UDC 519.21

S. Albeverio (Inst. Angew. Math.; Univ. Bonn, Germany),
M. Pratsiovytyi (Nat. Ped. Univ., Kyiv),
G. Torbin (Nat. Ped. Univ., Kyiv; Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

SINGULAR PROBABILITY DISTRIBUTIONS AND FRACTAL PROPERTIES OF SETS OF REAL NUMBERS DEFINED BY THE ASYMPTOTIC FREQUENCIES OF THEIR s-ADIC DIGITS

СИНГУЛЯРНІ ЙМОВІРНІСНІ РОЗПОДІЛИ ТА ФРАКТАЛЬНІ ВЛАСТИВОСТІ МНОЖИН ДІЙСНИХ ЧИСЕЛ, ЩО ЗАДАНІ АСИМПТОТИЧНОЮ ЧАСТОТОЮ ЇХ s-АДИЧНИХ ЦИФР

Dedicated to V. S. Korolyuk
on occasion of his 80-th birthday

Properties of the set T_s of "particularly nonnormal numbers" of the unit interval are studied in details (T_s consists of real numbers x, some of whose s-adic digits have the asymptotic frequencies in the nonterminating s-adic expansion of x, and some do not). It is proven that the set T_s is residual in the topological sense (i.e., it is of the first Baire category) and it is generic in the sense of fractal geometry (T_s is a superfractal set, i.e., its Hausdorff – Besicovitch dimension is equal to 1). A topological and fractal classification of sets of real numbers via analysis of asymptotic frequencies of digits in their s-adic expansions is presented.

Детально вивчаються властивості множини T_s „особливо ненормальних чисел" одиничного інтервалу (тобто множини чисел x, для яких немає асимптотичної частоти деяких цифр в s-адичному зображенні, а деякі цифри мають асимптотичні частоти). Доведено, що множина T_s є нехтуваною в топологічному сенсі (першої категорії Бера) та загальною в сенсі фрактальної геометрії (T_s є суперфрактальною множиною, розмірність Хаусдорфа – Безиковича якої дорівнює одиниці). Наведено топологічну і фрактальну класифікацію множин дійсних чисел через аналіз асимптотичної частоти їх s-адичних зображень.

1. Introduction. Let us consider the classical s-adic expansion of $x \in [0,1]$:

$$x = \sum_{n=1}^{\infty} s^{-n}\alpha_n(x) = \Delta^s \alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)\ldots, \quad \alpha_k(x) \in A = \{0,1,\ldots,(s-1)\},$$

and let $N_i(x,k)$ be the number of digits "i" among the first k digits of the s-adic expansion of x , $i \in A$. If the limit $\nu_i(x) = \lim\limits_{k\to\infty} \dfrac{N_i(x,k)}{k}$ exists, then the number $\nu_i(x)$ is said to be the frequency of the digit "i" (or the asymptotic frequency of "i") in the s-adic expansion of x .

A property of an element $x \in M$ is usually said to be "normal" if "almost all" elements of M have this property. There exist many mathematical notions (e.g., cardinality, measure, Hausdorff – Besicovitch dimension, Baire category) allowing us to interpret the words "almost all" in a rigorous mathematical sense. "Normal" properties of real numbers are deeply connected with the asymptotic frequencies of their digits in some systems of representation.

The set

$$N_s = \left\{ x \,\middle|\, (\forall i \in A) \ \lim_{k\to\infty} \frac{N_i(x,k)}{k} = \frac{1}{s} \right\}$$

is said to be the set of s-*normal* numbers (or the set of real numbers which are normal with respect to the base s). It is well known (E. Borel, 1909), that the sets N_s and the set $N^* = \bigcap\limits_{s=2}^{\infty} N_s$ are of full Lebesgue measure (i.e., they have Lebesgue measure 1).

The unit interval $[0,1]$ can be decomposed in the following way:

$$[0,1] = E_s \bigcup D_s,$$

where

$$E_s = \{x | \nu_i(x) \quad \text{exists} \quad \forall i \in A\},$$

$$D_s = \left\{ x \middle| \exists i \in A, \ \lim_{k\to\infty} \frac{N_i(x,k)}{k} \ \text{does not exist} \right\}.$$

The set D_s is said to be the set of nonnormal real numbers. Each of the subsets E_s and D_s can be decomposed in the following natural way.

The set

$$W_s = \left\{ x \middle| (\forall i \in A): \ \nu_i(x) \ \text{exists and} \ (\exists j \in A): \ \nu_j(x) \neq \frac{1}{s} \right\}$$

is said to be the set of *quasinormal* numbers. It is evident that

$$E_s = W_s \bigcup N_s, \qquad W_s \bigcap N_s = \varnothing.$$

The set

$$L_s = \left\{ x \middle| (\forall i \in A) \ \lim_{k\to\infty} \frac{N_i(x,k)}{k} \ \text{does not exist} \right\}$$

is said to be the set of *essentially nonnormal* numbers.

The set

$$T_s = \left\{ x \middle| (\exists i \in A): \ \lim_{k\to\infty} \frac{N_i(x,k)}{k} \ \text{does not exist, and} \right.$$
$$\left. (\exists j \in A): \ \lim_{k\to\infty} \frac{N_j(x,k)}{k} \ \text{exists} \right\}$$

is said to be the set of *particularly nonnormal* numbers.

It is evident that

$$D_s = L_s \bigcup T_s, \qquad L_s \bigcap T_s = \varnothing.$$

The sets N_s, W_s, T_s, L_s are everywhere dense sets, because the frequencies $\nu_i(x)$ do not depend on any finite number of s-adic symbols of x. It is also not hard to prove that these sets have the cardinality of the continuum.

The main purpose of the paper is to fill in completely the following table, which reflects the metric, topological and fractal properties of the corresponding sets:

	Lebesgue measure	Hausdorff dimension	Baire category
N_s			
W_s			
L_s			
T_s			

Let $\nu = (\nu_0, \nu_1, \ldots, \nu_{s-1})$ be a stochastic vector and let

$$W_s[\nu] = \Bigg\{ x : \quad x = \Delta^s \alpha_1(x)\alpha_2(x) \ldots \alpha_k(x) \ldots ,$$

$$\lim_{k\to\infty} \frac{N_i^*(x,k)}{k} = \nu_i \quad \forall i \in A \Bigg\}.$$

The well known Besicovitch – Eggleston's theorem (see, e.g., [1, 2]) gives the following formulae for the determination of the Hausdorff – Besicovitch dimension $\alpha_0 \left(W_s[\nu] \right)$ of the set $W_s[\nu]$:

$$\alpha_0 \left(W_s[\nu] \right) = \frac{\sum_{k=0}^{s-1} \nu_i \log \nu_i}{- \log s}.$$

From the latter formulae it easily follows that the set W_s of all quasinormal numbers is a superfractal set, i.e., W_s is a set of zero Lebesgue measure with full Hausdorff – Besicovitch dimension ($\alpha_0 \left(W_s \right) = 1$).

Properties of subsets of the set of nonnormal numbers have been intensively studied during recent years (see, e.g., [3 – 6] and references therein). Some interesting subsets of D_s were studied in [4] by using the techniques and results from the theory of multifractal divergence points. In [3] it has been proven that the set D_s is superfractal.

In the paper [7] of the authors it has been proven that the set L_s of essentially nonnormal numbers is also superfractal and it is of the second Baire category. Moreover, it has been proven that the set L_s contains an everywhere dense G_δ -set. So, the sets N_s, W_s, T_s are of the first Baire category. From these results it follows that essentially nonnormal numbers are generic in the topological sense as well as in the sense of fractal geometry; nevertheless, the set L_s is small from the point of view of Lebesgue measure.

The main goal of the present paper is the investigation of fractal properties of the set T_s of particularly nonnormal numbers. To this end we apply a probabilistic approach for the calculation of the Hausdorff dimension of subsets. More precisely, we apply the results of fine fractal analysis of singular continuous probability distributions.

The first step of the fractal analysis of a singular continuous measure ν is the investigation of metric, topological and fractal properties of the corresponding topological support S_ν (i.e., the minimal closed set supporting the measure). These are good characteristics only for the class of uniform Cantor-type singular measures. But, in general, they are only "external characteristics", because there exist essentially different singular continuous measures concentrating on the common topological support. The main idea of the paper [7] consisted in the construction of singular continuous measures whose topological supports coincide with some subsets of the set of essentially nonnormal numbers.

The second step of the fractal analysis of a singular continuous measure ν is the determination of the Hausdorff dimension $\alpha_0(\nu)$ (and the local Hausdorff dimension) of the measure, i.e., roughly speaking, finding the Hausdorff dimension of the minimal (in the fractal dimension sense) supports (which are not necessarily closed) of the measure. This problem is much more complicated than the previous one (see, e.g., [8]), especially in the case of essentially superfractal measures.

In Section 2 we prove that for all $s \geq 3$ the set T_s is of full Hausdorff dimension. To prove the main result we construct a sequence of singular continuous measures μ_p such that the corresponding minimal dimensional supports consist of only particularly nonnormal numbers, and apply the results of [8] to perform a fine fractal analysis of these supports.

2. Fractal properties of the set of particularly nonnormal numbers. Let us study the sets T_s of particularly nonnormal numbers which were defined in Section 1. It is easy to see that the set T_2 is empty, because from the existence of the asymptotic frequency $\nu_i(x)$ for some $i \in \{0,1\}$ the existence of another asymptotic frequency follows.

Theorem 1. *For any positive integer $s \geq 3$ the set T_s of particularly nonnormal real numbers is superfractal, i.e., the Hausdorff–Besicovitch dimension of the set T_s equals* 1.

Proof. To prove the theorem we shall construct a superfractal set $G \subset T_s$.

In the sequel we usually shall not use the indices s in the notation of the corresponding subsets, since s will be an arbitrary fixed natural number greater than 2. Let us consider the classical s-adic expansion of $x \in [0,1]$: $x = \sum_{n=1}^{\infty} s^{-n}\alpha_n(x) = \Delta^s\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)\ldots$. If x is an s-adic rational number, then we shall use the representation without the period $s-1$.

For a given $p \in N$ and for any $x \in [0,1)$ we define the following mapping φ_p :

$$\varphi_p(x) = \varphi_p\left(\Delta^s\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)\ldots\right) =$$

$$= \Delta^s\overbrace{00\ldots0}^{s-1}\overbrace{11\ldots1}^{s-1}\ldots\overbrace{(s-2)(s-2)\ldots(s-2)}^{s-1}(s-1)\alpha_1(x)\alpha_2(x)\ldots\alpha_{s^2p}(x)$$

$$\overbrace{00\ldots0}^{2(s-1)}\overbrace{11\ldots1}^{2(s-1)}\ldots\overbrace{(s-2)(s-2)\ldots(s-2)}^{2(s-1)}$$

$$(s-1)(s-1)\alpha_{s^2p+1}(x)\alpha_{s^2p+2}(x)\ldots\alpha_{s^2p+2s^2p}(x)\ldots$$

$$\ldots\ \overbrace{00\ldots0}^{2^{k-1}(s-1)}\ 11\ldots1\ldots\overbrace{(s-2)(s-2)\ldots(s-2)}^{2^{k-1}(s-1)}\overbrace{(s-1)(s-1)\ldots(s-1)}^{2^{k-1}}$$

$$\alpha_{(2^{k-1}-1)s^2p+1}(x)\ldots\alpha_{(2^k-1)s^2p}(x)\ldots.$$

Let us explain the construction of φ_p. First of all we divide the s-adic expansion of x into groups in the following way: the k-th group consists of the sequence $(\alpha_{(2^{k-1}-1)s^2p+1}(x)\ldots\alpha_{(2^k-1)s^2p}(x))$, $k \in N$. The s-adic expansion of $y = \varphi_p(x)$ is constructed from the s-adic expansion of x via inserting (before the k-th group) the following series of fixed symbols $(0\ldots01\ldots1\ldots(s-2)\ldots(s-2)(s-1)\ldots(s-1))$, where each symbol i $(0 \leq i \leq s-2)$ occurs $2^{k-1}(s-1)$ times, but the symbol $s-1$ occurs 2^{k-1} times.

Let $M_p = \varphi_p([0,1)) = \{y : y = \varphi_p(x), x \in [0,1)\}$.

For a given $p \in N$ and for any $y \in M_p$ we define the mapping $\psi_p(y)$ in the following way: if

$$y = \varphi_p(x) = \Delta^s\overbrace{00\ldots0}^{s-1}\ldots\overbrace{(s-2)\ldots(s-2)}^{s-1}(s-1)\alpha_1(x)\alpha_2(x)\ldots\alpha_{s^2p}(x)$$

$$\overbrace{0\ldots0}^{2(s-1)}\ldots\overbrace{(s-2)\ldots(s-2)}^{2(s-1)}(s-1)(s-1)$$

$$\alpha_{s^2p+1}(x)\alpha_{s^2p+2}(x)\ldots\alpha_{s^2p+2s^2p}(x)\ldots,$$

then

$$z = \psi_p(y) = \Delta^s \overbrace{0 \ldots 0}^{s-1}(s-1)\ldots \overbrace{(s-2)\ldots(s-2)}^{s-1}\ldots$$

$$\ldots(s-1)(s-1)\,(01\ldots(s-2))\,\alpha_1(x)\alpha_2(x)\ldots\alpha_{s^2p}(x)$$

$$\overbrace{00\ldots0}^{(s-1)}(s-1)\overbrace{00\ldots0}^{(s-1)}(s-1)\ldots\overbrace{(s-2)(s-2)\ldots(s-2)}^{(s-1)}$$

$$(s-1)\overbrace{(s-2)(s-2)\ldots(s-2)}^{(s-1)}(s-1)$$

$$(s-1)\,(01\ldots(s-2))\,(s-1)\,(01\ldots(s-2))\,\alpha_{s^2p+1}(x)\alpha_{s^2p+2}(x)\ldots$$

$$\ldots\alpha_{s^2p+2s^2p}(x)\ldots, \quad x \in [0,1),$$

i.e., the s-adic expansion of $z = \psi_p(y)$ can be obtained from the s-adic expansion of $y = \varphi(x)$ by using the following algorithm:

1) after any fixed symbol $(s-1)$ we insert the following series of symbols: $(01\ldots(s-$ $-2))$;

2) after any subseries consisting of $(s-1)$ fixed symbols i $(0 \le i \le s-2)$ we insert the symbol $s-1$.

Let $f_p = \psi_p(\varphi_p)$ and let

$$S_p = f_p([0,1)) = \{z : z = f_p(x),\ \ x \in [0,1)\} = \{z : z = \psi_p(y),\ \ y \in M_p\},$$

$$G_p = f_p([0,1)) = \{z : z = f_p(x),\ \ x \in N_s\}.$$

The following two lemmas will describe some properties of the constructed sets G_p.

Lemma 1. *For any* $z = \sum_{n=1}^{\infty} s^{-n}\alpha_n(z) \in G_p$ *the* $\lim\limits_{n\to\infty} \dfrac{N_i(z,n)}{n}$ *does not exist for any* $i \in \{0, 1, \ldots, s-2\}$, *and* $\lim\limits_{n\to\infty} \dfrac{N_{s-1}(z,n)}{n} = \dfrac{1}{s}$.

Proof. The set G_p has the following structure:

$$G_p =$$

$$= \Bigg\{ z{:}z{=}\Delta^s \underbrace{\overbrace{0\ldots0}^{s-1}(s-1)\ldots\overbrace{(s-2)\ldots(s-2)}^{s-1}(s-1)(s-1)(01\ldots(s-2))\alpha_1(x)\alpha_2(x)\ldots\alpha_{s^2p}(x)}_{\text{first group}}$$

$$\underbrace{0\ldots0(s-1)0\ldots0(s-1)\ldots(s-2)\ldots(s-2)(s-1)(s-2)\ldots(s-2)(s-1)(s-1)(01\ldots}_{\text{second}}$$

$$\underbrace{\ldots(s-2))(s-1)(01\ldots(s-2))\alpha_{s^2p+1}(x)\alpha_{s^2p+2}(x)\ldots\alpha_{s^2p+2s^2p}(x)}_{\text{group}}\ \ldots, \quad x{\in}N_s \Bigg\}.$$

From $x \in N_s$ it follows that the symbol $s-1$ has the asymptotic frequency $\dfrac{1}{s}$ in the sequence $\{\alpha_k(x)\}$ and the equality $\lim\limits_{n\to\infty} \dfrac{N_{s-1}(z,n)}{n} = \dfrac{1}{s}$ follows from the construction of the set G_p.

Let l_k be the number of the position at which the above k-th group of symbols ended, i.e., $l_k = s^2(p+1)(2^k - 1)$.

Let $m'_k(i)$ be the number of the position at which the k-th series of the fixed symbols i and $(s-1)$ $(0 \le i \le s-2)$ ended, i.e., $m'_{k+1}(i) = s^2(p+1)(2^k-1) + s(i+1)2^k$.

Let $m''_k(i)$ be the number of the position at which the k-th series of the fixed symbols i $(0 \le i \le s-2)$ started, i.e., $m''_{k+1}(i) = s^2(p+1)(2^k-1) + si2^k + 1$.

If $z \in G_p$, then there are $s(2^{k+1}-1) + d_k)$ symbols i $(0 \le i \le s-2)$ among the first $m'_{k+1}(i)$ symbols of the s-adic expansion of z, where d_k is the quantity of the symbol i among the first $(2^k-1)s^2p$ s-adic symbols $\alpha_i(x)$ in the expansion of $x = f_p^{-1}(z)$. Since x is an s-normal number, we have $d_k = (2^k-1)sp + \mathrm{o}(2^k)$.

So,

$$\lim_{n\to\infty} \frac{N_i(z, m'_{k+1}(i))}{m'_{k+1}(i)} =$$

$$= \lim_{n\to\infty} \frac{(2^{k+1}-1)s + (2^k-1)sp + s^{-1}\mathrm{o}(2^k)}{s^2(p+1)(2^k-1) + s(i+1)2^k} = \frac{p+2}{s(p+1)+i+1}.$$

If $z \in G_p$, then there are $s(2^k-1) + d_k$ symbols i $(0 \le i \le s-2)$ among the first $m''_{k+1}(i) - 1$ symbols of the s-adic expansion of z.

So,

$$\lim_{n\to\infty} \frac{N_i(z, m''_{k+1}(i)-1)}{m''_{k+1}(i)-1} = \lim_{n\to\infty} \frac{(2^k-1)s + (2^k-1)sp + s^{-1}\mathrm{o}(2^k)}{s^2(p+1)(2^k-1) + si2^k} =$$

$$= \frac{p+1}{s(p+1)+i} < \frac{p+2}{s(p+1)+i+1}.$$

Therefore, for any $z \in G_p$ and for any $i \in \{0, 1, \ldots, s-2\}$ the limit $\lim\limits_{n\to\infty} \dfrac{N_i(z,n)}{n}$ does not exist.

The lemma is proved.

The following Corollary is immediate, using the definitions of G_p, T_s and Lemma 1:

***Corollary* 1.** $G_p \subset T_s \ \ \forall p \in N$.

Lemma 2. *The Hausdorff–Besicovitch dimension of the set* G_p *is equal to* $\dfrac{p}{p+2}$.

Proof. Let $B_p(i)$ be the subset of N with the following property: $\forall k \in N$, $k \in B_p(i)$ if and only if $\alpha_k(f_p(x)) = i$ for any $x \in [0,1)$, i.e., $B_p(i)$ consists of the numbers of positions with the fixed symbols i in the s-adic expansion of any $z \in S_p$. Let $B_p = \bigcup\limits_{i=0}^{s-1} B_p(i)$, and let $C_p = N \setminus B_p$.

Let us consider the following random variable $\xi^{(p)}$ with independent s-adic digits:

$$\xi^{(p)} = \sum_{k=1}^{\infty} s^{-k}\, \xi_k^{(p)},$$

where $\xi_k^{(p)}$ are independent random variables with the following distributions: if $k \in B_p(i)$, then $\xi_k^{(p)}$ takes the value i with probability 1. If $k \in C_p$, then $\xi_k^{(p)}$ takes the values $0, 1, \ldots, (s-1)$ with probabilities $\dfrac{1}{s}, \dfrac{1}{s}, \ldots, \dfrac{1}{s}$.

It is evident that the set S_p is the topological support of the distribution of the random variable $\xi^{(p)}$. Actually, the corresponding probability measure $\mu_p = P_{\xi^{(p)}}$ is the image of Lebesgue measure on $[0, 1)$ under the mapping $f_p = \psi_p(\varphi_p)$, i.e., $\forall E \subset \mathcal{B}$: $\mu_p(E) = \mu_p(E \bigcap S_p) = \lambda(f_p^{-1}(E \bigcap S_p))$.

A. Firstly we prove that $\alpha_0(G_p) \leq \dfrac{p}{p+2}$. Since $G_p \subset S_p$, it is sufficient to show that $\alpha_0(S_p) \leq \dfrac{p}{p+2}$. To this end we consider the sequence $\{B_i^{(k)}\}$ $(k \in N,\ i \in \in \{1, 2, \ldots, s^{s^2p(2^{k-1}-1)}\})$ of special coverings of the set S_p by s-adic closed intervals of the rank $m_k = l_k - 2^{k-1}s^2p = s^2(p+1)(2^k - 1) - 2^{k-1}s^2p$. For any $k \in N$ the covering $\{B_i^{(k)}\}$ consists of the $s^{s^2p(2^{k-1}-1)}$ closed s-adic intervals of m_k-th rank with length $\varepsilon_k = s^{-(s^2(p+1)(2^k-1)-2^{k-1}s^2p)}$.

The α-volume of the covering $\{B_i^{(k)}\}$ is equal to

$$l^{\alpha}_{\varepsilon_k}(S_p) = s^{s^2p(2^{k-1}-1)} s^{-\alpha(s^2(p+1)(2^k-1)-2^{k-1}s^2p)} = s^{(p-\alpha(p+2))2^{k-1}s^2} s^{\alpha(p+1)-p}.$$

For the Hausdorff premeasure $h^{\alpha}_{\varepsilon_k}$ we have $h^{\alpha}_{\varepsilon_k}(S_p) \leq l^{\alpha}_{\varepsilon_k}(S_p)$ for any $k \in N$. So, for the Hausdorff measure H_α we have $H_\alpha(S_p) \leq \lim\limits_{k\to\infty} l^{\alpha}_{\varepsilon_k}(S_p) = 0$ if $\alpha > \dfrac{p}{p+2}$.

Hence, $\alpha_0(S_p) \leq \dfrac{p}{p+2}$.

B. Secondly we prove that $\alpha_0(G_p) \geq \dfrac{p}{p+2}$. To this end we shall analyze the internal fractal properties of the singular continuous measure μ_p.

For any probability measure ν one can introduce the notion of the Hausdorff dimension of the measure in the following way:

$$\alpha_0(\nu) = \inf_{E \in N(\nu)} \{\alpha_0(E), E \in \mathcal{B}\},$$

where $N(\nu)$ is the class of all “possible supports” of the measure ν, i.e.,

$$N(\nu) = \{E : E \in \mathcal{B}, \nu(E) = 1\}.$$

An explicit formula for the determination of the Hausdorff dimension of the measures with independent Q^*-symbols has been found in [8]. Applying this formula to our case $\left(q_{ik} = \dfrac{1}{s}, \forall k \in N, \forall i \in \{0, 1, \ldots, s-1\}\right)$, we have

$$\alpha_0(\mu_p) = \lim_{n\to\infty} \frac{H_n}{nlns},$$

where $H_n = \sum_{j=1}^{n} h_j$, and h_j are the entropies of the random variables $\xi_j^{(p)}$: $h_j = - \sum_{i=0}^{s-1} p_{ij} \ln p_{ij}$.

If $j \in B_p$, then $h_j = 0$. If $j \in C_p$, then $h_j = \ln s$.

So,

$$\alpha_0(\mu_p) = \lim_{n\to\infty} \frac{H_n}{nlns} = \lim_{k\to\infty} \frac{H_{m_k}}{m_k\, lns} =$$

$$= \lim_{k\to\infty} \frac{s^2p(2^{k-1}-1)\ln s}{(s^2(p+1)(2^k-1) - ps^2 2^{k-1})\ln s} = \frac{p}{p+2}.$$

The above defined set $G_p = f_p(N_s)$ is a support of the measure μ_p, because $\mu_p = \lambda(f_p^{-1})$ and the Lebesgue measure of the set N_s of s-normal numbers of the unit interval is equal to 1.

Since $G_p \in N(\mu_p)$ and $\alpha_0(\mu_p) = \dfrac{p}{p+2}$, we get $\alpha_0(\mu_p) \geq \dfrac{p}{p+2}$, which proves Lemma 2.

***Corollary* 2.** *The set G_p is the minimal dimensional support of the measure μ_p, i.e., $\alpha_0(G_p) \leq \alpha_0(E)$ for any other support E of the measure μ_p.*

Finally, let us consider the set $G = \bigcup\limits_{p=1}^{\infty} G_p$. From Lemma 1 it follows that $G \subset T_s$. From Lemma 2 and from the countable stability of the Hausdorff dimension it follows that $\alpha_0(G) = \sup\limits_p \alpha_0(G_p) = 1$. So, $\alpha_0(T_s) = 1$, which proves Theorem 1.

Summarizing the results of Sections 1 and 2, we have for $s > 2$:

	Lebesgue measure	Hausdorff dimension	Baire category
N_s	1	1	first
W_s	0	1	first
T_s	0	1	first
L_s	0	1	second

For the case $s = 2$ we have a corresponding result, but the Hausdorff dimension of the set T_s is equal to 0, because the set T_s is empty for $s = 2$.

Acknowledgment. This work was partly supported by DFG 436 UKR 113/78, DFG 436 UKR 113/80, INTAS 00-257 and SFB-611 projects. The last two named authors gratefully acknowledge the hospitality of the Institute of Applied Mathematics and of the IZKS of the University of Bonn.

1. *Besicovitch A.* On the sum of digits of real numbers represented in the dyadic systems // Math. Ann. – 1934. – **110**. – P. 321 – 330.
2. *Eggleston H. G.* The fractional dimension of a set defined by decimal properties // Quart. J. Math. Oxford Ser. – 1949. – **20**. – P. 31 – 36.
3. *Pratsiovytyi M., Torbin G.* Superfractality of the set of numbers having no frequency of n-adic digits, and fractal probability distributions // Ukr. Math. J. – 1995. – **47**, № 7. – P. 971 – 975.
4. *Olsen L.* Applications of multifractal divergence points to some sets of d-tuples of numbers defined by their N-adic expansion // Bull. Sci. Math. – 2004. – **128**. – P. 265 – 289.
5. *Olsen L.* Applications of multifractal divergence points to sets of numbers defined by their N-adic expansion // Math. Proc. Cambridge Phil. Soc. – 2004. – **136**, № 1. – P. 139 – 165.
6. *Olsen L., Winter S.* Normal and non-normal points of self-similar sets and divergence points of self-similar measures // J. London Math. Soc. – 2003. – **2(67)**, № 1. – P. 103 – 122.
7. *Albeverio S., Pratsiovytyi M., Torbin G.* Topological and fractal properties of subsets of real numbers which are not normal // Bull. Sci. Math. – 2004. – № 208.
8. *Albeverio S., Torbin G.* Fractal properties of singular continuous probability distributions with independent Q^*-digits // Ibid. – 2005. – **129**, № 4. – P. 356 – 367.

Received 17.06.2005