equations defined for model with the approximating Hamiltonian H_a , but the constant c does depends on

$$\text{temperature } c = \int v(p) \left[f_0(p)e^{-\beta E_0} + \int_{E_1}^{E_2} f_{E,0}(p)e^{-\beta(E-2\mu)} dE \right] dp.$$

4.5. Three phases of BCS model. The model Hamiltonian has two branches of eigenvalues and eigenfunctions. It has been well known ground state Φ_0^g (4.11) and its excitations (4.14) and constant c has been defined from condition of minimum of energy of ground state $C(c)$ per volume (for zero temperature). For different from zero temperature constant c is defined from condition of minimum of free energy per volume or, that is the same, from equations for correlation function. In this case constant c depend on temperature, it is different from zero for temperature less than critical temperature T_c , $T < T_c$, and vanish for temperature greater than T_c , $T > T_c$. This branch of spectra is associated with superconductivity.

We have showed [1–4] that the BCS hamiltonian H has the second branch of eigenvalues and eigenvectors, namely ground state Φ_0 (2.2) and excited states Φ_s (2.3) (or $\tilde{\Phi}_0, \tilde{\Phi}_s$ (4.7)).

We calculated the grand partition function and correlation functions, that coincide with correlation functions of the approximating Hamiltonian H_a but for second branch constant c (4.15) also depends on temperature and it is different from zero for arbitrary temperature. The gap in spectrum $\Delta = |c|^2 v^2(p)$ is different from zero for arbitrary temperature and it is known as “pseudogap.”

In fact, the BCS Hamiltonian has also the third branch of spectra, namely spectra of free system of electrons, that correspond to normal metal.

Summarizing above described three branches of spectra and eigenfunctions one can say that system with the BCS Hamiltonian can exist in three different phases:

- 1) superconducting phase with gap different from zero for $T < T_c$;
- 2) phase with ”pseudogap” different from zero for all the temperatures;
- 3) normal phase that corresponds to free system of electrons.

These phases correspond to three branches of spectra and eigenvectors. The first and third phases have been known, the second phase correspond to the second branch of spectra and eigenvectors has been recently discovered in our papers [1–6]. It is possible that for some temperature $T > T_c$ the system with Hamiltonian H is in the third normal phase and in this sense “pseudogap” disappears.

Remark 4.4. We calculated the grand partition function and the Green functions taking into account the all orbitel momente $l = 0, 1, 2, \dots$. From phisical point of view one should take into account only even $l = 0, 2, 4, \dots$.

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