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SOME COMMENTS ON REGULAR AND NORMAL BITOPOLOGICAL SPACES

ДЕЯКІ ЗАУВАЖЕННЯ ЩОДО РЕГУЛЯРНИХ І НОРМАЛЬНИХ БІТОПОЛОГІЧНИХ ПРОСТОРІВ

Some properties of regular and normal bitopological spaces are established. The classes of sets inheriting the bitopological properties of regularity and normality are found. The theorem is proved on a finite covering of pairwise normal spaces. Also, the behavior of individual multivalued mappings is studied taking the axioms of bitopological regularity and normality into account.

Встановлено деякі властивості регулярних і нормальних бітопологічних просторів. Знайдено класи множин, що успадковують бітопологічні властивості регулярності та нормальності. Доведено теорему про скінченне покриття попарно нормальних просторів. Також вивчено поведінку конкретних багатозначних відображень з урахуванням аксіом бітопологічної регулярності та нормальності.

New structural provisions of a basic (enveloping) set of the same logical nature frequently play the key role in the investigation of several mathematical objects. The consideration of two and more structures on one and the same set makes it possible to distinguish ordered sets of such objects and also stimulates us to study intermediate (simultaneously depending on two or several structures) constituent elements. One of the promising, intensively developing directions of general topology is the bitopological space theory for which fundamental concepts have already been constructed and which evokes much interest. Attempts to inculcate the asymmetry principle in general topology naturally led to the concept of a bitopological space. A bitopological space as a triple (X, τ_1, τ_2), where X is a basic set, while τ_1 and τ_2 are different topologies on X, was for the first time formulated by J. C. Kelly in [1]. The topics initiated by this concept not only became the object of a systematic study in asymmetrical topology, but were also realized in other mathematical disciplines (see, e.g., [2, 3]).

For a fixed topological space (X, τ), by the methods of classical topology, we can established, only a small part of topological properties of the subsets from family 2^X. The investigation of a bitopological structure yields more information and thus makes it possible to consider remaining members of the family 2^X, which previously could not be studied in the framework of general topology.

Like in general topology, in the theory of bitopological spaces the properties of regularity and normality are important for a complete investigation of many questions. In this paper, we study some bitopological properties, taking into account the factors of regularity and normality, and also consider the behavior of special multivalued mappings on these structures.

Our discussion rests on the topological concepts which can be found in the monograph [4], while for bitopological spaces we use [2].

For a fixed bitopological space (X, τ_1, τ_2) and any subset $A \subset X$, we denote by $\tau_i \operatorname{int} A$ and $\tau_j \operatorname{cl} A$ the interior and the closure of a set A with respect to the topologies τ_i and τ_j, respectively. Throughout the paper it is assumed that $i, j \in \{1; 2\}$, $i \neq j$. If $O \subset X$ is an open set and $F \subset X$ is a closed subset in the topology τ_i, then we use the notation $O \in \tau_i$ and $F \in \operatorname{co} \tau_i$. The family of all τ_i open (briefly, i-open) neighborhoods of any set $K \subset X$ is denoted by $\sum_i^X (K)$. Sets of the class $i-$ $-\operatorname{Clp}(X) = \tau_i \cap \operatorname{co} \tau_i$ are called clopens. An induced topology $\tau_i^*(A)$ on a fixed subset $A \subset X$ is defined usually as $\tau_i^*(A) = A \cap \tau_i$.

Recall that a bitopological space (X, τ_1, τ_2) is called to be (i, j)-regular if, for any point $x \in X$ and every subset $x \notin F \in \mathrm{co}\,\tau_i \backslash \varnothing$, there exist disjoint neighborhoods $U \in \sum_i^X (x)$ and $V \in \sum_j^X (F)$. While, for each pair of disjoint sets $F_1 \in \mathrm{co}\,\tau_i \backslash \varnothing$ and $F_2 \in \mathrm{co}\,\tau_j \backslash \varnothing$ there are disjoint $O_1 \in \sum_j^X (F_1)$ and $O_2 \in \sum_i^X (F_2)$, then (X, τ_1, τ_2) is called p-normal [1]. The properties of (i, j)-regularity and p-normality as well as various bitopological aspects of closely adjoined asymmetric objects are investigated in [5 – 9].

It should be noted that in our discussion it is sometimes convenient to use the following equivalent characterizations of (i, j)-regularity and p-normality. A bitopological space (X, τ_1, τ_2) is (i, j)-regular iff for each point $x \in X$ and any neighborhood $U \in \sum_i^X (x)$ there exists $V \in \sum_j^X (F)$ such that $\tau_j \,\mathrm{cl}\, V \subset U$. The property of p-normality can also be formulated as follows: a bitopological space (X, τ_1, τ_2) is p-normal iff for each set $F \in \mathrm{co}\,\tau_i \backslash \varnothing$ and any neighborhood $U \in \sum_j^X (F)$ there exists $G \in \sum_j^X (F)$ such that $F \subset G \subset \tau_i \,\mathrm{cl}\, G \subset U$.

The theorem below gives the types of sets which inherit the properties of (i, j)-regular and p-normal bitopological structures.

Theorem 1. *Let a bitopological space* (X, τ_1, τ_2) *contain a nonempty subset* $A \subset X$ *such that*: (a) $A \in j - \mathrm{Clp}\,(X)$, *then the* (i, j)*-regularity of* (X, τ_1, τ_2) *implies the* (i, j)*-regularity of a subspace* (A, τ_1^*, τ_2^*); (b) $A \in \mathrm{co}\,\tau_1 \bigcup \mathrm{co}\,\tau_2$, *then the* p*-normality of* (X, τ_1, τ_2) *implies the* p*-normality of a subspace* $\left(A, \tau_1^*, \tau_2^*\right)$.

Proof. (a) Let us take any point x_0 and its arbitrary neighborhood $U(x_0) \in \sum_i^A (x_0)$. Then there is $O(x_0) \in \sum_i^X (x_0)$ such that $U(x_0) = A \bigcap O(x_0)$. The (i, j)-regularity of (X, τ_1, τ_2) implies existence of $W(x_0) \in \sum_i^X (x_0)$ such that $W(x_0) \subset \tau_j \,\mathrm{cl}\, W(x_0) \subset O(x_0)$. It is easy to show that the inclusion $A \bigcap W(x_0) \subset A \bigcap$ $\bigcap \tau_j \,\mathrm{cl}\, W(x_0) \subset A \bigcap O(x_0) = U(x_0)$ is valid. Moreover, $A \bigcap W(x_0) \in \sum_i^A (x_0)$. Since $A \in j - \mathrm{Clp}\,(X)$, we have $A \bigcap \tau_j \,\mathrm{cl}\, W(x_0) = \tau_j \,\mathrm{cl}\left(A \bigcap W(x_0)\right)$. Denoting $G(x_0) = A \bigcap W(x_0) \in \sum_i^A (x_0)$, we obtain $G(x_0) \subset \tau_j^* \,\mathrm{cl}\, G(x_0) \subset U(x_0)$.

(b) Consider any set $B \in \tau_i^* \backslash \varnothing$ and any neighborhood $O \in \sum_j^A (B)$. Then there exists $M \in \sum_i^X (B)$ such that $B \subset O = M \bigcap A \subset M$. From the p-normality of (X, τ_1, τ_2) it follows existence of $V \in \sum_j^X (B)$ such that $B \subset V \subset \tau_j \,\mathrm{cl}\, V \subset M$, i. e., $B \subset$ $\subset A \bigcap V \subset \tau_i \,\mathrm{cl}\, V \bigcap A \subset A \bigcap M = O$. Taking into account $A \in \mathrm{co}\,\tau_i$, we can write $B \subset A \bigcap V \subset \tau_i \,\mathrm{cl}\left(A \bigcap V\right) \subset \tau_i \,\mathrm{cl}\, A \bigcap \tau_i \,\mathrm{cl}\, V = A \bigcap \tau_i \,\mathrm{cl}\, V \subset O$. Therefore, assuming $E = A \bigcap V \in \tau_i^* \backslash \varnothing$, we see that the inclusion $B \subset E \subset A \bigcap \tau_i \,\mathrm{cl}\, E \subset O$ is valid.

The theorem is proved.

Note that the existence of $A \in j - \mathrm{Clp}\,(X)$ is guaranteed in bitopological spaces (X, τ_1, τ_2) for which (X, τ_j) is disconnected.

Using doubly open coverings, in Theorem 2 we give the structural characterization of p-normality. It must to note that the Theorem 2 was presented without proof in [10].

Theorem 2. *Let a* p*-normal bitopological space* (X, τ_1, τ_2) *be covered by the family* $\mathcal{F} = \{ O_k \in \tau_1 \bigcap \tau_2 \}_{k=\overline{1;n}}$. *Then there exists the family*

$$\{ F_k \in \operatorname{co}\tau_1 \cap \operatorname{co}\tau_2 \}_{k=\overline{1;n}}$$

such that $X = \bigcup_{k=1}^{n} F_k$ *and* $F_k \subset O_k$, *for any* $k = \overline{1;n}$.

Proof. For $n = 1$, the theorem is trivial. We will prove the validity of the result for $n \geq 2$ by the method of mathematical induction. Assume that all p-normal spaces covered by doubly open sets, with a number of elements in this covering being less than n, satisfy the concluding part of the theorem. Following to the principles of induction we must consider the covering $\mathcal{F} = \{ O_k \in \tau_1 \cap \tau_2 \}_{k=\overline{1;n}}$. Then the sets $P_n \equiv \equiv X \setminus \bigcup_{k=1}^{n-1} O_k \in \operatorname{co}\tau_1 \cap \operatorname{co}\tau_2$ and $P_n \subset O_n$. Consequently, there is a nonempty set $V \in \tau_i$, such that $P_n \subset V \subset \tau_j \operatorname{cl} V \subset O_n$. Then $X_1 \equiv X \setminus V \in \operatorname{co}\tau_i$ and, by virtue of Theorem 1, $\left(X_1, \tau_1^*, \tau_2^*\right)$ is a p-normal space. Moreover, $X_1 = \bigcup_{k=1}^{n-1} G_k$, where $G_k \equiv \equiv O_k \cap X_1$ and $k = \overline{1; n-1}$. Hence there exists a family of j-closed subsets $\{ \tilde{F}_k \}_{k=\overline{1;n-1}}$ such that $X_1 = \bigcup_{k=1}^{n-1} \tilde{F}_k$ and $\tilde{F}_k \subset G_k$ for any $k = \overline{1; n-1}$. Finally, if $F_k = \tilde{F}_k$, where $k = \overline{1; n-1}$ and $F_n = \tau_j \operatorname{cl} V$, then the constructed family $\{ F_k \}_{k=\overline{1;n}}$ satisfies Theorem 2.

Below, we use some special types of the multivalued maps and consider their behaviors by the bitopological regularity and normality. For our further investigations recall some special notions on multivalued maps. Say that a map $F: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is multivalues if it has cardinality $|F(x)| > 1$, at any point $x \in X$. An opposite to multivalued map $F: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ usually denote by $F': (Y, \gamma_1, \gamma_2) \to \to (X, \tau_1, \tau_2)$ and define as $F'(y) = \{ x \in X \mid y \in F(x) \}$, for each $y \in Y$. It is clear, that $(F')' = F$. According [11], the sets $F(A) = \bigcup_{x \in A} F(x)$ and $F'(B) = = \{ x \in X \mid B \cap F(x) \neq \varnothing \}$ are called big image of A and big preimage of B. A multivalued map $F: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ call to be: (a) (i, j)-weakly upper semi-continuous (brief. w.u.s.c.) if for every $U \in \gamma_i \setminus \varnothing$ there exists such $V \in \tau_i \setminus \varnothing$ that $F(V) \subset \tau_j \operatorname{cl} U$; (b) $(i, j) - \Delta$ continuous at $x_0 \in X$, if for any $O \in \sum_j^Y (F(x_0))$ exists $U \in \sum_i^X (x_0)$, such that $F(U) \subset O$; (c) i-pointly closed if $F(x_0) \in \operatorname{co}\gamma_i$, for any $x_0 \in X$; (d) (i, j)-upper almost continuous (brief. u.a.c.) at $x_0 \in X$ if for any $V \in \sum_i^Y (F(x_0))$ exists $U \in \sum_i^X (x_0)$ such that $F(U) \subset \gamma_i \operatorname{int} \gamma_j \operatorname{cl} V$. From (b), it is obvious what we mean under i-continuity of F.

Moreover, in the Theorems 4, 5 and 6 we need the following important special sets: $T_i(F, y) = \Big\{ x \in X \;\Big|\;$ for every $U \in \sum_i^X (x)$ there exists a point $\xi \in U$ such that $y \in \gamma_i \operatorname{cl} F(\xi) \Big\}$, for a multivalued map $F: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ and any point $y \in Y$; suppose that are given the single valued map $f: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ and multivalued map $F: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$, then we put $K_i(f, F) = = \{ (a, b) \in X \times X \mid f(a) \in \gamma_i \operatorname{cl} F(b) \}$; also, it may be associate to the every pair of multivalued maps F, Φ : $(X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ the set $S_{X^2, Y}^{F, \Phi} \equiv \equiv \left\{ (x_1, x_2) \in X \times X \mid \gamma_i \operatorname{cl}(F(x_1)) \cap \gamma_j \operatorname{cl}(\Phi(x_2)) = \varnothing \right\}$. Taking into account these notations we obtain following results.

In the Theorem 3 are established conditions under which (i, j)-u.a.c. imply i-continuity of a multivalued map. Just as in [12], we say that a subset A in a bitopological space (X, τ_1, τ_2) is (i, j)-paracompact, if every i-open covering of A, contains j-locally finite i-open refinement, which covers A.

Theorem 3. *Let a multivalued map* $F: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ *be* (i, j)*-u.a.c. and* $F(x) \subset Y$ *be an* (i, j)*-paracompact subset, for each* $x \in X$*. If a bitopological space* (Y, γ_1, γ_2) *is* (i, j)*-regular, then* F *is i-continuous map.*

Proof. Consider arbitrary point $x_0 \in X$ and any $O \in \sum_i^Y (F(x_0))$. Then by (i, j)-regularity of (Y, γ_1, γ_2) follows that for each $y_k \in F(x_0)$ there exists $G_k \in \sum_i^Y (y_k)$, such that $\gamma_j \operatorname{cl} G_k \subset O$. It is obvious, that $F(x_0) \subset \bigcup_{y_k \in F(x_0)} G_k \subset \subset \bigcup_{y_k \in F(x_0)} \gamma_j \operatorname{cl} G_k \subset O$. Because $F(x_0) \subset Y$ is an (i, j)-paracompact, then there exists j-locally finite cover of X $\left\{ A_\xi \middle| A_\xi \in \gamma_i \right\}_{\xi \in \Omega}$, such that $A_\xi \subset G_k$ for some G_k. Note that takes place the implications $F(x_0) \subset \bigcup_{\xi \in \Omega} A_\xi \subset \bigcup_{y_k \in F(x_0)} G_k \subset \subset \bigcup_{y_k \in F(x_0)} \gamma_j \operatorname{cl} G_k \subset O$ and $F(x_0) \subset \bigcup_{\xi \in \Omega} A_\xi \subset \bigcup_{\xi \in \Omega} \gamma_j \operatorname{cl} A_\xi \subset \subset \bigcup_{y_k \in F(x_0)} \gamma_j \operatorname{cl} G_k \subset O$. Assume that $A \equiv \bigcup_{\xi \in \Omega} A_\xi$. Since $\left\{ A_\xi \right\}_{\xi \in \Omega}$ is j-locally finite family then $\gamma_j \operatorname{cl} A = \bigcup_{\xi \in \Omega} \gamma_j \operatorname{cl} A_\xi$ and respectively have $F(x_0) \subset A \subset \subset \gamma_j \operatorname{cl} A \subset O$. By taking into account (i, j)-u.a.c. of F, and the implication $F(x_0) \subset \subset A$ yields existence such $V \in \sum_i^X (x_0)$ that $F(x) \subset \gamma_i \operatorname{int} \gamma_j \operatorname{cl} A$ for each $x \in V$. Therefore the implication $F(x) \subset \gamma_i \operatorname{int} \gamma_j \operatorname{cl} A \subset \gamma_j \operatorname{cl} A \subset O$ is valid for any $x \in V$, i. e., F is i-continuous at $x_0 \in X$.

Is valid the following theorem.

Theorem 4. *Let* (X, τ_1, τ_2) *be a j-cofinite and* (Y, γ_1, γ_2) *be an* (i, j)*-regular bitopological space. If a multivalued map* $F: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ *is the* $(i, j) - \Delta$ *continuous, i-pointly closed and* $|F'(y)| < \aleph_0$*, at every point* $y \in Y$*. Then the set* $T_i(F, y) \in \operatorname{co} \tau_j$*.*

Proof. Indeed, suppose that F is an $(i, j) - \Delta$ continuous and $y \notin F(x_0) \in \operatorname{co} \gamma_i$ for some $x_0 \in T_i(F, y)$. Then by (i, j)-regularity of (Y, γ_1, γ_2) implies existence of the disjoint sets $V_1 \in \sum_i^Y (y)$ and $V_2 \in \sum_j^Y (F(x_0))$. Since F is $(i, j) - \Delta$ continuous, then there is such $U \in \sum_i^X (x_0)$ that $F(U) \subset V_2$. It is obvious that, if $x \in U$ then $F(x) \subset F(U) \subset V_2$ and $F(x) \cap V_1 = \varnothing$, this implies $x_0 \notin T_i(F, y)$, i. e., $T_i(F, y) \subset F'(y)$. Therefore $|T_i(F, y)| < \aleph_0$ and we have $T_i(F, y) \in \operatorname{co} \tau_j$.

It is well-known that a bitopological space (X, τ_1, τ_2) is p-extremally disconnected (brief. p-E.D.) iff $\tau_i \operatorname{cl} O_1 \cap \tau_j \operatorname{cl} O_2 = \varnothing$, for any pair of the disjoint sets $O_1 \in \tau_j$ and $O_2 \in \tau_i$ [13]. Using the notion of p-E.D., in the Theorem 5, we give a relation between the single valued and multivalued maps.

Theorem 5. *Let* $f: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ *be a single valued* $(i, j) - \Delta$ *continuous map and* $F: (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ *be* (i, j)*-w.u.s.c. multivalued map. If* (Y, γ_1, γ_2) *is the* (i, j)*-regular, p-E.D. bitopological space, then the set* $K_j(f, F)$ *is j-closed.*

Proof. Let us consider $(a, b) \in X \times X \setminus K_j(f, F)$, then $f(a) \notin \gamma_j \operatorname{cl} F(b)$. By (i, j)-regularity of (Y, γ_1, γ_2) follows existence the disjoint sets $V \in \sum_j^Y (f(a))$ and $W \in \sum_i^Y (\gamma_j \operatorname{cl} F(b))$. Obviously, from p-E.D. implies that $\gamma_i \operatorname{cl} V \cap \gamma_j \operatorname{cl} W = \varnothing$. From $(i, j) - \Delta$ continuity of f and (i, j)-weakly upper semicontinuity of F, follows existence of a pair of the sets $A \in \sum_i^X (a)$ and $B \in \sum_i^X (B)$, such that $f(A) \subset V$

and $F(B) \subset \gamma_j \operatorname{cl} W$. It is clear that $A \times B \in \sum_i^{X \times X} (a, b)$ and $(A \times B) \cap K_j(f, F) = \varnothing$, this implies that $K_j(f, F) \in \operatorname{co}(\tau_i \times \tau_i)$.

At the end of the paper we prove the following important result.

Theorem 6. *Consider the (j, i)-w.u.s.c. and (i, j)-w.u.s.c. multivalued maps F, $\Phi : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$, where (Y, γ_1, γ_2) is p-normal bitopological space. Then the set $S_{X^2, Y}^{F, \Phi} \in \operatorname{co}(\tau_j \times \tau_i)$.*

Proof. Suppose that $(x_1, x_2) \in X \times X \setminus S_{X^2, Y}^{F, \Phi}$, then $\gamma_i \operatorname{cl} F(x_1) \cap \gamma_j \operatorname{cl} \Phi(x_2) \neq \varnothing$. Since (Y, γ_1, γ_2) is p-normal, then there are the sets $V \in \sum_j^Y (\gamma_i \operatorname{cl} F(x_1))$ and $W \in \sum_i^Y (\gamma_j \operatorname{cl} \Phi(x_2))$ such that $\gamma_i \operatorname{cl} V \cap \gamma_j \operatorname{cl} W = \varnothing$. By the (j, i)-w.u.s.c. of F and (i, j)-w.u.s.c. of Φ, follows existence of the sets $A \in \sum_j^X (x_1)$ and $B \in \sum_i^X (x_2)$, such that $F(A) \subset \gamma_i \operatorname{cl} V$ and $\Phi(B) \subset \gamma_j \operatorname{cl} W$. Consequently, it takes place $A \times B \in \sum_{\tau_j \times \tau_i}^{X \times X} (x_1, x_2)$ and $(A \times B) \cap S_{X^2, Y}^{F, \Phi} = \varnothing$, i.e., $S_{X^2, Y}^{F, \Phi} \in \operatorname{co}(\tau_j \times \tau_i)$.

According [5], a space (X, τ_1, τ_2) is said to be (i, j)-stable if its every nonempty i-closed subset is j-compact. Several related properties such bitopological spaces are obtained in [6], too. Finally we note, that if in the Theorems 4, 5 and 6 the suitable bitopological spaces are stable, then the sets $T_i(F, y)$, $K_j(f, F)$ and $S_{X^2, Y}^{F, \Phi}$ respectively are compact.

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