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THE STONE – ČECH COMPACTIFICATION OF GROUPOIDS *

КОМПАКТИФІКАЦІЯ СТОУНА – ЧЕХА ДЛЯ ГРУПОЇДІВ

Let G be a discrete groupoid and consider the Stone – Čech compactification βG of G. We extend the operation on the set of composable elements $G^{(2)}$ of G to the operation “$*$” on a subset $(\beta G)^{(2)}$ of $\beta G \times \beta G$ such that the triple $(\beta G, (\beta G)^{(2)}, *)$ is a compact right topological semigroupoid.

Нехай G — дискретний групоїд. Розглянемо компактифікацію Стоуна – Чеха βG групоїда G. Розширимо операцію на множині $G^{(2)}$ елементів G, що компонуються, до операції “$*$” на підмножині $(\beta G)^{(2)}$ множини $\beta G \times \beta G$ такої, що трійка $(\beta G, (\beta G)^{(2)}, *)$ є компактним топологічним напівгрупоїдом.

1. Introduction. A compactification of a topological space X is a compact space K together with an embedding $e\colon X \longrightarrow K$ with $e(X)$ dense in K. We usually identify X with $e(X)$ and consider X as a subspace of K. There exists a very special type of compactification of X in which X is embedded in such a way that every bounded, real-valued (complex-valued) continuous function on X will extend continuously to the compactification. Such a compactification of X is called the Stone – Čech compactification and denoted by βX.

As known, the Stone – Čech compactification βG of an infinite discrete group G can be turned into a (compact) semigroup by an operation, extended from G [1, 4]. This operation can be taken in many ways depending on how we regard βG. We can regard βG as the maximal ideal space of $\mathcal{B}(G)$, the C*-algebra of all bounded complex-valued functions on G. In this case, the product of two elements $\theta, \eta \in \beta G$, is described by the following steps:

$$L_g(f)(h) = f(gh), \quad T_{\eta,f}(g) = \eta(L_g f), \quad \theta * \eta(f) = \theta(T_{\eta,f}).$$

Let $g \in G$. By using the universal property of βG (see S1 below), one can extend the continuous map $h \mapsto gh\colon G \longrightarrow \beta G$ to a continuous map $\eta \mapsto g * \eta\colon \beta G \longrightarrow \beta G$. Then the mappings $g \mapsto g * \eta\colon G \longrightarrow \beta G$ are in turn continuously extended to βG leading to a binary operation in βG. This operation in βG is associative, so βG is a compact right topological semigroup, that is, the map $\theta \mapsto \theta * \eta\colon \beta G \longrightarrow \beta G$ is continuous for every $\eta \in \beta G$. More generally, for any topological group, there are many compactifications. Each compactification can be described as the maximal ideal space of a function algebra.

In this paper, we deal with groupoids instead of groups. Unlike groups, in a groupoid G, the product is not defined for each two elements of G. But, the product defined on a subset of $G \times G$, the set of composable pairs. The product on composable elements is associative (see Definition 2.1 below). We will show that, like the group case, the operation of any groupoid G can be extend to βG such that this operation is still associative.

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2. Preliminaries. ***The Stone – Čech compactification.*** For the convenience of the reader we repeat the relevant material about βX, the Stone – Čech compactification of X, from [3, 8] without proofs, thus making our exposition self-contained.

Let X be a topological space. Then $C_b(X)$ stands for the algebra of all bounded continuous complex-valued functions on the topological space X. Also, a subset E of X is called C^*-embedded if every function in $C_b(E)$ can be extended to a function in $C_b(X)$. A subset E is called zero-set if there exists a continuous function f in $C_b(X)$ such that $E = \{x \in X : f(x) = 0\}$. Trivially, every subset E of a discrete space X is C*-embedded and also is zero-set. Every (completely regular) space X has a compactification βX, with the following properties:

S1. (Stone) Every continuous mapping T from X into any compact space Y has a continuous extension $\widetilde{T}$ from βX into Y.

S2. (Stone – Čech) Every function f in $C_b(X)$ has an extension to a function $\tilde{f}$ in $C(\beta X)$.

S3. (Čech) Any two disjoint zero-sets in X have disjoint closures in βX.

S4. For any two zero-sets Z_1 and Z_2 in X,

$$\overline{Z_1 \cap Z_2} = \overline{Z_1} \cap \overline{Z_2}.$$

S5. A subset S of X is C*-embedded in X if and only if $\beta S = \overline{S}$.

S6. If S is open-and-closed in X, then $\overline{S}$ is open-and-closed in βX.

Let E be a subset of a discrete space X. Then, by applying S1 and S5, we can deduce that every map f from E into compact space Y can be extended to a continuous map $\widetilde{f}$ from $\widetilde{E} = \beta E$ into Y(S1, S5).

Let X be a discrete space. It is customary to write $\mathcal{B}(X)$ rather than $C_b(X)$. So, $\mathcal{B}(X)$ by pointwise operations and the norm

$$||f||_X = \sup_{x \in X} |f(x)|$$

is a commutative unital C*-algebra. Since $\mathcal{B}(X)$ is isometrically isomorphism to $C(\beta X)$ (by S2), we can identify βX with the maximal ideal space of $\mathcal{B}(X)$. So, the topology of βX coincides with the Gelfand topology. Thus a net $\{\theta_i\}_{i \in I}$ converges to θ in βX if and only if for every $f \in \mathcal{B}(X)$ the net $\{\theta_i(f)\}_{i \in I}$ converges to $\theta(f)$.

Groupoids. Here is some elementary definitions in groupoid literatures. For more details we refer the reader to [5 – 7].

Definition 2.1. *A groupoid is a set G endowed with a product map $(g, h) \mapsto gh : G^{(2)} \longrightarrow G$ where $G^{(2)}$ is a subset of $G \times G$ called the set of composable pairs and an inverse map $g \mapsto g^{-1}$: $G \to G$ such that the following relation are satisfied:*

(1) $(g^{-1})^{-1} = g$;

(2) *if $(g, h) \in G^{(2)}$ and $(h, k) \in G^{(2)}$, then $(gh, k), (g, hk) \in G^{(2)}$ and we have*

$$(gh)k = g(hk);$$

(3) $(g^{-1}, g) \in G^{(2)}$ *and if $(g, h) \in G^{(2)}$, then $g^{-1}(gh) = h$;*

(4) $(g, g^{-1}) \in G^{(2)}$ *and if $(h, g) \in G^{(2)}$, then $(hg)g^{-1} = h$.*

The unit space G^0 is the subset of elements gg^{-1} where g ranges over G. The rang map r: $G \longrightarrow G^0$ and the source map d: $G \longrightarrow G^0$ is defined by $r(g) = gg^{-1}$ and $d(g) = g^{-1}g$. The pair (g, h) belongs to the set $G^{(2)}$ if and only if $d(g) = r(h)$. For each $u \in G^0$, the subsets G_u and G^u are given by $G_u = d^{-1}(\{u\})$, $G^u = r^{-1}(\{u\})$.

Definition 2.2. *A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure*:

(1) $(x, y) \mapsto xy\colon G^{(2)} \longrightarrow G$ *is continuous where* $G^{(2)}$ *has the induced topology from* $G \times G$;

(2) $g \mapsto g^{-1}\colon G \longrightarrow G$ *is continuous.*

If G is a topological groupoid, then the maps r, d are continuous. In addition, if G^0 is Hausdorff in the relative topology, then $G^{(2)}$ is closed in $G \times G$.

3. Discrete groupoids. Let G be a groupoid and $g \in G$. For any $f \in \mathcal{B}(G)$, we define the left g-translation and the right g-translation of f, respectively, by

$$L_g f(x) = \begin{cases} f(gx), & x \in G^{d(g)}, \\ 0, & x \notin G^{d(g)}, \end{cases} \qquad R_g f(x) = \begin{cases} f(xg), & x \in G_{r(g)}, \\ 0, & x \notin G_{r(g)}. \end{cases}$$

Since f is bounded, so are $L_g f$ and $R_g f$. Therefore, for any $\theta \in \beta G$ and $f \in \mathcal{B}(G)$, we can consider two new functions

$$T_{\theta,f}\colon G \longrightarrow \mathbb{C}, \quad S_{\theta,f}\colon G \longrightarrow \mathbb{C}$$

given by

$$T_{\theta,f}(g) = \theta(L_g f), \quad S_{\theta,f}(g) = \theta(R_g f).$$

It is clear that $T_{\theta,f}$ and $S_{\theta,f}$ are bounded. Next, we collect some elementary properties of these functions.

Lemma 3.1. *Let f be a bounded function on a groupoid G. Then for any $g, h \in G$ and any $\theta \in \beta G$*:

(1) *If $(g, h) \in G^{(2)}$, then $L_h(L_g f) = L_{gh} f$. Also, if $(g, h) \notin G^{(2)}$, then $L_h(L_g f) = 0$.*

(2) *$L_g(T_{\theta,f}) = T_{\theta,L_g f}$ and $R_g(S_{\theta,f}) = S_{\theta,R_g f}$.*

Proof. (1) Let $x \in G$ and $(g, h) \in G^{(2)}$. Suppose that $x \notin G^{d(h)} = G^{d(gh)}$. By the definition, $L_h(L_g f)(x) = 0 = L_{gh} f(x)$. Let $x \in G^{d(h)} = G^{d(gh)}$. Accordingly,

$$L_h(L_g f)(x) = L_g f(hx) = f(ghx) = L_{gh} f(x).$$

Therefore, $L_{gh} f = L_h(L_g f)$. Now, suppose that $(g, h) \notin G^{(2)}$. In this case, if $x \notin G^{d(h)}$, then $L_h(L_g f)(x) = 0$. If $x \in G^{d(h)}$, then $L_h(L_g f)(x) = L_g f(hx)$. Since $g \notin G^{d(h)} = G^{d(hx)}$, we have $L_g f(hx) = 0$.

(2) We only prove the first identity, the proof of the second one is similar. Let h be any element in G with $h \notin G^{d(g)}$. Then $L_g(T_{\theta,f})(h) = 0$. On the other hand, $T_{\theta,L_g f}(h) = \theta(L_h(L_g f)) = 0$. Now, suppose that $h \in G^{d(g)}$. So,

$$L_g T_{\theta,f}(h) = T_{\theta,f}(gh) = \theta(L_{gh} f) = \theta(L_h(L_g f)) = T_{\theta,L_g f}(h).$$

Lemma 3.1 is proved.

According to G, we define the set of composable elements of βG by

$$(\beta G)^{(2)} = \bigcup_{u \in G^0} \overline{G_u} \times \overline{G^u} = \bigcup_{u \in G^0} \beta G_u \times \beta G^u.$$

It is trivial that $G^{(2)} \subseteq (\beta G)^{(2)}$. In the following result, we extend the operation of $G^{(2)}$ to $(\beta G)^{(2)}$.

Theorem 3.1. *Let G be a discrete groupoid. There is a unique operation $*$ on $(\beta G)^{(2)}$ satisfying the following conditions*:

(1) *For every $(g,h)\in G^{(2)}$, $g*h=gh$.*

(2) *For every $u\in G^0$ and $g\in G_u$, the map $\eta\mapsto g*\eta\colon \overline{G^u}\longrightarrow \beta G$ is continuous.*

(3) *For every $u\in G^0$ and $\eta\in\overline{G^u}$, the map $\theta\mapsto\theta*\eta\colon\overline{G_u}\longrightarrow\beta G$ is continuous.*

Proof. Let $u\in G^0$. Given any $g\in G_u$, define $\ell^u_g\colon G^u\longrightarrow G\subseteq\beta G$ by $\ell^u_g(x)=gx$. By S1, there is a continuous map $\widetilde{\ell^u_g}\colon\beta G^u=\overline{G^u}\longrightarrow\beta G$ such that $\widetilde{\ell^u_g}|_{G^u}=\ell^u_g$. Now, let $\eta\in\overline{G^u}$ and define $r^u_\eta\colon G_u\longrightarrow\beta G$ by $r^u_\eta(g)=\widetilde{\ell^u_g}(\eta)$. Then there is a continuous map $\widetilde{r^u_\eta}\colon\beta G_u=\overline{G_u}\longrightarrow\beta G$ such that $\widetilde{r^u_\eta}|_{G_u}=r^u_\eta$. For any $(\theta,\eta)\in\overline{G_u}\times\overline{G^u}$, set

$$\theta*\eta=\widetilde{r^u_\eta}(\theta).$$

For (1), suppose that $(g,h)\in G^{(2)}$. Then there is a $u\in G^0$ such that $(g,h)\in G_u\times G^u$. Therefore,

$$g*h=\widetilde{r^u_h}(g)=r^u_h(g)=\widetilde{\ell^u_g}(h)=\ell^u_g(h)=gh.$$

The map $\eta\mapsto g*\eta\colon\overline{G^u}\longrightarrow\beta G$ is just the map $\widetilde{\ell^u_g}$ and the map $\theta\mapsto\theta*\eta\colon\overline{G_u}\longrightarrow\beta G$, is just the map $\widetilde{r^u_\eta}$ and the continuity of these maps follow from the continuity of $\widetilde{r^u_\eta}$ and $\widetilde{\ell^u_g}$.

Theorem 3.1 is proved.

Theorem 3.2. *Suppose that G is a discrete groupoid and $(\theta,\eta)\in(\beta G)^{(2)}$.*

(1) *If $\{g_i\}_{i\in I}$ and $\{h_j\}_{j\in J}$ are nets in G such that $\lim_i g_i=\theta$ and $\lim_j h_j=\eta$, then $\theta*\eta=$ $=\lim_i\lim_j g_ih_j$.*

(2) $\theta*\eta(f)=\theta(T_{\eta,f})$.

Proof. Since $(\theta,\eta)\in(\beta G)^{(2)}$, there is a $u\in G^0$ such that $(\theta,\eta)\in\overline{G_u}\times\overline{G^u}$. As $\overline{G_u}$ and $\overline{G^u}$ are open, we can suppose that $\{g_i\}_{i\in I}$ is a net in $\overline{G_u}$ and $\{h_j\}_{j\in J}$ is a net in $\overline{G^u}$. We have

$$\theta*\eta=\widetilde{r^u_\eta}(\theta)=\lim_i\widetilde{r^u_\eta}(g_i)=\lim_i r^u_\eta(g_i)=\lim_i\widetilde{\ell^u_{g_i}}(\eta)=$$

$$=\lim_i\lim_j\widetilde{\ell^u_{g_i}}(h_j)=\lim_i\lim_j\ell^u_{g_i}(h_j)=\lim_i\lim_j g_ih_j.$$

For (2), suppose that $\{g_i\}_{i\in I}$ is a net in $\overline{G_u}$ and $\{h_j\}_{j\in J}$ is a net in $\overline{G^u}$ such that $\lim_i g_i=\theta$ and $\lim_j h_j=\eta$. Then for any f in $\mathcal{B}(G)$, we have

$$\theta*\eta(f)=\lim_i\lim_j f(g_ih_j)=L_{g_i}f(h_j)=$$

$$=\lim_i\eta(L_{g_i}f)=\lim_i T_{\eta,f}(g_i)=\theta(T_{\eta,f}).$$

Theorem 3.2 is proved.

One can consider the inversion map defined by $g\mapsto g^{-1}\colon G\longrightarrow G$. By S2, this map has a continuous extension $\widetilde{inv}\colon\beta G\longrightarrow\beta G$. We denote again the $\widetilde{inv}(\theta)$ by θ^{-1}. By the continuity, if $\{g_i\}_{i\in I}$ is any net in G converging to θ in βG, then $\{g_i^{-1}\}_{i\in I}$ converges to θ^{-1}. Consequently, $(\theta^{-1})^{-1}=\theta$ and if $\theta\in\overline{G_u}$, then $\theta^{-1}\in\overline{G^u}$. Let $f\in\mathcal{B}(G)$ and define the transformation $\hat f$ on G by $\hat f(g)=f(g^{-1})$. This relation can be extended to βG, that is, for any $\theta\in\beta G$, we have $\theta^{-1}(f)=\theta(\hat f)$. If G is a groupoid, then $(g,g^{-1})\in G^{(2)}$ for all $g\in G$. But this property does not hold for the Stone – Čech compactification βG, unless G^o is finite.

Theorem 3.3. *Let G be a discrete groupoid. For every $\theta \in \beta G$, $(\theta, \theta^{-1}) \in (\beta G)^{(2)}$ if and only if G^0 is finite.*

Proof. Assume that $(\theta, \theta^{-1}) \in (\beta G)^{(2)}$ for all $\theta \in \beta G$. Then, it follows that $\beta G = \bigcup_{u \in G^0} \overline{G_u}$. Since $\overline{G_u}$ is open in βG, by compactness of βG, there is $u_1, u_2 \ldots, u_n \in G^0$ such that $\beta G = = \bigcup_{k=1}^{n} \overline{G_{u_k}}$. Thus $G^0 = \{u_1, u_2, \ldots, u_n\}$. Conversely, suppose that G^0 is finite. Then, for every $\theta \in \beta G$, there exists $u \in G^0$ with $\theta \in \overline{G_u}$. Let $\{g_i\}_{i \in I}$ be a net in G_u converging to θ. So, $\{g_i^{-1}\}_{i \in I}$ is a net in G^u which converges to θ^{-1}. Thus $(\theta, \theta^{-1}) \in (\beta G)^{(2)}$.

Theorem 3.3 is proved.

Lemma 3.2. *Let G be a discrete groupoid, $(\theta, \eta) \in (\beta G)^{(2)}$ and let $v \in G^0$. Then*

(1) $\eta \in \overline{G_v}$ *if and only if* $\theta * \eta \in \overline{G_v}$;

(2) $\theta \in \overline{G^v}$ *if and only if* $\theta * \eta \in \overline{G^v}$.

Proof. (1) Let u be in G^0 such that $(\theta, \eta) \in \overline{G_u} \times \overline{G^u}$. Then there exist nets $\{g_i\}_{i \in I}$ and $\{h_j\}_{j \in J}$, respectively, in G_u and G^u such that $g_i \longrightarrow \theta$ and $h_j \longrightarrow \eta$. Since $\overline{G_v}$ is open set βG containing η, we can assume that $\{h_j\}_{j \in J}$ is also a net in G_v. By the Theorem 3.2, $\theta * \eta = \lim_i \lim_j g_i h_j$. Since for any i and j, $g_i h_j \in G_v$, by Theorem 3.2, we deduce that $\theta * \eta = \lim_i \lim_j g_i h_j \in \overline{G_v}$. Conversely, suppose that $\theta * \eta \in \overline{G^v}$. Let $\{g_i\}_{i \in I}$ and $\{h_j\}_{j \in J}$ are net respectively, in G_u and G^u such that $g_i \longrightarrow \theta$ and $h_j \longrightarrow \eta$. Since $\overline{G^v}$ is open and containing $\theta * \eta$, we can assume that $g_i h_j \in G_v$. Thus $h_j \in G_v$ and hence $\theta \in \overline{G^v}$.

(2) The proof is similar to (1).

Lemma 3.2 is proved.

Definition 3.1. *A semigroupoid is a triple $(\Lambda, \Lambda^{(2)}, *)$ such that Λ is a set, $\Lambda^{(2)}$ is a subset of $\Lambda \times \Lambda$, and*

$$* \colon \Lambda^{(2)} \longrightarrow \Lambda$$

is an operation which is associative in the following sense: if $f, g, h \in \Lambda$ are such that either

(i) $(f, g) \in \Lambda^{(2)}$ *and* $(g, h) \in \Lambda^{(2)}$, *or*

(ii) $(f, g) \in \Lambda^{(2)}$ *and* $(f * g, h) \in \Lambda^{(2)}$, *or*

(iii) $(g, h) \in \Lambda^{(2)}$ *and* $(f, g * h) \in \Lambda^{(2)}$,

then all $(f, g), (g, h), (f * g, h)$ and $(f, g * h)$ lie in $\Lambda^{(2)}$, and

$$(f * g) * h = f * (g * h).$$

Moreover, for $f \in \Lambda$, we will set

$$\Lambda^f = \{g \in \Lambda \colon (f, g) \in \Lambda^{(2)}\}, \quad \Lambda_f = \{g \in \Lambda \colon (g, f) \in \Lambda^{(2)}\}.$$

Let $(\Lambda, \Lambda^{(2)}, *)$ and $(\Lambda', \Lambda'^{(2)}, *')$ be semigroupoids. A map $T \colon \Lambda \longrightarrow \Lambda'$ is called homomorphism if $(f, g) \in \Lambda^{(2)}$, then $(T(f), T(g)) \in \Lambda'^{(2)}$ and $T(f * g) = T(f) *' T(g)$.

Definition 3.2. *Let $(\Lambda, \Lambda^{(2)}, *)$ be a semigroupoid and a topological space. Then*

(i) Λ *is called left topological semigroupoid if for every $f \in \Lambda$ the map $g \mapsto f * g \colon \Lambda^f \longrightarrow \Lambda$ is continuous.*

(ii) Λ *is called right topological semigroupoid if for every $f \in \Lambda$ the map $g \mapsto g * f \colon \Lambda_f \longrightarrow \Lambda$ is continuous.*

Let Λ be a right topological semigroupoid. The topological center of Λ is the set of all $f \in \Lambda$ such that the map $g \mapsto f * g \colon \Lambda^g \longrightarrow \Lambda$ is continuous.

Theorem 3.4. *If G is discrete groupoid, then $(\beta G, (\beta G)^{(2)}, *)$ is a compact right topological semigroupoid. Moreover, the topological center of βG contains G.*

Proof. Suppose that θ, η, γ are in βG. By Lemma 3.2, each one of conditions (i) – (iii) of Definition 3.1 implies that $(f, g), (g, h), (f * g, h)$ and $(f, g * h)$ lie in $\Lambda^{(2)}$. Therefore, it is enough to prove that if $(\theta, \eta) \in (\beta G)^{(2)}$ and $(\eta, \gamma) \in (\beta G)^{(2)}$, then $(\theta * \eta) * \gamma = \theta * (\eta * \gamma)$. For, first we show the following identity:

$$T_{\theta, T_{\eta, f}} = T_{\theta * \eta, f}.$$

Suppose that $g \in G$. Then

$$T_{\theta, T_{\eta, f}}(g) = \theta(L_g(T_{\eta, f})) = \theta(T_{\eta, L_g f}) = \theta * \eta(L_g f) = T_{\theta * \eta, f}(g).$$

Now, for any $f \in \mathcal{B}(G)$, we have

$$\theta * (\eta * \gamma)(f) = \theta(T_{\eta * \gamma, f}) = \theta(T_{\eta, T_{\gamma, f}}) = \theta * \eta(T_{\gamma, f}) = (\theta * \eta) * \gamma(f).$$

The above arguments show that βG is a semigroupoid. Also, by the Theorem 3.1, βG is a compact right topological semigroupoid such that the topological center of βG contains G.

Theorem 3.4 is proved.

Theorem 3.5. *Let G be a discrete groupoid and let $(K, K^{(2)}, \star)$ be a compact right topological semigroupoid which is such that the following properties are satisfied*:

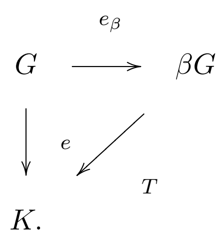
(1) *there is a morphism $e\colon G \to K$ such that $e(G)$ is dense in K*;

(2) *the topological center of K contains $e(G)$*;

(3) $\bigcup_{u \in G^0} e(\overline{G_u}) \times e(\overline{G^u}) \subseteq K^{(2)}$.

Then there exists a continuous surjective homomorphism $T\colon \beta G \longrightarrow K$ such that for each $g \in G$, $T(g) = e(g)$.

Proof. Since K is compact topological space, there exists a continuous surjective map $T\colon \beta G \longrightarrow K$ such that the following diagram is commutative:



Let $(\theta, \eta) \in (\beta G)^{(2)}$. By the definition, there exist $u \in G^0$ and a net $\{g_i\}_{i \in I}$ in G_u and a net $\{h_j\}_{j \in J}$ in G^u such that $g_i \longrightarrow \theta$ and $h_j \longrightarrow \eta$. Therefore, $e(g_i) \longrightarrow T(\theta)$ and $e(h_j) \longrightarrow T(\eta)$, and so the property (3) yields that $(T(\theta), T(\eta)) \in K^{(2)}$. Since $K^{(2)}$ is right topological semigroupoid and the topological center of K contains $e(G)$, we have

$$T(\theta * \eta) = T(\lim_i \lim_j g_i h_j) = \lim_i \lim_j T(g_i h_j) =$$

$$= \lim_i \lim_j e(g_i h_j) = \lim_i \lim_j e(g_i) \star e(h_j) =$$

$$= \lim_i \lim_j T(g_i) \star T(h_j) = T(\theta) \star T(\eta).$$

Theorem 3.5 is proved.

We can start the definition of the product on $(\beta G)^{(2)}$ by extending the h-right translation map $r_h^u:$ $x \mapsto xh$ (from G_u to βG) for $h \in G^u$ to the map $\widetilde{r_h^u}: \overline{G_u} \longrightarrow \beta G$. Then consider $\theta \in \overline{G_u}$ and define the map $\ell_\theta^u: G^u \longrightarrow \beta G$ by $\ell_\theta^u(h) = \widetilde{r_h^u}(\theta)$. We extend ℓ_θ^u to βG_u and denote it by $\widetilde{\ell_\theta^u}$. Now, define

$$\theta \,\square\, \eta = \widetilde{\ell_\theta^u}(\eta).$$

So, we have the following results:

(1) For every $g, h \in G$, $g \square h = gh$.

(2) For every $u \in G^0$ and $\theta \in \overline{G_u}$, the map $\eta \mapsto \theta \square \eta: G^u \longrightarrow \beta G$ is continuous.

(3) For every $u \in G^0$ and $h \in G^u$, the map $\theta \mapsto \theta \square h: G_u \longrightarrow \beta G$ is continuous.

(4) For every $u \in G^0$ and $(\theta, \eta) \in \overline{G_u} \times \overline{G^u}$:

$$\theta \,\square\, \eta = \lim_j \lim_i g_i h_j,$$

where $\{g_i\}_{i \in I}$ is a net in G_u and $\{h_j\}_{j \in J}$ is a net in G^u such that $g_i \longrightarrow \theta$ and $h_j \longrightarrow \eta$.

(5) For every $u \in G^0$, $(\theta, \eta) \in \overline{G_u} \times \overline{G^u}$ and $f \in \mathcal{B}(G)$:

$$\theta \,\square\, \eta(f) = \eta(S_{\theta,f}).$$

(6) $(\beta G, (\beta G)^{(2)}, \square)$ is a compact left topological semigroupoid.

Lemma 3.3. *Suppose that G is a groupoid, $g \in G$ and $\theta \in \beta G$. Then*

(1) $L_g \hat{f} = \widehat{R_{g^{-1}} f}$,

(2) $T_{\theta, \hat{f}} = \widehat{S_{\theta^{-1}, f}}$.

Proof. (1) Assume that $x \in G^{d(g)}$. Then $x^{-1} \in G_{r(g^{-1})}$ and we have

$$L_g \hat{f}(x) = \hat{f}(gx) = f(x^{-1} g^{-1}) = R_{g^{-1}} f(x^{-1}) = \widehat{R_{g^{-1}} f}(x).$$

Also, if $x \notin G^{d(g)}$, then $x^{-1} \notin G_{r(g^{-1})}$, and so

$$L_g \hat{f}(x) = 0 = R_{g^{-1}} f(x^{-1}) = \widehat{R_{g^{-1} f}}(x).$$

(2) Let x be any element in G. Then

$$T_{\theta, \hat{f}}(x) = \theta(L_x \hat{f}) = \theta(\widehat{R_{x^{-1}} f}) = \theta^{-1}(R_{x^{-1}} f) =$$

$$= S_{\theta^{-1}, f}(x^{-1}) = \widehat{S_{\theta^{-1}, f}}(x).$$

Lemma 3.3 is proved.

Theorem 3.6. *Let G be a discrete groupoid and $(\theta, \eta) \in (\beta G)^{(2)}$. Then $(\eta^{-1}, \theta^{-1}) \in (\beta G)^{(2)}$ and we have*

$$\eta^{-1} * \theta^{-1} = (\theta \,\square\, \eta)^{-1}.$$

Proof. Let $u \in G^0$ be such that $\theta \in \overline{G_u}$ and $\eta \in \overline{G^u}$. Then $\eta^{-1} \in \overline{G_u}$ and $\theta^{-1} \in \overline{G^u}$. Thus $(\eta^{-1}, \theta^{-1}) \in (\beta G)^{(2)}$, and so

$$\eta^{-1} * \theta^{-1}(f) = \eta^{-1}(T_{\theta^{-1}, f}) = \eta(\widehat{T_{\theta^{-1}, f}}) = \eta(S_{\theta, \hat{f}}) =$$

$$= \theta \square \eta(\hat{f}) = (\theta \,\square\, \eta)^{-1}(f).$$

Theorem 3.6 is proved.

***Example* 3.1.** An interesting example of a groupoid is an equivalence relation R on a set X. Here, $R^{(2)} = \{((g,h),(h,k))\colon (g,h),(h,k) \in R\}$ and the product map and inversion map are given by $(g,h)(h,k) = (g,k)$ and $(g,h)^{-1} = (h,g)$. So, the set of units is $\{(g,g)\colon g \in X\}$. Also, $G^{(g,g)} = \{g\} \times [g]$ and $G_{(g,g)} = [g] \times \{g\}$. Here $[g]$ denotes the equivalence class of g. In this case, the set of composable elements is

$$(\beta G)^{(2)} = \bigcup_{g \in G} \overline{[g] \times \{g\}} \times \overline{\{g\} \times [g]}.$$

To specify the product, let us first determine $\overline{[g] \times \{g\}}$ and $\overline{\{g\} \times [g]}$. Consider the bijection $\Pi_1 \colon [g] \times \{g\} \longrightarrow [g]$ defined by $\Pi_1((h,g)) = h$. Thus, there exists homeomorphism $\widetilde{\Pi_1} \colon \beta([g] \times \times \{g\}) \longrightarrow \beta[g]$ which is an extension of Π_1. If we repeat the argument for $\{g\} \times [g]$ we obtain the homeomorphism $\widetilde{\Pi_2} \colon \beta(\{g\} \times [g]) \longrightarrow \beta[g]$ which is an extension of the bijection $\Pi_2 \colon \{g\} \times [g] \longrightarrow \longrightarrow [g]$ defined by $\Pi_2((g,h)) = h$. Let $f \in \mathcal{B}(G)$ and $g \in X$ and define $f_g \colon [g] \longrightarrow \mathbb{C}$ by $f_g(h) = f(g,h)$. Also, for $\theta \in \beta G$ define

$$U_{\theta,f} \colon X \longrightarrow \mathbb{C}$$

by $U_{\theta,f}(g) = \theta(f_g)$. Let $\{(g_i,g)\}_{i \in I}$ and $\{(g,h_j)\}_{j \in J}$ be nets, respectively, in $\overline{[g] \times \{g\}}$ and $\overline{\{g\} \times [g]}$ such that $(g_i,g) \longrightarrow \theta'$ and $(g,h_j) \longrightarrow \eta'$ in βG. Since $\overline{[g] \times \{g\}}$ and $\overline{\{g\} \times [g]}$ are homeomorphic to βX, we can assume that there exist θ and η in βX such that $g_i \longrightarrow \theta$ and $h_j \longrightarrow \eta$ and $\widetilde{\Pi_1}(\theta') = \theta$ and $\widetilde{\Pi_2}(\eta') = \eta$. For any $f \in \mathcal{B}(G)$, we have

$$\theta' * \eta'(f) = \lim_i \lim_j f((g_i,g)(g,h_j)) = \lim_i \lim_j f(g_i,h_j) =$$

$$= \lim_i \lim_j f_{g_i}(h_j) = \lim_i \eta(f_{g_i}) =$$

$$= \lim_i U_{\eta,f}(g_i) = \theta(U_{\eta,f}).$$

Note that, we can deduce that the composable elements $(\beta G)^{(2)}$ is homeomorphic to the disjoint union of $\beta[g] \times \beta[g]$' s, that is,

$$\overset{\circ}{\bigsqcup_{g \in X}} \beta[g] \times \beta[g].$$

***Example* 3.2.** Another example of a groupoid is the transformation group groupoid. Suppose that the group S acts on a set U on the right. The image of the point u by the transformation s is denoted $u.s$. We let G be $U \times S$ and define the following groupoid structure: (u,s) and (v,t) are composable if and if $v = u.s$, $(u,s)(u.s,t) = (u,st)$, and $(u,s)^{-1} = (u.s,s^{-1})$. Then $r(u,s) = (u,e)$ and $d(u,s) = (u.s,e)$. The map $(u,e) \mapsto u$ identifies G^0 with U. It is easy to check that

$$G^u = \{u\} \times S, \quad G_u = \{(u.s^{-1},s)\colon s \in S\}.$$

By the same argument mentioned in Example 3.1, we can identify G^u with S and obtain a homeomorphism between βG^u and βS. Also, the map $(u.s^{-1},s) \mapsto s$ is a bijection from G_u onto S. So, this map has a continuous extension from βG_u onto βS. Thus, the set composable elements is homeomorphic to

$$\bigsqcup_{u\in U}^{\circ}\beta S\times\beta S.$$

Let $(\theta',\eta')\in\overline{G_u}\times\overline{G^u}$. Let $\{(u.s_i^{-1},s_i)\}_{i\in I}$ and $\{(u,t_j)\}_{j\in J}$ be nets in βG such that $(u.s_i^{-1},s_i)\longrightarrow$ $\longrightarrow\theta'$ and $(u,t_j)\longrightarrow\eta'$. Therefore, there exist η and θ in βS such that $s_i\longrightarrow\theta$ and $t_j\longrightarrow\eta$. For any $f\in\mathcal{B}(G)$, one has

$$\theta'*\eta'(f)=\lim_i\lim_j(u.s_i^{-1},s_i)(u,t_j)=$$

$$=\lim_i\lim_j(u,s_it_j)(f)=$$

$$=\lim_i\lim_j f_u(s_it_j)=\theta*\eta(f_u),$$

where f_u is a map from S into $\mathbb{C}$ defined by $f_u(s)=f(u,s)$.

4. Topological groupoids. Let G be a topological groupoid such that for every $u\in G^0$, G^u is C*-embedded. Let $\widetilde{l_g^u}$ be the extension of the map l_g^u mentioned in the previous section. Then for each fixed $\eta\in\overline{G^u}$, we may consider the mapping r_η^u from G_u into βG. Defined by $r_\eta^u(g)=\widetilde{l_g^u}(g)$. But unlike the discrete case, nothing guarantees that the mapping r_η^u is continuous for every $\eta\in\overline{G^u}$. Therefore we might not able to extend these mappings to βG leading to a continuous operation on βG.

We can start this process by extending the mappings $r_h^u\colon G_u\longrightarrow\beta G$ to mappings $\widetilde{r_h^u}\colon\overline{G_u}\longrightarrow$ $\longrightarrow\beta G$, where $h\in G^u$. If for every $\theta\in\overline{G_u}$ we define $l_\theta^u\colon G^u\longrightarrow\beta G$ by $l_\theta^u(h)=\widetilde{r_h^u}(\theta)$, again nothing guarantees the continuity of the mappings l_θ for each $\theta\in\overline{G_u}$.

Let G be a topological groupoid. As the previous section, we can define left g-translation L_gf and right g-translation. But these map are not continuous in general. As we know we can regard βG as the maximal ideal space of $C_b(G)$. It seems that we can not define the extended operation on βG in the term of elements of the maximal ideal space of $C_b(G)$. Because the mapping $g\mapsto\eta(L_gf)$ is not well-defined.

Lemma 4.1. *Suppose that $u\in G^0$, $\eta\in\overline{G^u}$ and $f\in C_b(G^u)$. Let $F_1,F_2\in C_b(G)$such that $F_1|_{G^u}=F_2|_{G^u}=f$. Then $\eta(F_1)=\eta(F_2)$.*

Let $u\in G^0$, $g\in G_u$ and let $h\in G^u$. For $f\in C_b(G)$ define $\mathcal{L}_g^uf\colon G^u\longrightarrow\mathbb{C}$ by $\mathcal{L}_g^uf(x)=f(gx)$. Also, define $\mathcal{R}_h^uf\colon G_u\longrightarrow\mathbb{C}$ by $\mathcal{R}_h^uf(x)=f(xh)$.

Definition 4.1. *Let $u\in G^0$, $\eta\in\overline{G^u}$, $\theta\in\overline{G_u}$. For $f\in C_b(G)$, set*

$$\mathcal{T}_{\eta,f}^u\colon G_u\longrightarrow\mathbb{C}\quad\mathcal{T}_{\eta,f}^u(g)=\eta(\widetilde{\mathcal{L}_g^uf}),$$

where $\widetilde{\mathcal{L}_g^uf}$ is an extension of $\mathcal{L}_g^uf$ in $C_b(G)$. By Lemma 4.1, *$\mathcal{T}_{\eta,f}^u$ is well-defined. Also defined*

$$\mathcal{S}_{\theta,f}^u\colon G^u\longrightarrow\mathbb{C},\quad\mathcal{S}_{\theta,f}^u(h)=\theta(\widetilde{\mathcal{R}_h^uf}),$$

where $\widetilde{\mathcal{R}_h^uf}$ is an extension of $\widetilde{\mathcal{R}_h^uf}$ in $C_b(G)$.

Theorem 4.1. *Let G be a topological groupoid and let $f\in C_b(G)$. Then the followings are equivalent*:

(1) *For every $u\in G^0$ and for every $\theta\in\overline{G_u}$, l_θ^u is continuous.*

(2) *For every $u\in G^0$ and for every $\eta\in\overline{G^u}$, r_η^u is continuous.*

(3) *For every* $u \in G^0$, *for every* $\eta \in \overline{G_u}$ *and every* $f \in C_b(G)$, $\mathcal{T}^u_h f$ *is continuous.*

(4) *For every* $u \in G^0$, *for every* $\theta \in \overline{G_u}$ *and every* $f \in C_b(G)$, $\mathcal{S}^u_{\theta,f}$ *is continuous.*

Proof. First, we prove the following identities, for any $u \in G^0$, $g \in G_u$, $h \in G^u$ and $f \in \mathcal{B}(G)$, are satisfied

$$f(r^u_\eta(g)) = \widehat{f}(l^u_{\eta^{-1}}(g^{-1})), \tag{4.1}$$

$$f(r^u_\eta(g)) = \mathcal{T}^u_{\eta,f}(g), \tag{4.2}$$

$$f(l^u_\theta(h)) = \mathcal{S}^u_{\theta,f}(h). \tag{4.3}$$

Let $u \in G^0$ and $\eta \in \overline{G^u}$. Let $\{h_j\}_{j\in J}$ be a net in G^u such that $h_j \longrightarrow \eta$. So, $\eta^{-1} \in \overline{G_u}$ and $h_j \longrightarrow \eta^{-1}$. For (4.1),

$$f(r^u_\eta(g)) = f(\widetilde{l^u_g}(\eta)) = \lim_j f(l^u_g(h_j)) = \lim_j f(gh_j) =$$

$$= \lim_j \widehat{f}(h_j^{-1}g^{-1}) = \lim_j f(r^u_{g^{-1}}(h_j^{-1})) =$$

$$= \widehat{f}(\widetilde{r^u_{g^{-1}}}(\eta^{-1})) = \widehat{f}(l^u_{\eta^{-1}}(g^{-1})).$$

On the other hand

$$f(r^u_\eta(g)) = f(\widetilde{l^u_g}(\eta)) = \lim_j f(l^u_g(h_j)) = \lim_j f(gh_j) =$$

$$= \lim_j \mathcal{L}^u_g f(h_j) = \eta(\widetilde{\mathcal{L}^u_g f}) = \mathcal{T}^u_{\eta,f}(g).$$

Similarly, one can prove (4.3). The identity (4.1) implies that (1) ⇔ (2) and the identity (4.2) implies that (2) ⇔ (3) and the identity (4.3) implies that (1) ⇔ (4) .

Theorem 4.1 is proved.

***Example* 4.1.** Let G be the groupoid $[0,\infty) \times [0,\infty)$. Consider the sequence $((1,n))_{n=1}^\infty$. So there exist a subnet $((1,n_k))_{k=1}^\infty$ and $\eta \in \beta G$ such that $(1,n_k) \longrightarrow \eta$ in βG. Let f be a function in $C_b([0,\infty))$ such that $f(n) = 1$ for every n and $f(t) = 0$ if $n + \dfrac{1}{n} < t < n + 1 - \dfrac{1}{n+1}$. Define $F : G \longrightarrow \mathbb{C}$ by $F(x,y) = f(x+y)$. Then for every $m \in \mathbb{N}$, $\mathcal{T}_{\eta,F}\left(\dfrac{1}{m}, 1\right) = \lim_k f\left(n_k + \dfrac{1}{m}\right) = 0$. But $\mathcal{T}_{\eta,F}(0,1) = f(n_k) = 1$. Therefore, $\mathcal{T}_{\eta,F}$ is not continuous.

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