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LONG TIME BEHAVIOR
OF NONAUTONOMOUS FOKKER – PLANCK EQUATIONS
AND THE COOLING OF GRANULAR GASES

ДОВГОЧАСОВА ПОВЕДІНКА
НЕАВТОНОМНИХ РІВНЯНЬ ФОККЕРА – ПЛАНКА
ТА ОХОЛОДЖЕННЯ ГРАНУЛЬОВАНИХ ГАЗІВ

We analyze the asymptotic behavior of linear Fokker – Planck equations with time-dependent coefficients. Relaxation towards a Maxwellian distribution with time-dependent temperature is shown under explicitly computable conditions. We apply this result to the study of Brownian motion in granular gases, by showing that the Homogenous Cooling State attracts any solution at an algebraic rate.

Проаналізовано асимптотичну поведінку лінійних рівнянь Фоккера – Планка з коефіцієнтами, залежними від часу. Показано, що за явно обчислюваних умов відбувається релаксація до розподілу Максвелла з залежною від часу температурою. Цей результат застосовано до вивчення броунівського руху в гранульованих газах і показано, що однорідний охолоджуючий стан притягує будь-який розв'язок з алгебраїчною швидкістю.

1. Introduction.   The linear Fokker – Planck equation (FPE)

$$\frac{\partial \varphi}{\partial t}(v,t) = \lambda \nabla \cdot \{v\varphi(v,t) + \theta \nabla \varphi(v,t)\}, \quad v \in \mathbb{R}^N, \quad N \geqslant 1, \qquad (1.1)$$

arises in many fields of applied sciences such that statistical mechanics, chemistry, mathematical finance (see the monographs [1] and [2] for a large account of applications to the FPE). Such a drift-diffusion equation can be derived from the Langevin equation to model the Brownian motion of particles in thermodynamical equilibrium. In this case, the parameters λ and θ are two positive constants which represent respectively the friction term and the temperature of the system. The qualitative analysis of equation (1.1) is well documented in the literature. We refer to [1] for a precise description of the hilbertian and spectral methods used in the study of (1.1) and to [3, 4] for the relatively recent approach to the L^1-theory by means of entropy-dissipation methods.

It is easy to notice that the set

$$\mathcal{M} = \left\{ g \in L^1(\mathbb{R}^N), \quad g \geqslant 0, \int\limits_{\mathbb{R}^N} g(v)dv = 1, \right.$$

$$\left. \int\limits_{\mathbb{R}^N} vg(v)dv = 0, \quad \int\limits_{\mathbb{R}^N} v^2 g(v)dv = \theta < \infty \right\}$$

is invariant under the action of the right-hand side of (1.1). Moreover, it is well-known that (1.1) admits a unique steady state in $\mathcal{M}$ given by the Gaussian distribution (*Maxwellian* function in the language of kinetic theory)

$$M_\theta(v) = (2\pi\theta)^{-N/2} \exp\left(-v^2/2\theta\right), \quad v \in \mathbb{R}^N.$$

Entropy-dissipation methods (see [5] for a review on recent results on the topic) provide a precise picture of the asymptotic behavior of the solution to (1.1) for initial data in $\mathcal{M}$. Given $f \in \mathcal{M}$, the (Boltzmann) *relative entropy* (finite or not) of f is defined as

$$H(f \,|\, M_f) = H(f) - H(M_f) = \int\limits_{\mathbb{R}^N} f \log\left(\frac{f}{M_f}\right) dv$$

where M_f is the unique Maxwellian distribution in $\mathcal{M}$ with the same temperature as f. Given $f_0 \in \mathcal{M}$, with the assumption of bounded initial relative entropy

$$H(f_0 \,|\, M_\theta) \, < \, \infty,$$

it has been proven in [3] that the unique mass-preserving solution $f(v,t)$ of (1.1) decays exponentially fast with rate 2λ to $M_\theta(v)$ in relative entropy, i.e, the estimate

$$H(f \,|\, M_\theta)(t) \le e^{-2\lambda t} H(f_0 \,|\, M_\theta)$$

holds. The classical *Csiszar – Kullback inequality* then allows to move convergence in relative entropy to the more standard L^1-setting. The following result is proved in [3]:

Theorem 1.1. *Let λ and θ be two positive constants. Let us assume that $f_0 \in \mathcal{M}$ has a finite relative entropy. Then, there exists a constant $C > 0$ depending only on the initial relative entropy such that the solution $f(v,t)$ to* (1.1) *fulfills*

$$\|f(\cdot, t) - M_\theta\|_{L^1(\mathbb{R}^N)} \leqslant C \exp(-\lambda t)$$

for any $t \geqslant 0$.

Main objective of this paper is to generalize this result allowing the friction term λ and the temperature θ to fluctuate with time. In this case, the Fokker – Planck equation reads:

$$\frac{\partial \varphi}{\partial t}(v,t) = \lambda(t)\nabla \cdot \{v\varphi(v,t) + \theta(t)\nabla\varphi(v,t)\} \tag{1.2}$$

where $\lambda(t)$ and $\theta(t)$ are positive functions of time.

Nonautonomous Fokker – Planck equations arise for instance in the study of a periodically driven Brownian rotor [6] and in this case $\lambda(t)$ and $\theta(t)$ are periodic functions of time. In statistical mechanics, equation (1.2) arises as a natural generalization of equation (1.1) in the context of *nonequilibrium* thermodynamics [7]. Among other models, equation (1.2) appears in the study of the tagged particle dynamics of a heavy particle in a gas of much lighter inelastic particles. As observed by J. J. Brey, W. Dufty and A. Santos [8], the large particles exhibit Brownian motion and the Boltzmann – Lorentz kinetic equation satisfied by the distribution function of large particles can be reduced to a Fokker – Planck equation whose coefficients depend on the temperature of the surrounding gas. Granular gases being nonequilibrium systems, this temperature turns out to be time-dependent and the Fokker – Planck equation derived in [8] is of the shape (1.2). Since the study of the long time behavior of the solution to the Brey – Dufty – Santos model is one of the main goals of our analysis, we will explain with much more details the approach of [8] in the next section.

Because of the time-dependence of both $\lambda(t)$ and $\theta(t)$, Equation (1.2) does not possess stationary states. Nevertheless, two natural questions arise:

Do they exist particular (self-similar) solutions to (1.2) which attract all other solutions (as the Maxwellian does in the autonomous case)?

If such self-similar solutions exist, is it possible to reckon the rate at which they attract the other solutions?

We answer positively to these two questions under some reasonable conditions on the time-behavior of friction and temperature. Our method is based upon suitable (time-dependent) scalings which allow us to transform the nonautonomous equation (1.2) into a Fokker – Planck equation of the form (1.1) and then to make use of Theorem 1.1. Clearly, the self-similar profile is a Maxwellian with time-dependent temperature. In this context, the Maxwellian distribution plays the role of the Barenblatt profile in the study of the porous medium equation [9].

Application of this abstract result to our motivating example in kinetic theory of granular gases, shows that the distribution of the Brownian particles relaxes towards a Maxwellian distribution with time-dependent temperature. While this fact was already been noticed by J. Javier Brey et al. [8] our result gives a precise *estimate the rate of convergence* towards this self-similar profile (the so-called *Homogeneous Cooling State*) which turns out to be only algebraic in time. Analogous results, which try to clarify the role of the Homogeneous Cooling State in kinetic models of granular gases, have been recently obtained for the case of the Boltzmann equation for inelastic Maxwell particles [10]. We postpone a detailed discussion on this point in the conclusions of this note.

The organization of the paper is the following. In the next section, we present in some details the derivation of the FPE in the context of Brownian motion for granular gases. In Section 3, we deals with a general nonautonomous FPE and we answer to the two aforementioned questions (Theorem 3.1). Finally, in Section 4 we turn back to our motivated example of Brownian particles and we show how the abstract result of Section 3 allows us to estimate the rate of convergence towards the Homogeneous cooling state.

2. The Brownian motion in granular gases. The motion of heavy granular particles of mass m embedded in a low density gas whose particles have mass m_g with $m_g \ll m$. has been considered in [8]. The particles under consideration are assumed to be hard-spheres of $\mathbb{R}^3$ and, for the sake of simplicity, the diameters of the particles of both species are assumed to be equal and normalized to unit. The case of particles with different diameter can be investigated as well, and does not lead to major supplementary difficulties [8]. The collisions between the heavy particles and the fluid ones are partially inelastic and are characterized by a coefficient of restitution $\epsilon \in (0,1)$. Assuming that the concentration of heavy particles is small, one neglects the collision phenomena between them. Let us denote by $f(\tilde{v}, t)$ the distribution function of the heavy particles having velocity $\tilde{v} \in \mathbb{R}^3$ at time $t > 0$ and by $g(\tilde{v}, t)$ the distribution function of the surrounding gas where, for simplicity, it is assumed that these two quantities are independent of the position. Then, the evolution of $f(\cdot, t)$ is given by the *Boltzmann – Lorentz* equation, which in weak form reads

$$\frac{d}{dt} \int_{\mathbb{R}^3} f(\tilde{v}, t)\psi(\tilde{v})d\tilde{v} = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} |\mathbf{q} \cdot \mathbf{n}| f(\tilde{v}, t)g(\tilde{w}, t)[\psi(\mathbf{v}^\star) - \psi(\tilde{v})]d\tilde{v}d\tilde{w}d\mathbf{n} \quad (2.1)$$

for any test-function $\psi(\tilde{v})$. Here $\mathbf{q} = \tilde{v} - \tilde{w}$ and $(\mathbf{v}^\star, \mathbf{w}^\star)$ are the post-collisional velocities:

$$\mathbf{v}^\star = \tilde{v} - \frac{\Delta(1+\epsilon)}{1+\Delta}(\mathbf{q} \cdot \mathbf{n})\mathbf{n}, \qquad \mathbf{w}_\star = \tilde{w} + \frac{\Delta(1+\epsilon)}{1+\Delta}(\mathbf{q} \cdot \mathbf{n})\mathbf{n} \qquad (2.2)$$

where Δ is the mass ratio $\Delta = m_g/m$. Note that, by assumption, $\Delta \ll 1$.

To solve the linear equation (2.1), one has to make explicit $g(\tilde{v}, t)$. Assuming that the binary collisions between the surrounding particles are inelastic and characterized

by a constant restitution coefficient $0 < \epsilon_g < 1$, $g(\tilde{v}, t)$ is given by a solution to the (nonlinear) Boltzmann equation for granular hard-spheres [11, 12]. Leaving details to the pertinent literature, the role of the equilibrium Maxwellian function in the elastic Boltzmann equation is here represented by the *Homogeneous Cooling State* (see References [12, 13]) which implies that

$$g(\tilde{v}, t) = v_g(t)^{-3} \Phi\left(\frac{\tilde{v}}{v_g(t)}\right)$$

where $v_g(t)$ is the thermal velocity of the gas particles defined as $v_g(t) = [2T_g(t)/m_g]^{1/2}$. The temperature $T_g(t)$ is defined in a standard way (see Section 3). The self-similar profile $\Phi(\cdot)$ is a stationary solution of some suitable steady Boltzmann equation (see [11, 13]) and is not explicitly known. However, an important fact to be noticed is that $\Phi(\tilde{v})$ is a function of ϵ_g which, in the quasielastic regime $\epsilon_g \to 1$, converges toward the Maxwellian distribution $\pi^{-3/2} \exp(-\tilde{v}^2)$. The temperature $T_g(t)$ is cooling because of the inelasticity of the collisions. Hereafter, we will assume $T_g(t)$ to obey the the so-called *Haff's law* [14]:

$$T_g(t) = T_g(0)\left(1 + t/\tau_0\right)^{-2} \tag{2.3}$$

where $\tau_0 > 0$ is the characteristic time [8]:

$$\tau_0^{-1} = \frac{\pi(1 - \epsilon_g^2)}{12}\sqrt{\frac{T_g(0)}{2m_g}} \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \Phi(\tilde{v})\Phi(\tilde{w})|\tilde{v} - \tilde{w}|^3 d\tilde{v}d\tilde{w}. \tag{2.4}$$

The Boltzmann – Lorentz equation (2.1) can be reduced to a FPE with time-dependent coefficients of the form (1.2) performing two asymptotic procedures:

The first procedure is a simple extension of the standard method for elastic particles (grazing collisions asymptotics [15], see also [16] in the context of the dissipative linear Boltzmann equation). Precisely, according to (2.2), one sees that, when a heavy particle collides with a small one, the velocity of the heavy particle is only slightly altered:

$$|\mathbf{v}^\star - \tilde{v}| = \left|\frac{\Delta(1 + \epsilon)}{1 + \Delta}(\mathbf{q} \cdot \mathbf{n})\mathbf{n}\right| \ll 1$$

so that $\mathbf{v}^\star \simeq \tilde{v}$. Therefore, performing in (2.1) a formal expansion to leading order in the mass ratio as $\Delta \to 0$, one obtains the following Fokker – Planck equation with time-dependent coefficients:

$$\frac{\partial f(\tilde{v}, t)}{\partial t} = \nabla_{\tilde{v}} \cdot \left[\mathbf{A}(\tilde{v}, t)f(\tilde{v}, t) + \frac{1}{2}\nabla_{\tilde{v}} \cdot (\mathbb{N}(\tilde{v}, t)f(\tilde{v}, t))\right] \tag{2.5}$$

where the vector $\mathbf{A}(\tilde{v}, t)$ and, respectively, the tensor $\mathbb{N}(\tilde{v}, t)$ are given by

$$\mathbf{A}(\tilde{v}, t) = \frac{(1 + \epsilon)\Delta}{1 + \Delta}\frac{\pi}{2}\int\limits_{\mathbb{R}^3} g(\tilde{w}, t)\mathbf{q}|\mathbf{q}|d\tilde{w},$$

and

$$\mathbb{N}_{ij}(\tilde{v}, t) = \left(\frac{1 + \epsilon\Delta}{1 + \Delta}\right)^2 \frac{\pi}{12}\int\limits_{\mathbb{R}^3} g(\tilde{w}, t)\left(|\mathbf{q}|^3\delta_{ij} + 3|\mathbf{q}|\mathbf{q}_i\mathbf{q}_j\right) d\tilde{w}, \quad i, j = 1, 2, 3.$$

We used above the standard notation

$$\nabla_{\tilde{v}} \cdot (\mathbb{N}(\tilde{v})f(\tilde{v})) = \left(\sum_{j=1}^{3} \frac{\partial}{\partial \tilde{v}_j} \mathbb{N}_{ij}(\tilde{v},t) f(\tilde{v},t) \right)_{i=1,2,3}$$

(see [8] (Appendix A) for a detailed derivation).

To simplify further the Fokker–Planck equation (2.5) one performs a second asymptotic procedure which consists in assuming that the thermal velocity of the heavy particles $v(t) = [2T(t)/m]^{1/2}$ is negligible with respect to the one of the surrounding particles $v_g(t)$: $v(t) \ll v_g(t)$. This leads to a formal expansion in $T(t)\Delta/T_g(t)$ where $T(t)$ is the temperature of the Brownian particles (see Remark 2.1). In this case, the vector $\mathbf{A}(\tilde{v},t)$ and the tensor $\mathbb{N}(\tilde{v},t)$ reduce to

$$\mathbf{A}(\tilde{v},t) \simeq \alpha(t)\tilde{v} \quad \text{and} \quad \mathbb{N}_{ij}(\tilde{v},t) \simeq 2\eta(t)\delta_{ij}, \quad i,j = 1,2,3,$$

where

$$\alpha(t) = \zeta T_g(t)^{1/2} \quad \text{and} \quad \eta(t) = \xi T_g(t)^{3/2}. \tag{2.6}$$

In (2.6) we used the notations

$$\zeta = \frac{2\sqrt{2}\pi}{3\sqrt{m_g}}(1+\epsilon)\Delta \int_{\mathbb{R}^3} \Phi(\tilde{w})|\tilde{w}|d\tilde{w}, \tag{2.7}$$

and

$$\xi = \frac{\pi m_g^{-3/2}}{3\sqrt{2}}(1+\epsilon)^2\Delta^2 \int_{\mathbb{R}^3} \Phi(\tilde{w})|\tilde{w}|^3 d\tilde{w}.$$

Taking these simplifications into account, equation (2.5) is replaced by the following

$$\frac{\partial f}{\partial t}(\tilde{v},t) = \nabla_{\tilde{v}} \cdot \{\alpha(t)\tilde{v}\,f(\tilde{v},t) + \eta(t)\nabla_{\tilde{v}}f(\tilde{v},t)\}. \tag{2.8}$$

The Fokker–Planck equation (2.8) is of the form (1.2). The drift and diffusion coefficient depend on time only through the surrounding gas temperature $T_g(t)$ (see (2.6)).

Remark **2.1.** The derivation of (2.8) form (2.1) is based upon a *time-dependent* asymptotic procedure where it is assumed that

$$\frac{T(t)\Delta}{T_g(t)} \ll 1. \tag{2.9}$$

As shown in [8] this assumption requires $\Delta \to 0$ and $\epsilon_g \to 1$ (quasielastic regime for the surrounding gas) simultaneously. A further consequence of these assumptions is that

$$\frac{1}{2\sqrt{2}\Delta}\frac{1-\epsilon_g^2}{1+\epsilon} < 1. \tag{2.10}$$

We will find again this condition hereafter.

3. Long time behavior of nonautonomous Fokker – Planck equations. In this section we consider the general Fokker – Planck collision operator with *time-dependent coefficients* written in the divergence form:

$$\mathcal{Q}_t(f)(v) = \lambda(t)\nabla_v \cdot \{v\,f(v,t) + \theta(t)\nabla_v f(v,t)\}, \quad v \in \mathbb{R}^N, \quad N \geqslant 1, \qquad (3.1)$$

where $\lambda(t)$ and $\theta(t)$ are two positive functions of time. We are concerned with the large-time asymptotic behavior of the solution to the Cauchy problem

$$\frac{\partial f}{\partial t}(v,t) = \mathcal{Q}_t(f)(v), \quad v \in \mathbb{R}^N, \quad t > 0,$$
$$f(v,0) = f_0(v), \qquad (3.2)$$

where the initial data f_0 is assumed to be nonnegative and integrable,

$$f_0 \geqslant 0, \quad \text{and} \quad f_0 \in L^1(\mathbb{R}^N).$$

In accordance with the language of kinetic theory, we define the mass density $\varrho(t)$, mean velocity $\mathbf{u}(t)$ and temperature $T(t)$ respectively as:

$$\varrho(t) = \int\limits_{\mathbb{R}^N} f(v,t)dv, \quad \mathbf{u}(t) = \frac{1}{\varrho(t)}\int\limits_{\mathbb{R}^N} vf(v,t)dv,$$

and

$$T(t) = \frac{1}{N\varrho(t)}\int\limits_{\mathbb{R}^N} |v - \mathbf{u}(t)|^2 f(v,t)dv.$$

The number density is preserved by the (nonautonomous) Fokker – Planck operator (3.1) while the mean velocity is preserved only if initially equal to zero. Precisely, if $\int\limits_{\mathbb{R}^n} vf_0(v)dv = 0$, then $\mathbf{u}(t) = 0$ for any $t > 0$. In this case, the evolution of the temperature is

$$\frac{dT(t)}{dt} = -2\lambda(t)\left(T(t) - \theta(t)\right), \quad t > 0. \qquad (3.3)$$

In order to find the intermediate asymptotic for (3.2), we look for a solution to (3.1) of the shape:

$$f(v,t) = \gamma(t)^{-N} F\left(v/\gamma(t), \tau(t)\right) = \gamma(t)^{-N} F(\tilde{v}, \tau)$$

where the new time scale $\tau = \tau(t)$ is nonnegative and such that $\tau(0) = 0$, the scaled velocity is

$$\tilde{v} = v/\gamma(t)$$

and $\gamma(\cdot)$ is positive. Without loss of generality, one may assume that $\gamma(0) = T_0 :=$
$:= \int\limits_{\mathbb{R}^3} v^2 f_0(v)dv$ so that

$$F(\tilde{v},0) = F(v/T_0,0) = f_0(v/T_0).$$

One sees immediately that

$$\frac{\partial f}{\partial t}(v,t) = \frac{\dot{\tau}(t)}{\gamma(t)^N}\frac{\partial F}{\partial \tau}(\tilde{v},\tau) - \frac{\dot{\gamma}(t)}{\gamma(t)^{N+1}}\nabla_{\tilde{v}} \cdot (\tilde{v}F(\tilde{v},\tau)), \qquad (3.4)$$

where the dot symbol stands for the time derivative. In the same way, one can show that

$$\mathcal{Q}_t(f)(v) = \frac{\lambda(t)}{\gamma(t)^N} \nabla_{\tilde{v}} \cdot (\tilde{v} F(\tilde{v}, \tau)) + \frac{\lambda(t)\theta(t)}{\gamma(t)^{N+2}} \nabla_{\tilde{v}}^2 F(\tilde{v}, \tau). \tag{3.5}$$

This leads to the following evolution equation for $F(\cdot, \tau)$:

$$\frac{\partial F}{\partial \tau}(\tilde{v}, \tau) = \frac{1}{\dot{\tau}(t)} \left[\lambda(t) + \frac{\dot{\gamma}(t)}{\gamma(t)} \right] \nabla_{\tilde{v}} \cdot (\tilde{v} F(\tilde{v}, \tau)) + \frac{\lambda(t)\theta(t)}{\dot{\tau}(t)\,\gamma(t)^2} \nabla_{\tilde{v}}^2 F(\tilde{v}, \tau). \tag{3.6}$$

One notes that (3.6) reduces to a *"good"* Fokker–Planck equation

$$\frac{\partial F}{\partial \tau}(\tilde{v}, \tau) = \nabla_{\tilde{v}} \cdot (\tilde{v}\, F(\tilde{v}, \tau) + \nabla_{\tilde{v}} F(\tilde{v}, \tau)), \qquad \tilde{v} \in \mathbb{R}^N, \quad \tau > 0, \tag{3.7}$$

provided there exists some $\sigma > 0$ such that

$$\frac{1}{\dot{\tau}(t)} \left[\lambda(t) + \frac{\dot{\gamma}(t)}{\gamma(t)} \right] = 1 \qquad \forall t > 0, \tag{3.8}$$

and

$$\frac{\lambda(t)\theta(t)}{\dot{\tau}(t)\,\gamma(t)^2} = 1 \qquad\qquad \forall t > 0. \tag{3.9}$$

Of course, to investigate the asymptotic behavior of $F(\cdot, \tau)$ and apply Theorem 1.1, one has to find conditions on $\lambda(\cdot)$ and $\theta(\cdot)$ insuring that the time scale τ verifies

$$\lim_{t \to \infty} \tau(t) = +\infty.$$

Solving equation (3.8), (3.9) leads to

$$\frac{\lambda(t)\theta(t)}{\gamma^2(t)} = \dot{\tau}(t) = \lambda(t) + \frac{\dot{\gamma}(t)}{\gamma(t)},$$

i.e.,

$$\lambda(t)\theta(t) = \gamma^2(t)\lambda(t) + \gamma(t)\dot{\gamma}(t) = \gamma^2(t)\lambda(t) + \frac{1}{2}\frac{d}{dt}\{\gamma^2(t)\}.$$

Since $\gamma(0) = \sqrt{T_0}$, one obtains

$$\gamma(t) = \exp\left(-\int_0^t \lambda(s)ds\right) \left\{ T_0 + 2\int_0^t \lambda(s)\theta(s) \exp\left(2\int_0^s \lambda(r)dr\right) ds \right\}^{1/2}, \quad t > 0. \tag{3.10}$$

Now, from (3.9),

$$\dot{\tau}(t) = \frac{\lambda(t)\theta(t)}{\gamma^2(t)} = \frac{\lambda(t)\theta(t) \exp\left(2\displaystyle\int_0^t \lambda(s)ds\right)}{T_0 + 2\displaystyle\int_0^t \lambda(s)\theta(s) \exp\left(2\displaystyle\int_0^s \lambda(r)dr\right) ds} =$$

$$= \frac{1}{2}\frac{d}{dt}\log\left[T_0 + 2\int_0^t \lambda(s)\theta(s) \exp\left(2\int_0^s \lambda(r)dr\right) ds \right].$$

Solving this equation with the initial datum $\tau(0) = 0$ one gets

$$\tau(t) = \frac{1}{2} \log \left[T_0 + 2 \int\limits_0^t \lambda(s)\theta(s) \exp \left(2 \int\limits_0^s \lambda(r)dr \right) ds \right], \quad t > 0. \qquad (3.11)$$

Clearly,

$$\lim_{t \to \infty} \tau(t) = +\infty \quad \text{if and only if} \quad \int\limits_0^\infty \lambda(t)\theta(t) \exp \left(2 \int\limits_0^t \lambda(s)ds \right) dt = \infty.$$

This leads to the following result.

Theorem 3.1. *Let us assume that* $\lambda(\cdot)$ *and* $\theta(\cdot)$ *are nonnegative functions on* $\mathbb{R}_+$ *satisfying*

$$\int\limits_0^\infty \lambda(t)\theta(t) \exp \left(2 \int\limits_0^t \lambda(s)ds \right) dt = \infty. \qquad (3.12)$$

Let us assume furthermore that $f_0 \in \mathcal{M}$ *has a finite relative entropy. Then, there exists a constant* $C > 0$ *such that*

$$\|f(\cdot, t) - f_\infty(\cdot, t)\|_{L^1(\mathbb{R}_v^N)} \leqslant$$
$$\leq \frac{C}{\left\{ T_0 + 2 \int_0^t \lambda(s)\theta(s) \exp \left(2 \int_0^s \lambda(r)dr \right) ds \right\}^{1/2}}, \quad t > 0. \qquad (3.13)$$

The intermediate asymptotic profile $f_\infty(v, t)$ *is given by*

$$f_\infty(v, t) = (2\pi\, T(t))^{-N/2} \exp\left\{ -v^2/2\, T(t) \right\} = M_{T(t)}(v)$$

where $T(t)$ *is the temperature of* $f(\cdot, t)$ *given by* (3.10).

Proof. The proof reduces to the study of (3.6). Clearly, one may choose $\sigma = 1$ (this is equivalent to change τ by (3.11)). By (3.12), $\tau(t) \to \infty$, and according to Theorem 1.1,

$$\|F(\cdot, \tau) - M_1(\cdot)\|_{L^1(\mathbb{R}_{\tilde{v}}^N)} \leqslant C \exp\left\{ -\tau \right\}, \quad \tau > 0, \qquad (3.14)$$

provided $F(\tilde{v}, 0)$ is of finite relative entropy. This is the case since $F(\tilde{v}, 0) = f_0(v/T_0)$. Now, turning back to the original variables, one gets the conclusion using the fact that

$$\exp\{-\tau(t)\} = \left\{ 1 + 2 \int\limits_0^t \lambda(s)\theta(s) \exp \left(2 \int\limits_0^s \lambda(r)dr \right) ds \right\}^{-1/2}, \quad t > 0,$$

by virtue of (3.11).

The theorem is proved.

Remark 3.1. The intermediate asymptotic is given by the Maxwellian distribution with the same temperature $T(t)$ as the one of $f(\cdot, t)$. Of course, this Maxwellian distribution is a particular solution to (3.2). The most important feature of Theorem 3.1 is that it provides the rate of convergence of any solution $f(v, t)$ towards the self-similar profile $f_\infty(v, t)$. This rate is explicit in terms of the known coefficients $\lambda(t)$ and $\theta(t)$.

4. The homogeneous cooling state for the Brownian particles. We apply here the results of Section 3 to the study of the so-called *homogeneous cooling state* for equation (2.8). Let $f_0(\tilde{v})$ be an element of $\mathcal{M}$ and let us consider the Cauchy problem

$$\frac{\partial f}{\partial t}(\tilde{v}, t) = \lambda(t)\nabla_{\tilde{v}} \cdot \{\tilde{v} f(\tilde{v}, t) + \theta(t)\nabla_{\tilde{v}}f(\tilde{v}, t)\}, \quad \tilde{v} \in \mathbb{R}^3, \quad t > 0,$$
$$f(\tilde{v}, 0) = f_0(\tilde{v}), \tag{4.1}$$

where we transformed the right-hand side of equation (2.8) into a nonautonomous Fokker–Planck operator of the form (3.1) by setting, according to (2.6),

$$\lambda(t) = \alpha(t) = \zeta T_g(t)^{1/2} \quad \text{and} \quad \theta(t) = \frac{\eta(t)}{\alpha(t)} = \xi\zeta^{-1}T_g(t). \tag{4.2}$$

Accordingly, for any $t > 0$ and with the notations of Section 3

$$\int_0^t \lambda(s)\theta(s)\exp\left(2\int_0^s \lambda(r)dr\right)ds = \xi\int_0^t T_g(s)^{3/2}\exp\left(2\zeta\int_0^s \sqrt{T_g(r)}dr\right)ds =$$

$$= \xi T_g(0)^{3/2}\int_0^t (1 + s/\tau_0)^{-3}\exp\left(2\zeta\sqrt{T_g(0)}\int_0^s \frac{dr}{1 + r/\tau_0}\right)ds.$$

Note that the last equality follows from Haff's law (2.3). Clearly, for any $s > 0$

$$(1 + s/\tau_0)^{-3}\exp\left(2\zeta\sqrt{T_g(0)}\int_0^s \frac{dr}{1 + r/\tau_0}\right) = (1 + s/\tau_0)^{\nu}$$

with $\nu = 2\zeta\tau_0\sqrt{T_g(0)} - 3$. Consequently, condition (3.12) of Theorem 3.1 is verified provided $\nu \geqslant -1$. Note however that, if $\nu = -1$, then

$$T(t) = \exp\left(-2\int_0^t \lambda(s)ds\right)\left\{T_0 + 2\int_0^t \lambda(s)\theta(s)\exp\left(2\int_0^s \lambda(r)dr\right)ds\right\} =$$

$$= (1 + t/\tau_0)^{-2}\left(T_0 + \xi T_g(0)^{3/2}\tau_0\log(1 + t/\tau_0)\right),$$

so that

$$\frac{T(t)}{T_g(t)} \longrightarrow \infty \quad \text{as} \quad t \to \infty.$$

In this case, assumption (2.9) is violated, and the Fokker–Planck equation (2.8) is meaningless. Hence, condition (3.12) of Theorem 3.1 reduces to $\nu > 1$, i.e.,

$$\zeta\tau_0\sqrt{T_g(0)} > 1. \tag{4.3}$$

Using (2.4) and (2.7) the above condition reads

$$16\frac{(1 + \epsilon)\Delta}{1 - \epsilon_g^2}\frac{\displaystyle\int \Phi(\tilde{v})|\tilde{v}|d\tilde{v}}{\displaystyle\int \Phi(\tilde{v})\Phi(\tilde{w})|\tilde{v} - \tilde{w}|^3d\tilde{v}d\tilde{w}} > 1.$$

Now, recall that the Fokker – Planck equation (3.1) turns out to be valid only for nearly elastic surrounding particles (see Remark 2.1). Since in this quasielastic regime $\Phi(\tilde{v})$ approaches the Maxwellian distribution $\pi^{-3/2}\exp(-\tilde{v}^2)$, one can reasonably approximate the two moments $\int_{\mathbb{R}^3}\Phi(\tilde{v})|\tilde{v}|d\tilde{v}$ and $\int_{\mathbb{R}^3\times\mathbb{R}^3}\Phi(\tilde{v})\Phi(\tilde{w})|\tilde{v}-\tilde{w}|^3d\tilde{v}d\tilde{w}$ by their limits as $\epsilon_g\to 1$, obtaining

$$\int_{\mathbb{R}^3}\pi^{-3/2}\exp\left(-\tilde{v}^2\right)|\tilde{v}|d\tilde{v}=2/\sqrt{\pi}$$

and

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\pi^{-3}\exp\left(-\tilde{v}^2-\tilde{w}^2\right)|\tilde{v}-\tilde{w}|^3d\tilde{v}d\tilde{w}=16/\sqrt{2\pi}.$$

Using these approximation, equation (4.3) turns out to be equivalent to

$$2\sqrt{2}\Delta\frac{1+\epsilon}{1-\epsilon_g^2}>1.$$

Once again, we find the same condition (2.10) of validity of the nonautonomous Fokker – Planck equation (2.8).

Remark 4.1. Note that, here again, we only assume that the surrounding gas particles suffer nearly elastic collisions (i.e., $1-\epsilon_g\ll 1$) but we not assume ϵ_g to be equal to one. As a consequence, we do not replace the cooling state profile Φ by the Maxwellian distribution, but we only assume that its moments do not differ to much from the ones of the Maxwellian distribution.

The previous reasoning leads to the following theorem.

Theorem 4.1. *Let us assume that*

$$\frac{1}{2\sqrt{2}\Delta}\frac{1-\epsilon_g^2}{1+\epsilon}\longrightarrow\beta^{-1}<1\quad as\quad\Delta\to 0,\quad\epsilon_g\to 1.$$

Let $f_0\in\mathcal{M}$ be of finite relative entropy. Then, the solution $f(\tilde{v},t)$ to the Fokker – Planck equation (3.1) converges towards the cooling Maxwellian

$$f_\infty(\tilde{v},t)=(2\pi T(t))^{-3/2}\exp\left\{-\tilde{v}^2/2T(t)\right\},$$

and the following bound holds

$$\|f(\cdot,t)-f_\infty(\cdot,t)\|_{L^1(\mathbb{R}^3)}=O(t^{1-\beta}),\quad t\to\infty.$$

The temperature of the Maxwellian is given by

$$T(t)=\frac{(1+\epsilon)\Delta T_g(0)}{2m_g(1-\beta^{-1})}\left(1+t/\tau_0\right)^{-2}+\left(T(0)-\frac{(1+\epsilon)\Delta T_g(0)}{2m_g(1-\beta^{-1})}\right)\left(1+t/\tau_0\right)^{-2\beta}.$$

Proof. The proof is a straightforward application of Theorem 3.1. Here we use the fact that

$$\int\limits_0^t \lambda(s)\theta(s)\exp\left(2\int\limits_0^s \lambda(r)dr\right)ds = \xi T_g(0)^{3/2}\int\limits_0^t (1+s/\tau_0)^{2\beta-3}ds =$$

$$= \frac{\tau_0}{2\beta-2}\xi T_g(0)^{3/2}\left\{(1+t/\tau_0)^{2\beta-2}-1\right\}.$$

Moreover, by (3.10), the temperature of the heavy particles is

$$T(t) = \exp\left(-2\int\limits_0^t \lambda(s)ds\right)\left\{T(0) + 2\int\limits_0^t \lambda(s)\theta(s)\exp\left(2\int\limits_0^s \lambda(r)dr\right)ds\right\}.$$

This last quantity is explicitly computable using the expressions of τ_0 and ξ.

Remark 4.2. One notes that $T(t)$ obeys asymptotically Haff's law since the decay of temperature of the heavy particles for large t is in $O(1+t/\tau_0)^{-2}$. Actually,

$$\frac{T(t)}{T_g(t)} \longrightarrow \frac{(1+\epsilon)\Delta}{2m_g(1-\beta^{-1})} \quad \text{as} \quad t \to \infty.$$

An interesting feature is that, depending on the values of the parameters ϵ, Δ, m_g and β, the temperature of the heavy particles is greater or smaller than the one of the surrounding gas. This contrasts the classical case of elastic particles in equilibrium. Indeed, in this case, according to Theorem 1.1, the distribution function relaxes to a Maxwellian distribution whose temperature θ is exactly the one of the surrounding bath (see [3] for more details). We refer the reader to [8] for a discussion of the competing effects which imply the asymptotic difference between $T(t)$ and $T_g(t)$.

Remark 4.3. We point out that, whereas for systems in equilibrium, the relaxation rate is exponential (Theorem 1.1), one notes here that the *Homogeneous Cooling State* $f_\infty(\tilde{v}, t)$ attracts the distribution function $f(\tilde{v}, t)$ only with an algebraic rate.

5. Concluding remarks. We discussed in this paper the intermediate asymptotics of a linear Fokker–Planck equation with time-dependent coefficients of the form (1.2). We showed that, under some reasonable assumptions on the drift and diffusion coefficients, any solution $f(v, t)$ to (1.2) relaxes towards a Maxwellian distribution function whose (time-dependent) temperature is the one of $f(v, t)$. More important is the fact that the rate of convergence towards this self-similar solution is explicitly computable in terms of the coefficients and the initial temperature.

We applied our result to the motivating example of Brownian motion in granular fluids, already addressed in [8]. For such a model, the Fokker–Planck equation (1.2) is an approximation of the Boltzmann–Lorentz equation. According to our general result (Theorem 3.1), the so-called Homogeneous Cooling State for this model is a Maxwellian distribution whose temperature obeys asymptotically the Haff's law. Moreover, the rate of convergence towards this self-similar solution is algebraic in time. We wish to emphasize here the fact that the question of the rate of convergence towards the Homogenous Cooling State (HCS) is of primary importance in the kinetic theory of gases. We recall here that any solution to the nonlinear Boltzmann equation for inelastic interactions relaxes towards a Dirac mass because of the dissipation of the kinetic energy. It has been conjectured however by Ernst and Brito [13, 17] that the HCS attracts any solution faster than the Dirac mass does. For hard-spheres interactions, only few results support this conjecture. For nearly elastic flows in one-dimension, the nonlinear Boltzmann equation reduces to a nonlinear friction equation [18, 19] and it has been shown in [20] that, in this

case, the HCS does not attract much faster than the Dirac mass since the improvement in the rate of convergence is only logarithmic in time. This question has also been addressed recently in [21].

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