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KINETIC EQUATIONS AND THE INTEGRABLE HAMILTONIAN SYSTEMS

КІНЕТИЧНІ РІВНЯННЯ ТА ІНТЕГРОВНІ ГАМІЛЬТОНОВІ СИСТЕМИ

A survey of interrelations between kinetic equations and integrable systems is presented. We discuss common origin of special classes of solutions of the Boltzmann kinetic equation for Maxvellian particles and the special solutions for integrable evolution equations. The thermodynamic limit and the soliton kinetic equation for the integrable Korteweg – de Vries equation are considered. The existence of decaying and degenerate dispersion laws and the appearance of additional integrals of motion for the interacting waves is discussed.

Наведено огляд взаємозв'язків між кінетичними рівняннями та інтегровними системами. Обговорено загальне походження спеціальних класів розв'язків кінетичного рівняння Больцмана для максвеллівських частинок і спеціальних розв'язків інтегровних еволюційних рівнянь. Розглянуто термодинамічну границю та солітонне кінетичне рівняння для інтегровного рівняння Кортевега – де Фріза. Обговорено існування розпадних і вироджених законів дисперсії та виникнення додаткових інтегралів руху для взаємодіючих хвиль.

1. Introduction. There are different rich, interesting and deep interrelations between kinetic equations and integrable systems. Our aim here is to present a survey of appropriate studies in this field. We formulate results which are related to this question but do not prove them and give only references.

Our paper is organized as follows. The second section presents a short discussion of a Poincaré linearization method for nonlinear equations. In the third section we describe a construction by A. V. Bobylev the exact solutions of the Boltzmann kinetic equation for Maxwellian particles. In the fourth section we present the inverse spectral transform method for solution of integrable nonlinear partial differential equations as a linearization method. In the fifth section we consider results by V. E. Zakharov and E. I. Shul'man on the kinetic equation, integrals of motion and degenerative dispersion laws. Then we deduce the kinetic equation for solitons of the Korteweg – de Vries equation.

2. A linearization method for nonlinear equations. According to Poincaré one can transform any nonlinear nonresonance vector field to its linear part by means of a formal diffeomorphism. By means of a Poincaré – Dulac transformation we can also take in account resonance terms. If a convex hull of eigenvalues of a linear part of the vector field does not contain a zero then this vector field is reduced to a polynomial normal form by means of above diffeomorphism (see e.g. [1]).

2.1. Nonlinear integro-differential equation. Let us apply this theorem on the normal form to a case of function

$$u(x,t), \quad -\infty < x < +\infty, \quad 0 < t < +\infty,$$

satisfying the nonlinear (quadratic, for simplicity) integro-differential equation

$$u_t(x,t) + \int_0^\infty d\theta \Lambda(\theta) u(x+\theta,t) =$$

$$= \int_0^\infty d\theta_1 \int_0^\infty d\theta_2 H(\theta_1,\theta_2) u(x+\theta_1,t) u(x+\theta_2,t).$$

Using the Poincaré theorem it is possible to transform this nonlinear equation to its linear part. We give an exact formulation of the above statement.

Theorem 2.1. *If $v(x,t)$ is a solution of the linear equation*

$$v_t(x,t) + \int_0^\infty d\theta \Lambda(\theta) v(x+\theta,t) = 0, \tag{1}$$

and a following nonresonance condition for the spectrum of the linear equation is fulfilled,

$$\Big|\Delta_n(p_1,\ldots,p_n)\Big| > 0, \quad n = 2,3,\ldots, \tag{2}$$

then the function

$$u(x,t) = \hat{R}v(x,t) = \sum_{n=1}^\infty \int_0^\infty d\theta_1 \ldots \int_0^\infty d\theta_n R_n(\theta_1 \ldots \theta_n) \prod_{j=1}^n v(x+\theta_j,t) \tag{3}$$

is a formal solution of the nonlinear equation

$$u_t(x,t) + \int_0^\infty d\theta \Lambda(\theta) u(x+\theta,t) =$$

$$= \int_0^\infty d\theta_1 \int_0^\infty d\theta_2 H(\theta_1,\theta_2) u(x+\theta_1,t) u(x+\theta_2,t). \tag{4}$$

Here the coefficient functions $R_n(\theta_1 \ldots \theta_n)$, are expressed in terms of its Laplace images $r_n(p_1,\ldots,p_n)$,

$$r_n(p_1,\ldots,p_n) = \int_0^\infty d\theta_1 \ldots \int_0^\infty d\theta_n R_n(\theta_1 \ldots \theta_n) \exp\left(-\sum_{j=1}^n p_j\theta_j\right),$$

that satisfy the recurrent relations

$$r_1(p_1) = 1,$$

$$r_n(p_1,\ldots,p_n) = \Delta_n^{-1}(p_1,\ldots,p_n) \sum_{k=1}^{n-1} h(p_1+\ldots+p_k, p_{k+1}+\ldots+p_n) \times$$

$$\times r_k(p_1,\ldots,p_k) r_{n-k}(p_{k+1},\ldots,p_n), \quad n = 2,3,\ldots,$$

where

$$\lambda(p) = \int_0^\infty d\theta \Lambda(\theta) \exp(-p\theta),$$

$$h(p_1, p_2) = \int_0^\infty d\theta_1 \int_0^\infty d\theta_2 H(\theta_1, \theta_2) \exp\left(-(p_1\theta_1 + p_2\theta_2)\right),$$

$$\Delta_n(p_1, \ldots, p_n) = \lambda\left(\sum_{j=1}^n p_j\right) - \sum_{j=1}^n \lambda(p_j).$$

In order to prove the theorem we have just to substitute the above expression for the function $u(x,t)$ in terms of the $v(x,t)$ into the nonlinear equation (4). Obviously we can generalize the theorem from quadratic to higher type of nonlinearity.

2.2. Special case. Let us study a special case of the functions $r_n(p_1, \ldots, p_n)$ when

$$r_1(p_1) = 1,$$

$$\langle r_n(p_1, \ldots, p_n)\rangle = \left\langle \prod_{j=1}^{n-1} r_2(p_j, p_{j+1})\right\rangle, \quad n = 2, 3, \ldots,$$

where $\langle\ \rangle$ means averaging over all permutations of arguments. Under this assumption we can prove the following theorem.

Theorem 2.2. *The function*

$$K(x,y) = (1-A)^{-1} v\left(\frac{1}{2}(x+y)\right),$$

with the operator A acting on the function of two variables $f(x,y)$ in such a way,

$$(Af)(x,y) = \int_0^\infty d\theta_1 f(x, x+2\theta_1) \int_0^\infty d\theta_2 R_2(\theta_1, \theta_2) v\left(\frac{1}{2}(x+y) + \theta_2\right),$$

defines a solution of the nonlinear equation (4) *as follows:*

$$u(x) = K(x,x).$$

2.3. The N-mode solutions. It is possible to prove that the following solutions of the linear equation:

$$v_N(x,t) = \sum_{i=1}^N \frac{\gamma_i}{\Gamma(\alpha_i+1)} \exp[-(\alpha_i x + \lambda(\alpha_i)t)], \quad N = 1, 2, \ldots,$$

with real parameters

$$\gamma_i; \quad \alpha_i : p_0 < \alpha_1 < \ldots < \alpha_i < \alpha_{i+1} < \ldots < \alpha_{N-1},$$

correspond to the following solutions of the nonlinear equation:

$$u_N(x,t) = \sum_{n=1}^\infty \sum_{m_1=1}^N \ldots \sum_{m_n=1}^N r_n(\alpha_{m_1}, \ldots, \alpha_{m_n}) \times$$

$$\times \prod_{j=1}^n \frac{\gamma_{m_j}}{\Gamma(\alpha_{m_j}+1)} \exp[-(\alpha_{m_j} x + \lambda(\alpha_{m_j})t)], \quad N = 1, 2, \ldots,$$

or

$$u_N(x,t) = F_N(z_1, \ldots, z_N), \quad z_j = \exp[-(\alpha_i x + \lambda(\alpha_i)t)], \quad i = 1, \ldots, N,$$

where $F_N(z_1, \ldots, z_N)$ is a formal power series with linear terms coinciding with $v_N(x,t)$. These solutions are called the N-mode solutions with parameters $\alpha_1, \ldots, \alpha_N$.

3. Exact solutions of the Boltzmann equation for Maxwellian particles. By means of the normal form theorem we can construct a class of exact solutions of the Boltzmann kinetic equation for Maxwellian particles. These solutions were discovered by A. V. Bobylev [2, 3] and M. Krook and T. T. Wu [4, 5].

The famous Boltzmann equation for the particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$ looks as follows:

$$(\partial_t + \mathbf{v}\nabla_{\mathbf{r}} + \mathbf{a}\nabla_{\mathbf{v}})\, f(\mathbf{r}, \mathbf{v}, t) =$$

$$= \int d\mathbf{w} \int d\mathbf{n}\; g\, \sigma(g, \chi)\, [f(\mathbf{r}, \mathbf{v}', t) f(\mathbf{r}, \mathbf{w}', t) - f(\mathbf{r}, \mathbf{v}, t) f(\mathbf{r}, \mathbf{w}, t)].$$

Here $\mathbf{g} = \mathbf{v} - \mathbf{w}$ is a relative velocity, $g = |\mathbf{g}|$, $\mathbf{n}$ is a unit vector in the scattering direction, $\chi = \arccos(\mathbf{g} \cdot \mathbf{n}/g)$ is a scattering angle, $\sigma(g, \chi)$ is a differential cross section and

$$\mathbf{v}' = \frac{1}{2}(\mathbf{v} + \mathbf{w}) + \mathbf{n}|\mathbf{v} - \mathbf{w}|, \qquad \mathbf{w}' = \frac{1}{2}(\mathbf{v} + \mathbf{w}) - \mathbf{n}|\mathbf{v} - \mathbf{w}|.$$

If the interaction potential is of the form

$$U(r) = \kappa r^{-s}, \quad \kappa > 0,$$

then transport cross section

$$g\, \sigma(g, \chi) = g^{1-(4/s)} \alpha(\cos\chi).$$

For the so called Maxvellian particles

$$s = 4,$$

and therefore in this case the transport cross section $g\sigma(g, \chi)$ does not depend on velocity,

$$g\, \sigma(g, \chi) = \alpha(\cos\chi).$$

Further we study a homogeneous and isotropic distribution functions,

$$f(\mathbf{r}, \mathbf{v}, t) = f(v, t),$$

when the Boltzmann equation has the well known equilibrium Maxwellian solution

$$f_0(v) = (2\pi)^{-d/2} \exp\left(-v^2/2\right).$$

In this case the Fourier transformation of the distribution function looks as follows:

$$f(\mathbf{v}, t) = \int \phi(\mathbf{k}, t) e^{i\mathbf{k}\mathbf{v}} \frac{d\mathbf{k}}{(2\pi)^3},$$

$$\phi(\mathbf{k}, t) = \int f(\mathbf{v}, t) e^{-i\mathbf{k}\mathbf{v}} d\mathbf{v}.$$

The Fourier transformation of the Boltzmann equation for the Maxwellian particles attains such a form,

$$\phi_t(\mathbf{k}, t) = \int dn\, \alpha\left(\frac{\mathbf{k}\mathbf{n}}{k}\right) \left\{ \phi\left(\frac{\mathbf{k} + k\mathbf{n}}{2}, t\right) \phi\left(\frac{\mathbf{k} - k\mathbf{n}}{2}, t\right) - \phi(\mathbf{k}, t)\phi(0, t) \right\}.$$

Using new variables

$$s = \frac{1}{2}\left(1 - \frac{\mathbf{k}\mathbf{n}}{k}\right), \quad x = -\ln\frac{k^2}{2},$$

and new functions

$$\phi(k, t) = e^{-k^2/2}[1 + b(x, t)], \quad b(x, t) = o(e^{-x}),$$

$$\rho(s) = \alpha(1-2s), \quad 0 \le s \le 1,$$

we can present the Boltzmann equation for the Maxwellian particles as the normal form problem, considered above, with the following identification:

$$H(\theta_1, \theta_2) = \rho(e^{-\theta_1}) e^{-(\theta_1+\theta_2)} \delta(e^{-\theta_1} + e^{-\theta_2} - 1),$$

$$\Lambda(\theta) = \int_0^\infty d\theta_1 \int_0^\infty d\theta_2 H(\theta_1, \theta_2)[\delta(\theta) - \delta(\theta - \theta_1) - \delta(\theta - \theta_2)].$$

Using the Poincaré linearization method for this equation we can prove the following theorem.

Theorem 3.3. *The Boltzmann equation for the Maxwellian particles has exact solutions*

$$f(\boldsymbol{v}, t) = \int \phi(\boldsymbol{k}, t) e^{i\boldsymbol{k}\boldsymbol{v}} \frac{d\boldsymbol{k}}{(2\pi)^3},$$

$$\phi(\boldsymbol{k}, t) = e^{-k^2/2} \Bigg\{ 1 + \sum_{n=1}^{\infty} \sum_{m_1=1}^{N} \ldots \sum_{m_n=1}^{N} r_n(p_{m_1}, \ldots, p_{m_n}) \times$$

$$\times \prod_{j=1}^{n} \gamma_{m_j} \frac{(k^2/2)^{p_{m_j}}}{\Gamma(p_{m_j}+1)} e^{-\lambda(p_{m_j})t} \Bigg\},$$

where the quantities $r_n(p_{m_1}, \ldots, p_{m_n})$ *are defined as it is described in the previous section and*

$$\lambda(p) = \int_0^1 ds \rho(s)[1 - s^p - (1-s)^p],$$

$$h(p_1, p_2) = \int_0^1 ds \rho(s) s^{p_1} (1-s)^{p_2}.$$

By means of the same theorem we can construct for the Boltzmann equation a countable set of conserved quantities, study the Cauchy problem for general class of initial data, consider asymptotic behavior of the distribution function at $|\mathbf{v}| \to \infty$ (formation of the so called Maxvellian tails) and at $t \to \infty$ (rate of relaxation) etc.

4. The linearization method and the inverse spectral transform. The normal form theorem appears to be equivalent to the famous inverse spectral transform method used with big success in the theory of integrable nonlinear evolution equations. This wonderful connection was discussed by A. V. Bobylev [6], V. A. Marchenko [7], V. E. Zakharov and E. I. Shulman [8] and others. In fact we consider often the integrable evolution equations as special weak perturbations of the linear ones.

Here we demonstrate the equivalence of these two methods on an example of the Korteweg – de Vries equation

$$w_t + w_{xxx} - 6ww_x = 0.$$

By means of the relation $w = -u_x$ we present this equation as follows,

$$u_t + u_{xxx} + 3u_x^2 = 0, \tag{5}$$

with a linearized form

$$v_t + v_{xxx} = 0.$$

Theorem 4.4. *The nonlinear evolution equation* (5) *for the function* $u(x,t)$ *is equivalent to the nonlinear integral equation considered in the previous section if*

$$\Lambda(\theta) = \delta'''(\theta), \quad H(\theta_1, \theta_2) = -3\delta'(\theta_1)\delta'(\theta_2).$$

Under this assumption the function

$$K(x,y) = (1-A)^{-1} v\left(\frac{x+y}{2}\right),$$

with the operator A *of the form*

$$(Af)(x,y) = \frac{1}{2}\int_x^\infty ds f(x,s) v\left(\frac{s+y}{2}\right),$$

satisfies the Gelfand – Levitan – Marchenko equation

$$K(x,y) + F(x+y) + \int_x^\infty K(x,s)F(s,y)ds = 0, \quad y > x,$$

and

$$u(x) = K(x,x).$$

The solution

$$u_N(x,t) = -\int_x^\infty w_N(s,t)ds,$$

where $w_N(x,t)$ is a N-soliton solution of the KdV equation, coincides with an analytic continuation of the N-modal solution, introduced above, to a domain

$$\gamma_i < 0, \quad i = 1, \ldots, N.$$

In terms of this approach we can prove the integrability of the equation (5), construct a countable set of integrals of motion, study the appropriate symplectic structure and Poisson brackets (of hydrodynamic type), solve the Cauchy problem for initial data of general type, etc. This is also valid for other integrable evolution equations.

The tight connection of the Poincaré linearizing method and the inverse spectral transform method allows to create new approaches in the theory of integrable evolution equations (see e.g. [9]).

5. Integrability and dispersion laws. It is interesting to understand what properties of a nonlinear evolution equation makes it integrable. The Poincaré linearizing procedure can help to discover these properties [8, 10 – 12].

Let us consider an arbitrary nonlinear evolution for the function $u(\mathbf{x},t)$, $\mathbf{x} \in \mathbb{R}^d$, and assume that this equation in terms of Fourier amplitude $a_{\mathbf{k}}(t)$ of this function attains the Hamiltonian form

$$i\frac{d}{dt}a_{\mathbf{k}}(t) = \frac{\delta H}{\delta a_{\mathbf{k}}(t)},$$

where

$$H = H_0 + H_{int},$$

$$H_0 = \int \omega_{\mathbf{k}} |a_{\mathbf{k}}|^2 d\mathbf{k},$$

$$H_{int} = \frac{1}{3!} \sum_{s,s_1,s_2} \int V^{s,s_1,s_2}_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} a^s_{\mathbf{k}} a^{s_1}_{\mathbf{k}} a^{s_2}_{\mathbf{k}} \delta(s\mathbf{k} + s_1\mathbf{k}_1 + s_2\mathbf{k}_2) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2,$$

$$s,\ s_1,\ s_2 = \pm 1, \quad a^1_{\mathbf{k}} = a_{\mathbf{k}}, \quad a^{-1}_{\mathbf{k}} = \overline{a}_{\mathbf{k}}.$$

We can interpret these equations as the equations for scattering of classical waves. Asymptotic expressions for the amplitudes of these waves $a_{\mathbf{k}}(t)$,

$$a^{\pm}_{\mathbf{k}}(t) = c^{\pm}_{\mathbf{k}} \exp(i\omega_{\mathbf{k}} t) = \lim_{t \to \pm\infty} a_{\mathbf{k}}(t),$$

are related by the classical scattering matrix S,

$$a^{+s}_{\mathbf{k}}(t) = (I + S) a^{-s}_{\mathbf{k}}(t).$$

Matrix elements of the scattering matrix have singularities of a type

$$(\Delta_{q_1} \ldots \Delta_{q_l})^{-1}$$

on the resonance manifolds defined by a set of following equations:

$$\Delta_{q_j} := \sum_{\alpha}^{q} s_\alpha \omega_{\mathbf{k}_\alpha} = 0, \quad P_q := \sum_{\alpha}^{q} s_\alpha \mathbf{k}_\alpha, \quad q := \sum_{j=1}^{l} q_j.$$

Properties of the classical scattering matrix are important in order to understand what kind of restrictions we should impose on the Hamiltonian in order to get additional integrals of motion.

A basic characteristics of the Hamiltonian is a dispersion law $\omega_{\mathbf{k}}$. The dispersion law is called decaying or nondecaying depending on whether the equation

$$\omega_{\mathbf{k}_1+\mathbf{k}_2} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}$$

has real solution or not. For decaying laws the above equation defines a real surface Γ. The decaying law is called degenerate if there exists a function $f_{\mathbf{k}}$ which satisfies on the surface Γ the equation

$$f_{\mathbf{k}_1+\mathbf{k}_2} = f_{\mathbf{k}_1} + f_{\mathbf{k}_2}$$

but is not a linear superposition of the $\omega_{\mathbf{k}}$ and **k**.

Let us mention here that the well known KP-1 equation is characterized by the dispersion law

$$\omega_{\mathbf{k}} = \omega_{p,q} = p^3 + 3q^2/p, \quad \mathbf{k} = (p, q)$$

which is decaying and degenerate, and the KP-2 equation is characterized by the dispersion law

$$\omega_{\mathbf{k}} = \omega_{p,q} = p^3 - 3q^2/p, \quad \mathbf{k} = (p, q)$$

which is nondecaying.

It appears that the above properties of dispersion laws are very important for solution of the following the problem: do there exist any additional, besides of linear momentum and energy, integrals of motion or not?

According to V. E. Zakharov and E. I. Shulman for the existence of an additional motion integral of motion of the form

$$I[a] = I[a]_0 + \ldots = \int \sum_{\alpha_j} f^{\alpha_j}_{\mathbf{k}} |a^{\alpha_j}_{\mathbf{k}}|^2 d\mathbf{k} + \ldots,$$

it is necessary that on each resonance manyfold

$$\Delta_{q_j} := \sum_{\alpha}^{q} s_\alpha \omega_{\mathbf{k}_\alpha} = 0, \qquad P_q := \sum_{\alpha}^{q} s_\alpha \mathbf{k}_\alpha,$$

in the generic situation the following alternative occurs: 1) either the amplitude of the scattering matrix on the manifold is equal zero, or 2) the following condition holds,

$$\sum_{j=1}^{l} s_j f_{\mathbf{k}}^{\alpha_j} = 0.$$

6. Kinetic equation for solitons of the Korteweg – de Vries equation. In connection to integrable evolution equations the kinetic equations naturally appears when one study many-soliton solutions. As an example let us consider a solitons kinetic equation for the Korteweg – de Vries equation.

6.1. The N-phase nonlinear wave. The N-phase solution of the KdV equation with phases ϕ_j, wave vectors k_j and frequencies ω_j looks as follows (see e.g. [13]):

$$u_N = \theta_N(\phi_1, \ldots, \phi_j, \ldots, \phi_N),$$
$$\phi_j = k_j x + \omega_j t + \phi_j^0, \quad j = 1, \ldots, N.$$

Here θ_N is N-dimensional theta-function,

$$\theta_N(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbb{Z}^N} \exp\{-i\pi(\mathbf{m}, B\mathbf{m}) + i2\pi(\mathbf{z}, \mathbf{m})\}, \quad \mathbf{z} \in \mathbb{C}^N,$$

with the $N \times N$ period matrix B_{jk}. This matrix is defined by equality

$$B_{jk} = \int_{b_k} \psi_j,$$

where ψ_j, $j = 1, \ldots, N$, are the normalized holomorphic differentials,

$$\psi_j = \sum_{k=1}^{N} a_{jk} \frac{E^k}{R(E)}\, dE, \quad \int_{a_k} \psi_j = \delta_{jk}, \quad j = 1, \ldots, N,$$

and a_k, b_k, $k = 1, \ldots, N$, is a basis of cycles for the Riemann surface Γ of the genus N,

$$\Gamma: \quad R^2(E) = \prod_{j=1}^{2N+1} (E - E_j), \quad E \in \mathbb{C},$$

$$E_1 < E_2 < \ldots < E_{2N} < E_{2N+1}, \quad E_j \in \mathbb{R}.$$

The wave vectors and frequencies k_m, ω_m, $m = 1, \ldots, N$, are solutions of a system of linear equations

$$B_{lm} k_m = -4\pi\imath a_{N-1,l},$$

$$B_{lm}\omega_m = -8\pi\imath \left(a_{N-1,l} \sum_{j=1}^{2n+1} E_j + 2a_{N-2,l} \right).$$

The wave vector meromorphic differential $dp_N(E)$ and the frequency meromorphic differential $dq_N(E)$ are defined as follows:

$$dp_N(E) = \frac{E^N + b^{N-1}E^{N-1} + \ldots + b_0}{R(E)}, \qquad \int_{b_l} dp_N(E) = 0, \quad l = 1, \ldots, N,$$

$$dq_N(E) = 12\frac{E^{N+1} + c^N E^N + \ldots + c_0}{R(E)}, \qquad \int\limits_{b_l} dq_N(E) = 0, \quad l = 1, \ldots, N.$$

In terms of above differentials we can present the space rotation number $\alpha_N(E)$ and and the time rotation number $\beta_N(E)$,

$$\alpha_N(E) = \operatorname{Re}\frac{1}{\pi}\int\limits_{-1}^{E} dp_N(E'), \quad E \in (-1, 0),$$

$$\beta_N(E) = \operatorname{Re}\frac{1}{\pi}\int\limits_{-1}^{E} dq_N(E'), \quad E \in (-1, 0).$$

6.2. Slow modulations. Modulations of the N-phase waves

$$u_N(x, t; X, T) = \theta_N(\phi_1, \ldots, \phi_j, \ldots, \phi_N | B_{lm}(X, T)),$$
$$\phi_j = k_j(X, T)x + \omega_j(X, T)t + \phi_j^0(X, T), \quad j = 1, \ldots, N,$$
$$X = \varepsilon x, \quad T = \varepsilon t, \quad \varepsilon \ll 1,$$

resulting due to modulations of the branchng points through the slow variables X and T are formulated as a conservation law equation [14]:

$$\partial_T dp_N = \partial_X dq_N.$$

6.3. The soliton limit. 6.3 Now let us consider a soliton limit of the N-phase wave [15]

$$|\mathrm{gap}E| \sim \frac{1}{N}, \quad |\mathrm{band}E| \sim e^{-N}, \quad N \gg 1, \quad E \in (-1, 0).$$

We shall designate further the centers of spectral bands as

$$-\eta_j^2 = \frac{1}{2}(E_{2j-1} + E_{2j}),$$

and assume that

$$1 \simeq \eta_1 > \eta_2 > \cdots \eta_N \simeq 0.$$

In soliton limit we obtain a following expression for the period matrix

$$B_{lm} \simeq -\frac{\imath}{\pi}\left(\ln\frac{\eta_l - \eta_m}{\eta_l + \eta_m} + n\gamma(\eta_l)\delta_{lm}\right), \quad l, m = 1, \ldots, N,$$

and the normalization coefficients of holomorphic differentials attains the form,

$$a_{N-1,l} \simeq -\frac{\eta_l}{2\pi}, \quad a_{N-1,l}\sum_{j=1}^{2N+1} E_j + 2a_{N-2,l} \simeq \frac{\eta_l^3}{\pi}, \quad l = 1, \ldots, N.$$

As a result of that we can define now the wave vectors k_l and frequencies ω_l in terms of the functions $\kappa(\eta_l)$ and $\omega(\eta_l)$

$$k_l = \frac{1}{N}\kappa(\eta_l), \quad \omega_l = \frac{1}{N}\omega(\eta_l),$$

satisfying the following equations:

$$\frac{1}{N}\sum_{j=1}^{N}\log\left|\frac{\eta_I - \eta_j}{\eta_I + \eta_j}\right| k(\eta_j) + \gamma(\eta_j)k(\eta_j) = -2\pi\eta_i,$$

$$\frac{1}{N}\sum_{j=1}^{N}\log\Big|\frac{\eta_I-\eta_j}{\eta_I+\eta_j}\Big|\omega(\eta_j)+\gamma(\eta_j)\omega(\eta_j)=8\eta_i^3.$$

6.4. Thermodynamic limit and kinetic equation. Let us consider a thermodynamic limit

$$N, L\to\infty, \quad N/L=\mathrm{const},$$

where N is a number of solitons and L is a space length. In this case we can define the distribution functions for the wave vectors and frequencies $\alpha(\nu)$, $\beta(\nu)$ in such a way:

$$d\alpha(-\eta^2)=\frac{1}{2}\phi(\eta)\kappa(\eta)d(\eta), \quad d\beta(-\eta^2)=\frac{1}{2}\phi(\eta)\omega(\eta)d(\eta),$$

where

$$\phi(\eta_l)\simeq\frac{1}{N(\eta_l-\eta_{l-1})}, \qquad \int_0^1\phi(\eta)d\eta=1, \quad \eta^2=-E\in[-1,0],$$

$$\gamma(\eta_l)\simeq-\frac{1}{N}\ln\delta_l+O\left(\frac{1}{N}\right), \quad \delta_l=E_{2l}-E_{2l-1}.$$

These distribution functions satisfy the following integral equation:

$$\int_0^1\ln\Big|\frac{\eta-\mu}{\eta+\mu}\Big|\frac{\mu}{\eta}\alpha'(-\mu^2)d\mu+\sigma(\eta)\alpha'(-\eta^2)=\frac{\pi}{2},$$

$$\int_0^1\ln\Big|\frac{\eta-\mu}{\eta+\mu}\Big|\frac{\mu}{\eta}\beta'(-\mu^2)d\mu+\sigma(\eta)\beta'(-\eta^2)=-2\pi\eta^2,$$

where

$$\sigma(\eta)=\gamma(\eta)/\phi(\eta).$$

Let us introduce the distribution function of solitons

$$f(\eta)=\frac{1}{\pi}\alpha(-\eta^2), \qquad \int_0^1 f(\eta)d\eta=\kappa,$$

and the velocity of soliton

$$s(\eta)=\frac{\beta'(-\eta^2)}{\alpha'(-\eta^2)}.$$

Then we can write down the kinetic equation for the distribution function of solitons,

$$s=-4\eta^2+\frac{1}{\eta}\int_0^1\ln\Big|\frac{\eta-\mu}{\eta+\mu}\Big|f(\mu)[s(\mu)-s(\eta)]d\mu.$$

Under assumption of small density of solitons we have [16]

$$s\simeq-4\eta^2-\frac{4}{\eta}\int_0^1\ln\Big|\frac{\eta-\mu}{\eta+\mu}\Big|f(\mu)(\mu^2-\eta^2)d\mu.$$

6.5. Kinetic equation for the solitons with sources. In 1988 V. K. Mel'nikov [17] studied for the first time the soliton equations with sources. It is possible to generalize the kinetic equation for solitons on the case of presence of sources. In this case we should add an additional term to the kinetic equation. This question deserves a special consideration.

7. Conclusion. We gather in the paper a number of results on the connections of kinetic equations and integrable systems. It appears that they are related very closely. They have common problems and common methods to study them. We believe that considerations of these two domains of mathematical physics from a single point of view may be helpful for both of them.

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