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ON CENTRALIZING AND STRONG COMMUTATIVITY PRESERVING MAPS OF SEMIPRIME RINGS*

ПРО ЦЕНТРАЛІЗУЮЧІ ТА СИЛЬНІ ВІДОБРАЖЕННЯ НАПІВПРОСТИХ КІЛЕЦЬ, ЩО ЗБЕРІГАЮТЬ КОМУТАТИВНІСТЬ

We study some properties of centralizing and strong commutativity preserving maps of semiprime rings.

Вивчаються деякі властивості централізуючих та сильних відображень напівпростих кілець, що зберігають комутативність.

1. Introduction. Let R will be an associative ring with center Z. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol xoy denotes for the anticommutator $xy + yx$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$ and R is semiprime if $xRx = 0$ implies $x = 0$. A prime ring obviously semiprime. An additive mapping d from R into itself is called derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let S be a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$, for all $x \in S$. Also, F is called commutativity preserving on a subset S of R if $[x, y] = 0$ implies $[F(x), F(y)] = 0$, for all $x, y \in S$. The mapping F is called strong commutativity preserving (simply, SCP) on S if $[x, y] = [F(x), F(y)]$, for all $x, y \in S$. The study of centralizing and commuting mappings was initiated by Posner in [14]. Over the last fifteen years, several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations which are centralizing and commuting on appropriate subsets of R (see, e.g., [4, 6, 11, 12] and references therein). On the other hand, there is also a growing literature SCP maps and derivations. For more information on SCP maps and derivations, we refer to [5, 7, 10]. In [8], M. A. Chaudhry and A. B. Thaheem showed that if R is a semiprime ring and f is an endomorphism of R, g is an epimorphism of R such that $[f(x), g(x)] \in Z$, then $[f(x), g(x)] = 0$ holds for all $x \in R$. In [1], A. Ali, M. Yasen and M. Anwar showed that if R is a semiprime ring, f is an endomorphism which is a strong commutativity preserving map on a nonzero ideal U of R, then f is commuting on U. In [13], M. S. Samman proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing.

Morever, in [2], M. Asraf and N. Rehman showed that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d satisfying $d(xy) \pm xy \in Z$, for all $x, y \in R$. In [3], the authors explored this result for a generalized derivation of R.

In this paper, we prove some results of centralizing and strong commutativity preserving maps of semiprime rings. In Theorem 1, we extend a result of M. A. Chaudhry and A. B. Thaheem [8] (Theorem 2.2). In Theorem 2 is an analogues of [1] (Theorem 1) and Theorem 4 is an extension of

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[3] (Theorem 2.5). Also, we shall make use of the following basic identities without any specific mention:

i) $[x, yz] = y[x, z] + [x, y]z$,

ii) $[xy, z] = [x, z]y + x[y, z]$,

iii) $xyoz = (xoz)y + x[y, z] = x(yoz) - [x, z]y$,

iv) $xoyz = y(xoz) + [x, y]z = (xoy)z + y[z, x]$.

2. Results.

Lemma 1 ([9], Lemma 1)**.** *Let R be a semiprime ring and U a nonzero ideal of R. If z in R centralizes the set $[U, U]$, then z centralizes U.*

Theorem 1. *Let R be a semiprime ring with* $\operatorname{char} R \neq 2$, *$f$ and g be two endomorphisms of R and U is a nonzero right ideal of R. If $[f(u), g(u)] \in Z$ for all $u \in U$, then $[f(u), g(u)] = 0$ for all $u \in U$.*

Proof. A linearization of $[f(u), g(u)] \in Z$ yields that

$$[f(u), g(v)] + [f(v), g(u)] \in Z \qquad \text{for all} \quad u, v \in U.$$

Replacing v by u^2 in this equation, we get

$$[f(u), g(u)]\, g(u) + g(u)\, [f(u), g(u)] + f(u)\, [f(u), g(u)] + [f(u), g(u)]\, f(u) \in Z.$$

Using the hypothesis and $\operatorname{char} R \neq 2$, we obtain that

$$g(u)\, [f(u), g(u)] + f(u)\, [f(u), g(u)] \in Z \qquad \text{for all} \quad u \in U.$$

Commuting this term with $f(u)$, we arrive at

$$[f(u), g(u)]^2 = 0 \qquad \text{for all} \quad u \in U.$$

Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that $[f(u), g(u)] = 0$ for all $u \in U$.

Theorem 1 is proved.

In particular, if we take $g = I$, where $I\colon R \to R$ is an identity endomorphism, then we have the following corollary which is a generalization of [4] (Lemma 2) for the case when characteristic is different from two.

***Corollary* 1.** *Let R be a semiprime ring with* $\operatorname{char} R \neq 2$, *$f$ be an endomorphism of R and U is a nonzero right ideal of R. If f is centralizing on U, then f is commuting on U.*

Theorem 2. *Let R be a semiprime ring, f and g be two endomorphisms of R and U is a nonzero ideal of R. If $[f(u), g(v)] = [u, v]$ for all $u, v \in U$, then g is commuting on U.*

Proof. By the hyphothesis, we have

$$[f(u), g(v)] = [u, v] \qquad \text{for all} \quad u, v \in U.$$

Substituting vw, $w \in U$ for v in the above equation, we obtain that

$$[f(u), g(v)]\, g(w) + g(v)\, [f(u), g(w)] = [u, v]\, w + v\, [u, w].$$

Using the hyphothesis, we arrive at

$$[u, v]\, g(w) + g(v)\, [u, w] = [u, v]\, w + v\, [u, w],$$

and so

$$[u,v]\left(g\left(w\right)-w\right)+\left(g\left(v\right)-v\right)[u,w]=0 \qquad \text{for all} \quad u,v,w\in U. \tag{1}$$

Replacing w by u in (1), we get

$$[u,v]\left(g\left(u\right)-u\right)=0 \qquad \text{for all} \quad u,v\in U.$$

Taking v by rv, $r\in R$ in the last equation and using this equation, we see that

$$[u,r]\,v\left(g\left(u\right)-u\right)=0 \qquad \text{for all} \quad u,v\in U,\ r\in R,$$

and so

$$[u,r]\,RU\left(g\left(u\right)-u\right)=0 \qquad \text{for all} \quad u\in U,\ r\in R.$$

Now, we let $\wp=\{P_\alpha\mid\alpha\in\Lambda\}$ be a family of prime ideals with $\cap P_\alpha=(0)$. If P is a typical member of $\wp$ and $u\in U$, then the last equation shows that

$$[u,R]\subseteq P \qquad \text{or} \quad U\left(g\left(u\right)-u\right)\subseteq P.$$

Suppose that $\exists v\in U$ such that $[v,R]\nsubseteq P$. Thus $U\left(g\left(v\right)-v\right)\subseteq P$. Let w is any element of U such that $[v+w,R]\subseteq P$. Hence $[w,R]\nsubseteq P$. Indeed, if $[w,R]\subseteq P$, then $[v,R]\subseteq P$. It contradicts $[v,R]\nsubseteq P$. Therefore we get $[w,R]\nsubseteq P$. This implies that $U\left(g\left(w\right)-w\right)\subseteq P$ for all $w\in U$. If $[v+w,R]\nsubseteq P$, then $U\left(g\left(v+w\right)-\left(v+w\right)\right)\subseteq P$ for all $w\in U$, and so $U\left(g\left(w\right)-w\right)\subseteq P$ for all $w\in U$. Hence we obtain that $U\left(g\left(w\right)-w\right)\subseteq P$ for all $w\in U$, for any cases. Therefore $[U,U]\left(g\left(w\right)-w\right)\subseteq P$ for all $w\in U$.

Since $\cap P_\alpha=(0)$, we have

$$[U,U]\left(g\left(w\right)-w\right)=(0) \qquad \text{for all} \quad w\in U. \tag{2}$$

On the other hand, taking u instead of v in (1), we obtain

$$\left(g\left(u\right)-u\right)[u,w]=0 \qquad \text{for all} \quad u,w\in U.$$

Again appliying similar arguments as above, we get

$$\left(g\left(w\right)-w\right)[U,U]=(0) \qquad \text{for all} \quad w\in U. \tag{3}$$

Using (2) and (3), we conclude that $\left(g\left(w\right)-w\right)\in C_R\left([U,U]\right)$ for all $w\in U$. By Lemma 1, we obtain that $\left(g\left(w\right)-w\right)\in C_R\left(U\right)$ for all $w\in U$. Thus $[g\left(w\right)-w,w]=0$ for all $w\in U$. This implies that $[g\left(w\right),w]=0$ for all $w\in U$, and so g is commuting on U.

Theorem 2 is proved.

If we have $f=g$, then we can give the following corollary which is a generalization of [1] (Theorem 1).

***Corollary* 2.** *Let R be a semiprime ring, f be an endomorphism of R and U is a nonzero ideal of R. If f is strong commutativity preserving on U, then f is commuting on U.*

***Corollary* 3.** *Let R be a semiprime ring, f be an endomorphism of R and U is a nonzero ideal of R. If f satisfies one of the following conditions*:

(i) $f(uv)=uv$ *for all* $u,v\in U$,

(ii) $f(uv)=-uv$ *for all* $u,v\in U$,

(iii) *for each* $u,v\in U$, *either* $f(uv)=uv$ *or* $f(uv)=-uv$,

then f is commuting on U.

Proof. (i) By the hypothesis, we get $f(uv) = uv$ for all $u, v \in U$. Thus, we have

$$f(uv - vu) = f(uv) - f(vu) = uv - vu.$$

Therefore $[f(u), f(v)] = [u, v]$, for all $u, v \in U$. By Corollary 2, we arrive at f is commuting on U.

(ii) Using the same arguments in the proof of (i), we find the required result.

(iii) For each $u \in U$, we put $U_u = \{v \in U \mid f(uv) = uv\}$ and $U_u^* = \{v \in U \mid f(uv) = -uv\}$. Then $(U, +) = U_u \cup U_u^*$. But a group cannot be the union of proper subgroups. Hence we get $U = U_u$ or $U = U_u^*$. By the same method in (i) or (ii), we complete the proof.

Theorem 3. *Let R be a semiprime ring, f and g be two endomorphisms of R and U is a nonzero ideal of R. If $f(u)g(v) - uv = 0$ for all $u, v \in U$, then g is commuting on U.*

Proof. By the hypothesis, we have $f(u)g(v) = uv$ for all $u, v \in U$. Replacing v by vw, we find that

$$f(u)g(v)g(w) = uvw \qquad \text{for all} \quad u, v \in U.$$

Using the hypothesis in this equation, we get $uvg(w) = uvw$, and so $uv(g(w) - w) = 0$. This can be written as $U^2(g(w) - w) = (0)$ and implies that

$$[U, U](g(w) - w) = 0 \qquad \text{for all} \quad w \in W. \tag{4}$$

Substituting uw for u in the hypothesis and using this, we find that $f(u)wv = uwv$. Taking $g(t)w$ instead of w in this equation, we get

$$f(u)g(t)wv = ug(t)wv,$$

and so

$$utwv = ug(t)wv \qquad \text{for all} \quad u, v, w, t \in U.$$

The above expression implies that $u(g(t) - t)U^2 = (0)$. Replacing u by $ur, r \in R$ in this equation, we obtain that

$$uR(g(t) - t)U^2 = (0) \qquad \text{for all} \quad u, t \in U.$$

Now as in the proof of Theorem 2, we let $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ be a family of prime ideals with $\cap P_\alpha = (0)$. If P is a typical member of $\wp$ and $u \in U$, then the last equation shows that

$$(g(t) - t)U^2 = (0) \qquad \text{for all} \quad t \in U,$$

and so

$$(g(t) - t)[U, U] = (0) \qquad \text{for all} \quad t \in U. \tag{5}$$

Using (4) and (5), we conclude that $(g(t) - t) \in C_R([U, U])$ for all $t \in U$. By Lemma 1, we obtain that $(g(t) - t) \in C_R(U)$ for all $t \in U$. Thus $[g(t) - t, t] = 0$ for all $t \in U$. This implies that $[g(t), t] = 0$ for all $t \in U$, and so g is commuting on U.

Theorem 3 is proved.

Theorem 4. *Let R be a semiprime ring, f and g be two endomorphisms of R and U is a nonzero ideal of R. If $f(u)g(v) - uv \in Z$ for all $u, v \in U$, then g is commuting on U.*

Proof. Replacing v by vw, $w \in U$ in the hypothesis, we get

$$f(u)\,g(v)\,g(w) - uvw \in Z \qquad \text{for all} \quad u, v, w \in U.$$

Let $f(u)\,g(v) - uv = \alpha$, $\alpha \in Z$. Writing $f(u)\,g(v) = \alpha + uv$ in the above equation, we have

$$(\alpha + uv)\,g(w) - uvw \in Z,$$

and so

$$uv\,(g(w) - w) + \alpha g(w) \in Z \qquad \text{for all} \quad u, v, w \in U. \tag{6}$$

Commuting this term with $g(w)$, we arrive at

$$-uv\,[w, g(w)] + u\,[v, g(w)]\,(g(w) - w) + [u, g(w)]\,v\,(g(w) - w) = 0.$$

Substituting ru, $r \in R$ for u in the last equation, we obtain that

$$-ruv\,[w, g(w)] + ru\,[v, g(w)]\,(g(w) - w) + r\,[u, g(w)]\,v\,(g(w) - w) + [r, g(w)]\,uv\,(g(w) - w) = 0.$$

That is

$$[r, g(w)]\,uv\,(g(w) - w) = 0,$$

and so

$$[r, g(w)]\,RU^2\,(g(w) - w) = 0 \qquad \text{for all} \quad w \in U,\ r \in R.$$

Let $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ be a family of prime ideals with $\cap P_\alpha = (0)$, P a typical member of $\wp$ and $u \in U$. For each $w \in U$ either $U^2\,(g(w) - w) \subseteq P$ or $[r, g(w)] \in P$ for all $r \in R$. We assume that $\exists v \in U$ such that $[r, g(v)] \notin P$. Let u is any element of U such that $[r, g(u + v)] \in P$. This implies that $[r, g(u)] \notin P$. Thus $U^2\,(g(u) - u) \subseteq P$ for all $u \in U$. If $[r, g(u + v)] \notin P$, then $U^2\,(g(u + v) - (u + v)) \subseteq P$ for all $u \in U$, and so $U^2\,(g(u) - u) \subseteq P$ for all $u \in U$. Hence we get $U^2\,(g(u) - u) \subseteq P$ for all $u \in U$.

Since P arbitrary and $\cap P_\alpha = (0)$, we arrive at

$$U^2\,(g(u) - u) = (0) \qquad \text{for all} \quad u \in U. \tag{7}$$

That is

$$[U, U]\,(g(u) - u) = (0) \qquad \text{for all} \quad u \in U. \tag{8}$$

Multipliying (7) on the left by $(g(u) - u)$, we have

$$(g(u) - u)U^2\,(g(u) - u) = (0) \qquad \text{for all} \quad u \in U.$$

Again multipliying this equation on the right by U^2, we obtain that

$$(g(u) - u)U^2\,(g(u) - u)\,U^2 = (0) \qquad \text{for all} \quad u \in U,$$

and so

$$(g(u) - u)U^2R\,(g(u) - u)\,U^2 = (0) \qquad \text{for all} \quad u \in U.$$

By the semipimeness of R, we conclude that

$$(g(u) - u)\,U^2 = (0) \qquad \text{for all} \quad u \in U,$$

and so

$$(g(u)-u)[U,U]=(0) \qquad \text{for all} \quad u\in U. \tag{9}$$

By (8) and (9), one easily checks that $(g(u)-u)\in C_R([U,U])$, for all $u\in U$. By Lemma 1, we get $(g(u)-u)\in C_R(U)$ for all $u\in U$. Thus $[g(u)-u,u]=0$ for all $t\in U$, and so g is commuting on U.

Theorem 4 is proved.

We can give a following corollary which is a generalization of Corollary 3 (i).

***Corollary* 4.** *Let R be a semiprime ring, f be an endomorphism of R and U is a nonzero ideal of R. If $f(uv)-uv\in Z$ for all $u,v\in U$, then f is commuting on U.*

Theorem 5. *Let R be a semiprime ring, f be an endomorphism, g an epimorphism of R and U is a nonzero ideal of R. If $f(u)og(v)=uov$ for all $u,v\in U$, then g is commuting on U.*

Proof. Writing v by vr, $r\in R$ in the hypothesis, we have

$$(f(u)og(v))g(r)+g(v)[g(r),f(u)]=(uov)r+v[r,u] \qquad \text{for all} \quad u,v\in U,\ r\in R.$$

Using the hypothesis, we obtain that

$$(uov)(g(r)-r)+g(v)[g(r),f(u)]=v[r,u] \qquad \text{for all} \quad u,v\in U,\ r\in R. \tag{10}$$

Replacing v by uv in (10), we get

$$u(uov)(g(r)-r)+[u,u]v(g(r)-r)+g(u)g(v)[g(r),f(u)]=uv[r,u],$$

and so

$$u((uov)(g(r)-r)-v[r,u])+g(u)g(v)[g(r),f(u)]=0 \qquad \text{for all} \quad u,v\in U,\ r\in R.$$

Using (10) in the above equation, we see that

$$-ug(v)[g(r),f(u)]+g(u)g(v)[g(r),f(u)]=0.$$

That is

$$(g(u)-u)g(v)[g(r),f(u)]=0 \qquad \text{for all} \quad u,v\in U,\ r\in R.$$

Since g is onto, we arrive at

$$(g(u)-u)V[r,f(u)]=0 \qquad \text{for all} \quad u\in U,\ r\in R,$$

where V is an ideal of R. Thus $(g(u)-u)VR[r,f(u)]=0$, for all $u\in U$. By the semiprimeness of R, it must contain a family $\wp=\{P_\alpha\,|\,\alpha\in\Lambda\}$ of prime ideals such that $\cap P_\alpha=(0)$. Let P denote a fixed one of the P_α. From the last equation, it follows that for each $u\in U$ either $(g(u)-u)V\subseteq P$ or $[r,f(u)]\subseteq P$ for all $r\in R$. Assume that $\exists v\in U$ such that $[r,f(v)]\nsubseteq P$. Therefore $(g(v)-v)V\subseteq P$.

Suppose w is any element of U. If $[r,f(v+w)]\in P$, then $[r,f(w)]\notin P$. Indeed, if $[r,f(w)]\in P$, then $[r,f(v)]\in P$. It contradicts $[r,f(v)]\notin P$. Hence we get $[r,f(w)]\notin P$. That is $(g(w)-w)V\subseteq P$ for all $w\in U$. On the other hand, if $[r,f(v+w)]\notin P$, then $(g(v+w)-(v+w))V\subseteq P$. This implies that $(g(w)-w)V\subseteq P$ for all $w\in U$. For any cases $(g(w)-w)V\subseteq P$ for all $w\in U$ and so $(g(w)-w)[V,V]\subseteq P$ for all $w\in U$. Since P is arbitrary and $\cap P_\alpha=(0)$, we obtain that

$$(g(w)-w)[V,V]=(0) \qquad \text{for all} \quad w\in U. \tag{11}$$

Taking rv instead of $v, r\in R$ in the hypothesis, we find that

$$f(u)og(r)g(v) = uorv,$$

$$g(r)(f(u)og(v)) + [f(u), g(r)]g(v) = r(uov) + [u, r]v,$$

and so

$$(g(r) - r)(uov) + [f(u), g(r)]g(v) = [u, r]v \qquad \text{for all} \quad u, v \in U, r \in R. \tag{12}$$

Replacing v by vu in the last equation, we get

$$((g(r) - r)(uov) - [u, r]v)u + [f(u), g(r)]g(v)g(u) = 0.$$

Using (12) in the above equation, we arrive at

$$[f(u), g(r)]g(v)(g(u) - u) = 0 \qquad \text{for all} \quad u, v \in U, r \in R.$$

Since g is onto, we have

$$[f(u), r]V(g(u) - u) = 0 \qquad \text{for all} \quad u \in U, r \in R,$$

where V is an ideal of R. Using similar arguments as above, we can prove that

$$[V, V](g(w) - w) = (0) \qquad \text{for all} \quad w \in U. \tag{13}$$

By equations (11) and (13), one easily checks that $(g(w) - w) \in C_R([V, V])$ for all $w \in U$. By Lemma 1, we obtain that $(g(w) - w) \in C_R(V)$ for all $w \in U$. Since $V = g(U)$, we have $(g(w) - w) \in C_R(g(U))$ for all $w \in U$. Thus $[g(w) - w, g(w)] = 0$ for all $w \in U$. This implies that $[g(w), w] = 0$ for all $w \in U$, and so g is commuting on U.

Theorem 5 is proved.

***Corollary* 5.** *Let R be a semiprime ring, f be an epimorphism of R and U is a nonzero ideal of R. If $f(uov) = uov$ for all $u, v \in U$, then f is commuting on U.*

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