UDC 517.9

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A NOTE ON THE ASYMPTOTIC STABILITY OF FUZZY DIFFERENTIAL EQUATIONS

ПРО АСИМПТОТИЧНУ СТІЙКІСТЬ НЕЧІТКИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

In this paper, we study the stability of solutions of fuzzy differential equations by Lyapunov's second method. By using scale equations and comparison principle for Lyapunov-like functions, we give some sufficient criteria for the stability and asymptotic stability of solutions of fuzzy differential equations.

Вивчено стійкість розв'язків нечітких диференціальних рівнянь за допомогою другого методу Ляпунова. За допомогою масштабних рівнянь та принципу порівняння для рівнянь типу Ляпунова встановлено достатні умови стабільності та асимптотичної стабільності розв'язків нечітких диференціальних рівнянь.

1. Introductions. The investigation of stability of solutions is the most important problem in the qualitative theory of differential equations. It has been widely applied in Physics, Mechanics, Control, ...

Recently, the industrial interest in fuzzy control and logic [1] has dramatically increased the study of fuzzy systems. The calculus of fuzzy-valued functions has recently developed [2 – 6] and the investigation of fuzzy differential equations has been initiated in [7 – 11].

In this paper, we study the stability theory which corresponds to Lyapunov stability theory for fuzzy differential equations.

2. Preliminaries. Let $P_K(\mathbb{R}^n)$ denote the family of all nonempty compact, convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_K(\mathbb{R}^n)$ as usual. Let A, B be two nonempty subsets in $\mathbb{R}^n$. The distance between A and B is defined by Haussdorff metric:

$$d_H(A, B) = \max\left[\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right],$$

where $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$. Then it is clear that $(P_K(\mathbb{R}^n), d_H)$ becomes a metric space. Moreover, the metric space $(P_K(\mathbb{R}^n), d_H)$ is complete and separable (see [12]). Let $T = [a; b]$, $a \geq 0$, be an interval in $\mathbb{R}$ and denote $\varepsilon^n = \{u\colon \mathbb{R}^n \to \to [0;1] \mid u$ satisfies (i) to (iv) below$\}$:

(i) u is normal, that is, there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$:

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)];$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 = \overline{\{x \in \mathbb{R}^n\colon u(x) > 0\}}$ is a compact subset in $\mathbb{R}^n$.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{x \in \mathbb{R}^n\colon u(x) \geq \alpha\}$, then from (i) to (iv) it follows that the α-level $[u]^\alpha \in P_K(\mathbb{R}^n)$ for all $\alpha \in [0;1]$. For later purpose, we define $\hat{o} \in \varepsilon^n$ as $\hat{o}(x) = \chi_{\{0\}}(x) = 1$ if $x = 0$ and $\hat{o}(x) = 0$ if $x \neq 0$. Define a metric function $d\colon \varepsilon^n \times \varepsilon^n \to \mathbb{R}^n$ by

$$d[u, v] = \sup_{0\leq\alpha\leq 1} d_H([u]^\alpha, [v]^\alpha);$$

then (ε^n, d) becomes a complete metric space (see [12]). We list here some properties of metric $d[u, v]$ (see [7, 10, 12]):

(i) $d[u, v] = d[v, u]$; $d[u, v] = 0 \Leftrightarrow u = v$;

(ii) $d[u, w] \le d[u, v] + d[v, w]$;

(iii) $d[\lambda u, \lambda v] = |\lambda| d[u, v]$;

(iv) $d[u + w, v + w] = d[u, v]$, u, v, $w \in \varepsilon^n$, $\lambda \in \mathbb{R}$.

For x, $y \in \varepsilon^n$, if there exists $z \in \varepsilon^n$ such that $x = y + z$, then z is called H-difference of x and y and is denoted by $x - y$.

A mapping $F\colon T \to \varepsilon^n$ is differentiable at $t_0 \in T$ if for small $h > 0$, there exist H-differences $P(t_0 + h) - F(t_0)$; $F(t_0) - P(t_0 - h)$ and there exists $F'(t_0) \in \varepsilon^n$ such that the limits

$$\lim_{h \to 0+} \frac{F(t_0 + h) - F(t_0)}{h}, \qquad \lim_{h \to 0+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and equal $F'(t_0)$.

If F, G are differentiable at t, then $(F + G)'(t) = F'(t) + G'(t)$ and $(\lambda F)'(t) = \lambda F'(t)$, $\lambda \in \mathbb{R}$ (see [6, 7, 10]).

If $F\colon T \to \varepsilon^n$ is strongly measurable and integrable bounded, then it is integrable on T and $\int_T F(t)\,dt \in \varepsilon^n$,

$$\left[\int_T F(t)\,dt\right]^\alpha = \int_T F_\alpha(t)\,dt, \quad 0 < \alpha \le 1, \qquad F_\alpha(t) = [F(t)]^\alpha,$$

where $\int_T F_\alpha(t)\,dt$ is an *Aumann integral.* It is well known that $\left[\int_T F(t)\,dt\right]^0 = \int_T F_0(t)\,dt$ (see [7], Remark 4.1). Also the following properties of integral are valid (see [3, 4, 7, 10]). If $F, G\colon T \to \varepsilon^n$ are integrable on T and $\lambda \in \mathbb{R}$, then:

(i) $\int_T (F + G)(t)\,dt = \int_T F(t)\,dt + \int_T G(t)\,dt$;

(ii) $\int_T (\lambda F)(t)\,dt = \lambda \int_T F(t)\,dt$;

(iii) $d[F(\cdot), G(\cdot)]\colon T \to \mathbb{R}_+$ is integrable;

(iv) $d\left[\int_T F(t)\,dt, \int_T G(t)\,dt\right] \le \int_T d[F(t), G(t)]\,dt$;

(v) $\int_a^b F(t)\,dt = \int_a^c F(t)\,dt + \int_c^b F(t)\,dt$, $a \le c \le b$.

If F is continuous, then $G(t) = \int_a^t F(\tau)\,d\tau$ is differentiable on T and $G'(t) = F(t)$ $\forall\, t \in T$. Moreover, if F is differentiable on T and $F'(\cdot)$ is integrable on T, then for all $t \in T$ we have $F(t) = F(t_0) + \int_{t_0}^t F(\tau)\,d\tau$, $a \le t_0 \le t \le b$. If F is continuous on T and $G(t) = \int_a^t F(\tau)\,d\tau$, then for $t_1 \le t_2$ we have (see [7])

$$d[G(t_1), G(t_2)] \le (t_2 - t_1) \sup_{[t_1, t_2]} d[F(t), \hat{o}].$$

3. Stability. Consider fuzzy differentiable equation:

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0, \tag{1}$$

where $f \in C[\mathbb{R}_+ \times S(\rho), \varepsilon^n]$, $S(\rho) = \{x \in \varepsilon^n\colon d[x, \hat{o}] < \rho\}$, $f(t, \hat{o}) \equiv \hat{o}$.

In this section, we shall discuss the stability, especially, asymptotically stability of solutions of Eq. (1) by Lyapunov's second method. First, we give some notions of stability which are used in the sequel. Let $x(t) = x(t; t_0, x_0)$ be any solution of (1) existing on $[t_0, \infty]$. Denote $\mathcal{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+]\}$, $a(0) = 0$, $a(\cdot)$ is increasing$\}$.

Definition 1. *The trivial solution* $x = \hat{o}$ *of* (1) *is stable if for any* $\varepsilon > 0$, $t_0 \in \mathbb{R}_+$, *there exists* $\delta = \delta(t_0, \varepsilon) > 0$ *such that if* $d[x_0, \hat{o}] < \delta$, *then* $d[x(t), \hat{o}] < \varepsilon \;\; \forall\, t \geq t_0$.

Definition 2. *The trivial solution* $x = \hat{o}$ *of* (1) *is uniform-stable if* δ *in Definition* 1 *is independent of* t_0, *i.e., for any* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that if* $d[x_0, \hat{o}] < \delta$, $t_0 \in \mathbb{R}_+$, *then* $d[x(t; t_0, x_0), \hat{o}] < \varepsilon \;\; \forall\, t \geq t_0$.

Definition 3. *The trivial solution* $x = \hat{o}$ *of* (1) *is asymptotically stable if* $x = \hat{o}$ *is stable and for any* $t_0 \in \mathbb{R}_+$, *there exists* $\Delta = \Delta(t_0) > 0$ *such that if* $d[x_0, \hat{o}] < \Delta$, *then* $\lim_{t \to \infty} d[x(t; t_0, x_0), \hat{o}] = 0$.

Definition 4. *The trivial solution* $x = \hat{o}$ *of* (1) *is uniform-asymptotically stable if for any* $\varepsilon > 0$, *there exist* $\delta_0 = \delta_0(\varepsilon) > 0$, $T(\varepsilon) \geq 0$ *such that if* $d[x_0, \hat{o}] < \delta_0$, $t_0 \in \mathbb{R}_+$, *then* $d[x(t; t_0, x_0), \hat{o}] < \varepsilon \;\; \forall\, t \geq t_0 + T(\varepsilon)$.

Definition 5. *The trivial solution* $x = \hat{o}$ *is exponential stable if there exist* $\delta > 0$, $\alpha > 0$ *such that for any solution* $x(t) = x(t; t_0, x_0)$ *of* (1) *defines on* $[t_0, \infty)$:

$$d[x(t), \hat{o}] \leq \beta(d[x_0, \hat{o}])e^{-\alpha(t - t_0)}, \quad t \geq t_0,$$

where $\beta(\cdot): [0, \delta) \to \mathbb{R}_+$ *increasing in* $h \in [0, \delta)$.

Before discuss the stability of solutions of (1), we need the following lemma which corresponds to Comparison Principle (see [10] for detail).

Lemma 1. *Suppose that for Eq.* (1) *there exists a function* $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *satisfying*

$$|V(t, x) - V(t, y)| \leq Ld[x, y] \quad \forall\, (t, x),\ (t, y) \in \mathbb{R}_+ \times S(\rho);$$

$$D^+V(t, x) = \limsup_{h \to 0+} \frac{1}{h}[V(t + h, x + hf(t, x)) - V(t, x)] \leq g(t, V(t, x)),$$

$$g(\cdot, \cdot) \in C\left[\mathbb{R}_+^2, \mathbb{R}\right].$$

Let $r(t) = r(t; t_0, w_0)$ *be the maximal solution of equation*

$$w' = g(t, w) \tag{2}$$

existing on $[t_0, \infty)$ *and* $x(t) = x(t; t_0, x_0)$ *be any solution of* (1).

Then $V(t_0, x_0) \leq w_0$ *implies* $V(t, x(t)) \leq r(t) \;\; \forall\, t \geq t_0$.

Theorem 1. *Suppose that for Eq.* (1) *there exists a function* $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *which satisfies the following conditions:*

(i) $|V(t, x) - V(t, y)| \leq Ld[x, y] \;\; \forall\, (t, x),\ (t, y) \in \mathbb{R}_+ \times S(\rho)$;

(ii) $a(d[x, \hat{o}]) \leq V(t, x)$, $V(t, \hat{o}) = 0$, *where* $\alpha(\cdot) \in \mathcal{K}$ *class;*

(iii) $D^+V(t, x) \leq g(t, V(t, x))$, $g \in C\left[\mathbb{R}_+^2, \mathbb{R}\right]$, $g(t, 0) = 0$.

If the solution $w = 0$ *of* (2) *is stable* (*asymptotically stable*), *then the trivial solution* $x = \hat{o}$ *of* (1) *is stable* (*asymptotically stable*).

Proof. Let $x(t) = x(t; t_0, x_0)$, $t_0 \in \mathbb{R}_+$, be any solution of Eq. (1) existing on $[t_0, \infty)$ and solution $w = 0$ of (2) be stable. Then, for any $0 < \varepsilon < \rho$, there exists $\delta_0 = \delta_0(t_0, \varepsilon) > 0$ such that if $0 \leq w_0 < \delta_0$, then $|w(t; t_0, w_0)| < a(\varepsilon) \;\; \forall\, t \geq t_0$.

From (ii), it follows that there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $V(t_0, x) < \delta_0$ if $d[x, \hat{o}] < \delta$. We will show that if $d[x, \hat{o}] < \delta$, then $d[x(t), \hat{o}] < \varepsilon \ \forall t \geq t_0$.

Suppose that $d[x(t), \hat{o}] \geq \varepsilon$ for some $t_* > t_0$; then there exists $t_1 > t_0$ such that

$$d[x(t_1), \hat{o}] = \varepsilon; \qquad d[x(t), \hat{o}] < \varepsilon \quad \forall t \in [t_0, t_1).$$

Let $m(t) = V(t, x(t))$, $t \geq t_0$, then we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t, x(t)) =$$

$$= V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t))) +$$

$$+ V(t+h, x(t) + hf(t, x(t))) - V(t, x(t)) \leq$$

$$\leq Ld[x(t+h), x(t) + hf(t, x(t))] + V(t+h, x(t) + hf(t, x(t))) - V(t, x(t)).$$

For small $h > 0$, H-differences of $x(t+h)$ and $x(t)$ are assumed to exist. Let $x(t+h) = x(t) + z(t)$. Using the properties of metric $d[x, y]$, we have

$$d[x(t+h), x(t) + hf(t, x(t))] = hd\left[\frac{x(t+h) - x(t)}{h}, \ f(t, x(t))\right].$$

Hence,

$$D^+ m(t) = \limsup_{h \to 0+} [m(t+h) - m(t)] \leq$$

$$\leq L \limsup_{h \to 0+} d\left[\frac{x(t+h) - x(t)}{h}, \ f(t, x(t))\right] +$$

$$+ \limsup_{h \to 0+} \frac{1}{h}[V(t+h, x(t) + hf(t, x(t))) - V(t, x(t))] =$$

$$= Ld\left[x'(t), f(t, x(t))\right] + D^+ V(t, x(t)) = D^+ V(t, x(t)) =$$

$$= D^+ m(t) \leq g(t, m(t)), \quad t_0 \leq t \leq t_1.$$

Applying Lemma 1, $m(t) \leq r(t; t_0, w_0)$, $w_0 = V(t_0, x_0)$, $t \in [t_0, t_1]$. On the other hand, $V(t_0, x_0) < \delta_0$, so, $r(t; t_0, w_0) < a(\varepsilon)$, $t \in [t_0, t_1]$, and therefore

$$m(t_1) \leq r(t_1; t_0, w_0) < a(\varepsilon).$$

By the choice of t_1, we have $a(\varepsilon) = a(d[x(t_1), \hat{o}]) \leq V(t_1, x(t_1)) = m(t_1) < a(\varepsilon)$. This is a contradiction, whence

$$d[x(t), \hat{o}] < \varepsilon \quad \forall t \geq t_0.$$

This shows that the trivial solution $x = \hat{o}$ of (1) is stable.

If $w = 0$ of (2) is asymptotically stable, then it's stable, therefore $x = \hat{o}$ of (1) is stable. For $t_0 \in \mathbb{R}_+$, there exist $\delta = \delta(t_0) > 0$, $\Delta_1(t_0) > 0$ such that $d[x(t), \hat{o}] < \rho \ \forall t \geq t_0$ if $d[x_0, \hat{o}] < \delta$ and if $0 \leq w_0 < \Delta_1(t_0)$, then $\lim_{t \to \infty} w(t; t_0, w_0) = 0$. From hypothesises of function $V(t, x)$, we can find $\Delta_2 > 0$ such that if $d[x, \hat{o}] < \Delta_2$, then $V(t_0, x) < \Delta_1(t_0)$. Put $\Delta = \min[\delta, \Delta_2]$. Let $x(t)$ be any solution of (1), $t_0 \in \mathbb{R}_+$, $d[x_0, \hat{o}] < \Delta$. Define $m(t) = V(t, x(t))$, $t \geq t_0$. By the first part of this proof we see that $D^+ m(t) \leq g(t, m(t))$. Apply Lemma 1, $m(t) \leq r(t; t_0, w_0)$, $w_0 = V(t_0, x_0)$, $t \geq t_0$. Since $w_0 = V(t_0, x_0) < \Delta_1(t_0)$, we have $\lim_{t \to \infty} r(t; t_0, w_0) = 0$. From $a(d[x(t), \hat{o}]) \leq V(t, x(t)) = m(t) \leq r(t; t_0, w_0)$, $a(\cdot) \in \mathcal{K}$, it follows that

$\lim_{t\to\infty} d[x(t), \hat{o}] = 0$. This shows that $x = \hat{o}$ is asymptotically stable. The proof is completed.

Theorem 2. *Suppose that for Eq.* (1) *there exists a function* $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *which satisfies the following conditions:*

(i) $|V(t, x) - V(t, y)| \le Ld[x, y] \quad \forall\, (t, x),\ (t, y) \in \mathbb{R}_+ \times S(\rho)$;

(ii) $a(d[x, \hat{o}]) \le V(t, x) \le b(d[x, \hat{o}]),\ a(\cdot),\ b(\cdot) \in \mathcal{K}$;

(iii) $D^+V(t, x) \le g(t, V(t, x)),\ g \in C[\mathbb{R}_+^2, \mathbb{R}],\ g(t, 0) = 0$.

If the solution $w = 0$ *of* (2) *is uniform-stable* (*uniform-asymptotically stable*), *then the trivial solution* $x = \hat{o}$ *of* (1) *is uniform-stable* (*uniform-asymptotically stable*).

Proof. If solution $w = 0$ of (2) is uniform-stable, then for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that if $t_0 \in \mathbb{R}_+$ and $0 \le w_0 < \delta_0$, then $|w(t; t_0, w_0)| < a(\varepsilon)\ \forall\, t \ge t_0$. By choosing $\delta = \delta(\varepsilon) > 0$ such that $b(\delta) < a(\delta_0)$ and by the same argument in the proof of Theorem 1, it can be proved that if $d[x, \hat{o}] < \delta$, then $d[x(t;\, t_0,\, x_0), \hat{o}] < \varepsilon,\ t \ge t_0$. This shows that $x = \hat{o}$ is uniform-stable.

Now, we assume that $w = 0$ is uniform-asymptotically stable, then by the first part of this proof, the trivial solution $x = \hat{o}$ is uniform-stable. Hence, there exists $\delta_0 > 0$ such that $t_0 \in \mathbb{R}_+$, $d[x_0, \hat{o}] < \delta_0$ implies $d[x(t;\, t_0,\, x_0), \hat{o}] < \rho\ \ \forall\, t \ge t_0$. Moreover, there exists $\delta_1 > 0$ such that for any $\varepsilon > 0$, exists $T = T(\varepsilon) \ge 0$ such that if $t \ge 0$, $0 \le w_0 < \delta_1$, then $|w(t; t_0, w_0)| < a(\varepsilon)\ \ \forall\, t \ge t_0 + T$. Put $\delta = \min[\delta_0, b^{-1}(\delta_1)]$. By the same argument in the proof of Theorem 1, it can be proved that if $d[x_0, \hat{o}] < \delta$, then $d[x(t;\, t_0,\, x_0), \hat{o}] < \varepsilon\ \ \forall\, t \ge t_0 + T(\varepsilon)$. This shows that $x = \hat{o}$ is uniform-asymptotically stable. The proof is completed.

***Example* 1.** *Consider a fuzzy-valued function* $f(t, x)$ *which satisfies*

$$d[f(t, x), \hat{o}] \le a(t)\, d[x, \hat{o}]; \qquad \int_0^\infty a(t)\, dt < \infty$$

(*for example,* $f(t, x) = \dfrac{1}{1 + t^2} x$, $a(t) = \dfrac{1}{1 + t^2}$ *satisfies all the above conditions*).

Then the trivial solution $x = \hat{o}$ *of* (1) *is uniform-stable.*

Proof. Consider a Lyapunov function $V(t, x) = d[x, \hat{o}]$.

Then $\dfrac{1}{2} d[x, \hat{o}] \le V(t, x) \le 2\, d[x, \hat{o}]$ and $|V(t, x) - V(t, y)| \le d[x, y] \quad \forall\, (t, x);$ $(t, y) \in \mathbb{R}_+ \times \varepsilon^n$. For $h > 0$, we have

$$V(t + h, x + hf(t, x)) = d[x + hf(t, x), \hat{o}] \le$$

$$\le d[x, \hat{o}] + hd[f(t, x), \hat{o}] \le d[x, \hat{o}] + ha(t)\, d[x, \hat{o}].$$

Hence, $D^+V(t, x) \le a(t)\, d[x, \hat{o}] = g(t, V(t, x))$, where $g(t, w) = a(t) w$. It's easy to show that the solution $w = 0$ of (2) is uniform-stable, so by Theorem 2, the trivial solution $x = \hat{o}$ of (1) is uniform-stable.

Theorem 3. *Suppose that:*

(i) $f(t, x)$ *is bounded on* $\mathbb{R}_+ \times S(\rho)$;

(ii) $\exists\, V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}]$: $|V(t, x) - V(t, y)| \le Ld[x, y]$; $a(d[x, \hat{o}]) \le V(t, x) \le a_0(t, d[x, \hat{o}])$, *where* $a(\cdot) \in \mathcal{K}$, $a_0(t, \cdot) \in \mathcal{K}$;

(iii) $D^+V(t,x) + V^*(t,x) \le g(t, V(t,x))$, *where* $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $g(t,\cdot)$ *is nondecreasing for each* $t \in \mathbb{R}_+$ *and* $V^* \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $V^*(t,x) \ge c(d[x,\hat{o}])$, $c(\cdot) \in \mathcal{K}$.

If solution $w = 0$ *of* (2) *is stable, then the trivial solution* $x = \hat{o}$ *of* (1) *is asymptotically stable.*

Proof. By Theorem 1, the trivial solution $x = \hat{o}$ is stable. Hence, for $t_0 \in \mathbb{R}_+$, there exists $\delta_1(t_0)$ such that $d[x_0, \hat{o}] < \delta_1$ implies $d[x(t; t_0, x_0), \hat{o}] < \rho$ $\forall t \ge t_0$. Moreover, for $t_0 \in \mathbb{R}_+$, there exists $\delta_2(t_0) > 0$ such that if $0 \le w_0 < \delta_2(t_0)$, then $r(t; t_0, w_0) < \rho$ $\forall t \ge t_0$, where $r(t; t_0, w_0)$ is the maximal solution of (2). Since $a_0(t_0, \cdot) \in \mathcal{K}$, then there exists $\delta_3(t_0) > 0$ such that $a_0(t_0, \delta_3) < \delta_2(t_0)$. Put $\delta = \delta(t_0) = \min\{\delta_1, \delta_2, \delta_3\}$. Let $x(t) = x(t; t_0, x_0)$ be any solution of (1), $d[x_0, \hat{o}] < \delta$. We will show that

$$\lim_{t \to \infty} d[x(t), \hat{o}] = 0.$$

Suppose that $\limsup_{t \to \infty} d[x(t), \hat{o}] > 0$. Then there exists $\eta > 0$ and a sequence $\{t_n\} \to \infty$ such that $d[x(t_n), \hat{o}] \ge \eta$, $n = 0, 1, 2, \ldots$.

By the boundedness of $f(t,x)$ and by taking a subsequence of $\{t_n\}$, we can assume that there exists $M > 0$, $\{t_n\} \to \infty$ such that $t_{n+1} - t_n \ge \dfrac{\eta}{2M}$, $n \ge 0$.

For $t \in \left[t_n, t_n + \dfrac{\eta}{2M}\right]$, we have $x(t) = x(t_n) + \int_{t_n}^{t} f(\tau, x(\tau))d\tau$. Hence

$$d[x(t), \hat{o}] \ \ge\ d[x(t_n), \hat{o}] - \int_{t_n}^{t} d[f(\tau, x(\tau)), \hat{o}]d\tau \ \ge\ \eta - M\frac{\eta}{2M} \ =\ \frac{\eta}{2}.$$

Define $m(t) = V(t, x(t)) + \int_{t_0}^{t} V^*(\tau, x(\tau))d\tau$, $t \ge t_0$. Then

$$D^+m(t) \ \le\ D^+V(t, x(t)) + V^*(t, x(t)) \le g(t, V(t, x(t))) \ \le\ g(t, m(t)), \quad t \ge t_0.$$

Applying Lemma 1, it follows that $m(t) \le r(t; t_0, w_0)$, where $w_0 = V(t_0, x_0)$. Since $V(t_0, x_0) \le a_0(t_0, d[x_0, \hat{o}]) < a_0(t_0, \delta) < \delta_2(t_0)$, we have $|r(t; t_0, w_0)| < \rho$ $\forall t \ge t_0$. Therefore,

$$V\left(t_n + \frac{\eta}{2M}, x\left(t_n + \frac{\eta}{2M}\right)\right) \ \le\ r(t; t_0, w_0) - \sum_{k=0}^{n} \int_{t_k}^{t_k + \frac{\eta}{2M}} V^*(\tau, x(\tau))d\tau \ \le$$

$$\le\ r(t; t_0, w_0) - c\left(\frac{\eta}{2}\right)\frac{\eta}{2M}n \ <\ \rho - c\left(\frac{\eta}{2}\right)\frac{\eta}{2M}n \ <\ 0$$

for n sufficiently large. This is a contradiction and, therefore,

$$\lim_{t \to \infty} d[x(t), \hat{o}] = 0.$$

The proof is completed.

Theorem 4. *Let the assumptions* (i), (ii) *of Theorem* 2 *hold and*

(iii′) $D^+V(t,x) + V^*(t,x) \le g(t, V(t,x))$, $g(\cdot,\cdot)$ *as in Theorem* 3, $V^* \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $V^*(t,x) \ge c(d[x,\hat{o}])$, $c(\cdot) \in \mathcal{K}$.

If solution $w = 0$ *of* (2) *is uniform-stable, then the trivial solution* $x = \hat{o}$ *of* (1) *is uniform-asymptotically stable.*

Proof. By Theorem 2, $x = \hat{o}$ of (1) is uniform-stable. Hence, for $\varepsilon = \rho$ there exists $\delta_0 > 0$ such that if $t_0 \in \mathbb{R}_+$, $d[x_0, \hat{o}] < \delta_0$, then

$$d[x(t; t_0, x_0), \hat{o}] < \rho, \quad t \geq t_0.$$

We can assume that δ_0 satisfies if $0 \leq w_0 < b(\delta_0)$ then $|r(t; t_0, w_0)| < a(\rho)$ $\forall\, t \geq t_0$. By the uniform-stability of $x = \hat{o}$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d[x_0, \hat{o}] < \delta$, $t_0 \in \mathbb{R}_+$, then $d[x(t;\, t_0,\, x_0),\, \hat{o}] < \varepsilon$ $\forall\, t \geq t_0$. Let's put $T = T(\varepsilon) = 1 + \dfrac{a(\rho)}{c(\delta)}$. Let $x(t) = x(t; t_0, x_0)$ be any solution of (1), $d[x_0, \hat{o}] < \delta_0$. We will show that $d[x_0, \hat{o}] < \delta$ for some $t_* \in [t_0,\, t_0 + T(\varepsilon)]$. Suppose that $d[x(t), \hat{o}] \geq \delta$ $\forall\, t \in [t_0,\, t_0 + T(\varepsilon)]$. Define $m(t) = V(t, x(t)) + \int_{t_0}^{t} V^*(\tau, x(\tau))d\tau$, $t \geq t_0$. By the same argument in the proof of Theorem 3, we have $m(t) \leq r(t;\, t_0,\, w_0)$, $t \geq t_0$, where $w_0 = V(t_0, x_0)$ and $r(t;\, t_0,\, w_0)$ is the maximal solution of (2). Therefore,

$$0 \leq V(t_0 + T, x(t_0 + T)) \leq$$

$$\leq r(t_0 + T; t_0, w_0) - \int_{t_0}^{t_0+T} V^*(\tau, x(\tau))d\tau \leq r(t_0 + T; t_0, w_0) - Tc(\delta).$$

Since $V(t_0, x_0) \leq b(d[x, \hat{o}]) < b(\delta_0)$, we have $w_0 = V(t_0, x_0) < b(\delta_0)$ and, hence, $r(t_0 + T;\, t_0,\, w_0) < a(\rho)$. Therefore, $0 \leq V(t_0 + T, x(t_0 + T)) < a(\rho) - Tc(\delta) < 0$.

This contradiction shows that there exists $t_1 \in [t_0,\, t_0 + T]$ such that $d[x(t_1), \hat{o}] < \delta$. On the other hand, $x(t;\, t_1, x(t_1;\, t_0,\, x_0)) = x(t; t_0, x_0)$ $\forall\, t \geq t_1$, hence,

$$d[x(t), \hat{o}] < \varepsilon \quad \forall\, t \geq t_0 + T(\varepsilon).$$

This shows that the trivial solution $x = \hat{o}$ of (1) is uniform-asymptotically stable. The proof is completed.

Theorem 5. *Suppose that for Eq.* (1) *there exists a function* $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}]$ *which satisfies the following conditions:*

(i) $|V(t, x) - V(t, y)| \leq Ld[x, y]$ $\forall\, (t, x), (t, y) \in \mathbb{R}_+ \times S(\rho)$;

(ii) $\lambda(d[x, \hat{o}])^p \leq V(t, x) \leq \Lambda(d[x, \hat{o}])^p$, $p > 0$, $\lambda, \Lambda > 0$;

(iii) $D^+V(t, x) \leq -c.(d[x, \hat{o}])^p + Ke^{-\alpha t}$, $t \geq 0$, $c > 0$.

If $\alpha > \dfrac{c}{\Lambda}$, *then the trivial solution* $x = \hat{o}$ *of* (1) *is exponential stable.*

Proof. By Theorem 1, $x = \hat{o}$ is uniform-stable. Hence, there exists δ such that $t_0 \in \mathbb{R}_+$, $d[x_0, \hat{o}] < \delta \Rightarrow d[x(t;\, t_0,\, x_0),\, \hat{o}] < \rho$ $\forall\, t \geq t_0$.

Let's put $M = \dfrac{c}{\Lambda}$, $m(t) = V(t, x(t))e^{M(t-t_0)}$, $t \geq t_0$. We have

$$D^+m(t) \leq MV(t, x(t))e^{M(t-t_0)} + e^{M(t-t_0)} D^+V(t, x(t)) \leq$$

$$\leq MV(t, x(t))e^{M(t-t_0)} + e^{M(t-t_0)}\left[Ke^{-\alpha t} - c(d[x, \hat{o}])^p\right] \leq$$

$$\leq MV(t, x(t))e^{M(t-t_0)} + Ke^{(M-\alpha)(t-t_0)} - \frac{c}{\Lambda}e^{M(t-t_0)}V(t, x(t)) =$$

$$= Ke^{(M-\alpha)(t-t_0)}.$$

Apply Lemma 1, $m(t) - m(t_0) \le K \int_{t_0}^{t} e^{(M-\alpha)(\tau - t_0)} d\tau = \dfrac{K}{M-\alpha}\left[e^{(M-\alpha)(t-t_0)} - 1\right]$. By hypothesises, $m(t_0) = V(t_0, x_0) \le \Lambda\left(d[x_0, \hat{o}]\right)^p$, we have

$$m(t) \le \frac{K}{M-\alpha} e^{(M-\alpha)(t-t_0)} - \frac{K}{M-\alpha} + \Lambda\left(d[x_0, \hat{o}]\right)^p.$$

Put $\alpha_1 = -(M - \alpha) > 0$, then

$$m(t) \le \Lambda\left(d[x_0, \hat{o}]\right)^p + \frac{K}{\alpha_1} - \frac{K}{\alpha_1} e^{-\alpha_1 (t-t_0)} \le \Lambda\left(d[x_0, \hat{o}]\right)^p + \frac{K}{\alpha_1}, \quad t \ge t_0.$$

Therefore, $V(t, x(t)) \le \beta_1(d[x_0, \hat{o}]) e^{-M(t-t_0)}$, $t \ge t_0$, where $\beta_1(d[x_0, \hat{o}]) = \Lambda\left(d[x_0, \hat{o}]\right)^p + \dfrac{K}{\alpha_1}$. On the other hand, $\Lambda\left(d[x(t), \hat{o}]\right)^p \le V(t, x(t))$, $t \ge t_0$, so finally we have

$$d[x(t), \hat{o}] \le \left[\frac{\beta_1(d[x_0, \hat{o}])}{\lambda}\right]^{\frac{1}{p}} e^{-\frac{M}{p}(t-t_0)}, \quad t \ge t_0.$$

Denote $\alpha = \dfrac{M}{p}$, $\beta(d[x_0, \hat{o}]) = \left[\dfrac{\beta_1(d[x_0, \hat{o}])}{\lambda}\right]^{\frac{1}{p}}$, then

$$d[x(t), \hat{o}] \le \beta(d[x_0, \hat{o}]) e^{-\alpha(t-t_0)}, \quad t \ge t_0.$$

This shows that the trivial solution $x = \hat{o}$ of (1) is exponential-stable. The proof is completed.

Acknowledgement. I wish to express my sincere thanks to Professor Vu Tuan for his encouragement and valuable ideas.

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Received 25.10.2004