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# LOCAL MAXIMA OF THE POTENTIAL ENERGY ON SPHERES

# ЛОКАЛЬНІ МАКСИМУМИ ПОТЕНЦІАЛЬНОЇ ЕНЕРГІЇ НА СФЕРАХ

Let $S^d$ be a unit sphere in $\mathbb{R}^{d+1}$, and let $\alpha$ be a positive real number. For pairwise different points $x_1, x_2, \ldots, x_N \in S^d$, we consider a functional $E_\alpha(x_1, x_2, \ldots, x_N) = \sum_{i\neq j} \|x_i - x_j\|^{-\alpha}$. The following theorem is proved: for $\alpha \geq d-2$, the functional $E_\alpha(x_1, x_2, \ldots, x_N)$ does not have local maxima.

Нехай $S^d$ — одинична сфера в $\mathbb{R}^{d+1}$, а $\alpha$ — додатне число. Для попарно різних точок $x_1, x_2, \ldots, x_N \in S^d$ розглядається функціонал $E_\alpha(x_1, x_2, \ldots, x_N) = \sum_{i\neq j} \|x_i - x_j\|^{-\alpha}$. Доведено, що при $\alpha \geq d-2$ функціонал $E_\alpha(x_1, x_2, \ldots, x_N)$ не має локальних максимумів.

**1. Introduction.** In this short note we will prove that certain potential energy functionals on sphere attain no local maxima. This partially answers the question that Professor Edward Saff asked on the conference “Optimal Configurations on the Sphere and Other Manifolds” at Vanderbilt University in 2010.

For $d \in \mathbb{N}$, denote by $S^d$ a unit sphere in $\mathbb{R}^{d+1}$. For $\alpha > 0$ and any configuration of $N \geq 2$ distinct points $x_1, x_2, \ldots, x_N \in S^d$ consider the following energy functional:

$$E_\alpha(x_1, x_2, \ldots, x_N) = \sum_{i\neq j} \frac{1}{\|x_i - x_j\|^\alpha},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{d+1}$.

For $d = 2$, $\alpha = 1$ this functional has a physical interpretation as the electrostatic potential energy of a system containing $N$ equally charged particles on the sphere.

The problem of finding configurations which minimize these functionals is closely related to the problem of finding uniformly distributed collections of points on sphere, in particular, of spherical designs (see [3, 1]), as well as to the problem of finding optimal spherical codes (see [2]).

It is clear that for each $d \in \mathbb{N}$, $N \geq 2$, and $\alpha > 0$ there exists a configuration of $N$ points on sphere $S^d$ at which $E_\alpha$ has a local (and even global) minimum. In his closing speech at the conference “Optimal Configurations on the Sphere and Other Manifolds” Professor Saff asked whether $E_\alpha$ can have local maxima. We prove the following theorem, which says that for sufficiently large $\alpha$ this is impossible.

**Theorem 1.** *For positive $\alpha \geq d-2$ the functional $E_\alpha(x_1, x_2, \ldots, x_N)$ has no local maxima.*

**2. Proof of Theorem 1.** For convenience, we rescale the energy by a factor of $2^{\alpha/2}$. Let $r = \alpha/2$ and denote $g_r(t) = (1-t)^{-r}$. Then

$$\frac{2^r}{\|x_i - x_j\|^\alpha} = g_r(\langle x_i, x_j \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^{d+1}$. Therefore, the energy functional can be written as

$$E_\alpha(x_1,\ldots,x_N)=\sum_{i\neq j} g_r(\langle x_i,x_j\rangle).$$

Introduce arbitrary vectors $h_1,h_2,\ldots,h_N$ orthogonal to corresponding $x_i$, (i.e. $\langle x_i,h_i\rangle=0$) and consider the function $f\colon \mathbb{R}\to\mathbb{R}$ defined by

$$f(t)=E_\alpha\left(\frac{x_1+th_1}{\|x_1+th_1\|},\ldots,\frac{x_N+th_N}{\|x_N+th_N\|}\right).$$

If $E_\alpha$ attains a local maximum at $x_1,\ldots,x_N$, then we must have $f'(0)=0$ and $f''(0)\le 0$. The expression for second derivative is

$$f''(0)=\sum_{i\neq j}\left[g_r''(\langle x_i,x_j\rangle)(\langle x_i,h_j\rangle+\langle x_j,h_i\rangle)^2+\right.$$

$$\left.+\,g_r'(\langle x_i,x_j\rangle)\bigl(2\langle h_i,h_j\rangle-(\|h_i\|^2+\|h_j\|^2)\langle x_i,x_j\rangle\bigr)\right]. \tag{1}$$

Therefore, in order to prove that our energy has no local maxima it is sufficient to find $h_i$ such that (1) is strictly positive. To do so, take $h_2=h_3=\ldots=h_N=0$ and $h_1=h$, where $\|h\|=1$. Then $f''(0)/2$ is equal to

$$\sum_{j=2}^{N}\left[g_r''(\langle x_1,x_j\rangle)\langle x_j,h\rangle^2-g_r'(\langle x_1,x_j\rangle)\langle x_1,x_j\rangle\right]. \tag{2}$$

Suppose that (2) is nonpositive for all $h$ orthogonal to $x_1$. Then the average value of (2) over all such $h$ is also nonpositive. More specifically, let $H=\{h\in S^d\colon \langle x_1,h\rangle=0\}$, then $H$ is a $(d-1)$-dimensional sphere, and we take $\mu_{d-1}$ to be the normalized Lebesgue measure on $H$. We have

$$\int\limits_H \langle x_j,h\rangle^2 d\mu_{d-1}(h)=\int\limits_H \langle x_j-x_1\langle x_1,x_j\rangle,h\rangle^2 d\mu_{d-1}(h)=\frac{1-\langle x_j,x_1\rangle^2}{d},$$

because $x_j'=x_j-x_1\langle x_1,x_j\rangle$ is orthogonal to $x_1$ and $\|x_j'\|^2=1-\langle x_j,x_1\rangle^2$. Therefore, integrating (2) over $H$ with respect to $\mu_{d-1}(h)$ gives us

$$\sum_{j=2}^{N}\left(g_r''(\langle x_1,x_j\rangle)\frac{1-\langle x_1,x_j\rangle^2}{d}-g_r'(\langle x_1,x_j\rangle)\langle x_1,x_j\rangle\right)\le 0. \tag{3}$$

After substituting $g_r(t)=(1-t)^{-r}$ into (3) we get

$$\sum_{j=2}^{N}\left(\frac{r(r+1)(1+\langle x_1,x_j\rangle)}{d(1-\langle x_1,x_j\rangle)^{r+1}}-\frac{r\langle x_1,x_j\rangle}{(1-\langle x_1,x_j\rangle)^{r+1}}\right)\le 0,$$

or, equivalently,

$$\sum_{j=2}^{N} r\,\frac{(r+1)+(r+1-d)\langle x_1,x_j\rangle}{d(1-\langle x_1,x_j\rangle)^{r+1}}\le 0. \tag{4}$$

Since $\alpha\ge d-2$, we have $|r+1-d|\le r+1$ and hence every term on the left of (4) is nonnegative.

In fact, we have $\langle x_1, x_j \rangle < 1$, so every term is strictly positive, and therefore the sum on the left of (4) must be strictly positive. This contradiction concludes the proof.

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