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BOUNDEDNESS OF WEAK SOLUTIONS TO NONDIAGONAL SINGULAR PARABOLIC SYSTEM OF THREE EQUATIONS

ОБМЕЖЕНІСТЬ СЛАБКИХ РОЗВ'ЯЗКІВ НЕДІАГОНАЛЬНОЇ СИНГУЛЯРНОЇ ПАРАБОЛІЧНОЇ СИСТЕМИ ТРЬОХ РІВНЯНЬ

L^∞-estimates of weak solutions are established for a quasilinear nondiagonal parabolic system of singular equations with matrix of coefficients satisfying special structure conditions. A technique based on estimating the linear combinations of unknowns is employed to this end.

L^∞-оцінки слабких розв'язків встановлено для квазілінійної недіагональної параболічної системи сингулярних рівнянь, матриця коефіцієнтів якої задовольняє спеціальні структурні умови. Використовується техніка, що базується на оцінці лінійних комбінацій невідомих.

1. Introduction. In the present paper we study the boundedness of weak solutions to the quasilinear nondiagonal parabolic system of three singular equations in divergence form under special assumptions upon its structure.

It is well-known that the De Giorgi – Nash – Moser estimates are no longer valid in general for an elliptic system, the latter can be regarded as a special case of the parabolic version. An example of an unbounded solution to the linear elliptic system with bounded coefficients was built up by De Giorgi in [1]. There is yet another example due to J. Nečas and J. Souček of nonlinear elliptic system with the coefficients sufficiently smooth, but the weak solution not belonging to $W^{2,2}$. These two and many other examples illustrate that the regularity problem for elliptic systems proves to be far more complicated than that for second order elliptic equations and that the smooth properties of solutions are not only determined by the smoothness of data, but strongly depend upon the structure of system.

Until now a priori estimates of De Giorgi type have been extended only to a special class of parabolic systems of equations, the so-called weakly coupled systems. The system is said to be weakly coupled if it is coupled only through the terms which are not differentiated, each equation containing derivatives of just one component.

There exists yet another approach to a priori estimates for a parabolic system of second order differential equations [2]. This applies for diagonal systems which on freezing the leading coefficients and discarding the right-hand sides and lower order terms reduce to just one single equation rewritten several times in turn for all the unknown functions; see also [3, p. 27; 4, p. 32, 33; 5].

The technique we are utilizing has been employed earlier in [6] for semilinear systems (see also [7 – 9]), and consists in switching to new functions, for each of which the estimate is established in a conventional way, whence the final conclusion about each component of the vector function solution follows. This technique allows for extension to nondiagonal systems with nonlinearities in the spatial derivatives also.

The main idea of our approach is as follows: instead of trying to establish estimates for each component of solution (u, v, w) rather to introduce some linear combinations of components of the solution:

$$\begin{aligned} H_1 &= \alpha_1 u + \beta_2 v + w, \\ H_2 &= \alpha_2 u + \beta_2 v + w, \\ H_3 &= \alpha_3 u + \beta_3 v + w, \end{aligned} \tag{1.1}$$

in general some functions H of t, x, u, v, w, for each of which the estimates hold and from whose estimates we shall be able to derive the estimates for the components of solution (u, v, w).

In the present paper although restricting ourselves to systems of second order equations in divergence form possessing special structure, we demonstrate boundedness of solution to quasilinear singular parabolic system of three equations in which coupling occurs in the leading derivatives and whose leading coefficients depend on x, u, v, w and u_x, v_x, w_x.

2. Basic notations and hypotheses. We shall be concerned with a system of three equations of the form:

$$u_t - \frac{\partial}{\partial x_i}\Big(A_i^{(1)}(x, u, v, w, u_x, v_x, w_x)\Big) = B^{(1)}(x, u, v, w, u_x, v_x, w_x),$$

$$v_t - \frac{\partial}{\partial x_i}\Big(A_i^{(2)}(x, u, v, w, u_x, v_x, w_x)\Big) = B^{(2)}(x, u, v, w, u_x, v_x, w_x), \tag{2.1}$$

$$w_t - \frac{\partial}{\partial x_i}\Big(A_i^{(3)}(x, u, v, w, u_x, v_x, w_x)\Big) = B^{(3)}(x, u, v, w, u_x, v_x, w_x), \qquad x \in Q.$$

The boundary conditions of the Dirichlet type are assigned:

$$(x - g_1, v - g_2, w - g_3)(x, t) \in W_0^{1,p}(\Omega) \quad \text{a.e.} \quad t \in (0, T),$$
$$(u, v, w)(x, 0) = (u_0, v_0, w_0)(x). \tag{2.2}$$

A solution to system (2.1) with Dirichlet data (2.2) is understood in the weak sense, as in [10].

Definition 2.1. *A measurable vector function* $(u^1, u^2, u^3) = (u, v, w)$ *is called a weak solution of problem* (2.1), (2.2) *if*

$$u^j \in C\big(0, T; L^2(\Omega)\big) \cap L^p\big(0, T; W^{1,p}(\Omega)\big)$$

and for all $t \in (0, T]$

$$\int_\Omega u^j \varphi_j(x, t)\, dx + \iint_{\Omega\times(0,t]} \left\{-u^j \varphi_{jt} + A_i^j \varphi_{jx_i}\right\} dx\, d\tau =$$

$$= \int_\Omega u_0^j \varphi_j(x, 0)\, dx + \iint_{\Omega\times(0,t]} B^j \varphi_j\, dx\, d\tau$$

for all testing functions

$$\varphi \in W^{1,2}\big(0, T; L^2(\Omega)\big) \cap L^p\big(0, T; W_0^{1,p}(\Omega)\big), \qquad \varphi \geq 0.$$

The boundary condition in (2.2) *is meant in the weak sense.*

Let us also define the boundary norms of functions that will come useful in the follow-up considerations.

Definition 2.2. *Let* Ω *be a domain in* $\mathbb{R}^n$ (*here* n *is any natural number*) *and* $\partial\Omega$ *a portion of its boundary*; $W(\Omega)$ *be any Sobolev space. For a function* $u(x)$ *specified on* $\partial\Omega$ *we define*

$$\|u\|_{W(\partial\Omega)} = \inf_\psi \|\psi\|_{W(\Omega)},$$

where the infimum is taken in all functions $\psi \in W(\Omega)$ *such that* $\psi(x) = u(x)$ *a. e. on* $\partial\Omega$. *We shall denote by* $W(\partial\Omega)$ *a functional space for which the aforementioned norm is finite.*

Let us describe the notions, quantities and functions that will appear in this paper.

Here and onward we accept the following notations: $Q = (0, T]$; $S = \partial\Omega \times (0, T]$; $\partial Q \equiv \{\Omega \times \{0\}\} \cup \{\partial\Omega \times (0, T]\}$; Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary; $x \in \Omega$; $T > 0$; $t \in (0, T]$; $1 < p < 2$; $p < n$; $i = 1, \ldots, n$; $j = 1, 2, 3$ and summation convention over repeated indices is assumed; u, v, $w \in C\big(0, T; L^2(\Omega)\big) \cap L^p\big(0, T; W^{1,p}(\Omega)\big)$; $W_0^{1,p}(\Omega)$ is a space of functions in $W^{1,p}(\Omega)$ vanishing on $\partial\Omega$ in the sense of traces for a.e. $t \in (0, T]$. Throughout the paper, for the sake of brevity, by $|s|$ and $|s_i|$ is denoted the distance in $3n$-dimensional and n-dimensional Euclidean space respectively, i.e.,

$$|s| = \sqrt{\sum_{j=1}^{3} \sum_{i=1}^{n} \left(s_i^j\right)^2}, \qquad |s_i| = \sqrt{\sum_{i=1}^{n} \left(s_i^j\right)^2},$$

where s_i^j stands for a $3n$-component vector.

By parabolicity of system (2.1) it is meant that the part without derivatives with respect to time is elliptic. The notion of ellipticity of a system of differential equations is understood in the following sense, as it is introduced in [11]:

$$\begin{gathered} \exists\, \lambda > 0 \quad 0 < F = F(x) \in L^{p/(p-1)}(Q) \,\big|\, \forall\, s_j^i \in \mathbb{R}^{3n} \\ \forall\, r_j \in \mathbb{R}^3 \quad \forall\, x \in \mathbb{R}^n\colon\ A_i^j(x, r, s) s_j^i \ \geq\ \lambda |s|^p - F. \end{gathered} \tag{2.3}$$

It should be emphasized that we impose neither the Legendre nor the Legendre – Hadamard condition. The Legendre condition stems from the calculus of variations, the problem of the minimization of functional, as a sufficient conditions for the existence of extremal. Since it is to be calculated on the extremal it bears no relation to the set-up of a problem. Its usage as the ellipticity condition in the theory of systems of equations is entirely technically motivated. The Legendre – Hadamard condition, being a weakened version of the Legendre one, has been regarded by many authors as a more natural ellipticity condition for systems. Both conditions produce an obstacle from the technical point of view in the approach used by ours, and that is why we dispense with them and accept the ellipticity condition (2.3) for quasilinear system as the most appropriate to our ends.

About $A_i^j(x, r, s)$ it is assumed that they are measurable $\Omega \times \mathbb{R}^3 \times \mathbb{R}^{3n} \to \mathbb{R}$ functions that satisfy the ellipticity conditions and are subject to the growth conditions:

$$\exists\, \Lambda_2 > 0 \quad \forall\, s_i^j \in \mathbb{R}^{3n} \quad \forall\, r^j \in \mathbb{R}^3 \quad \forall\, x \in \mathbb{R}^n\colon \left|A_i^j(x, r, s)\right| \ \leq\ \Lambda_2 |s|^{p-1}, \tag{2.4}$$

and to the following structure conditions:

$$\exists\, \alpha_j, \beta_j \in \mathbb{R};\quad \operatorname{Det}\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0 \quad \text{such that}$$

$$\forall\, s_i^j \in \mathbb{R}^{3n} \quad \forall\, r^j \in \mathbb{R}^3 \quad \forall\, x \in \mathbb{R}^n\colon$$

$$\begin{gathered} \left|\alpha_1 A_i^{(1)}(x, r, s) + \beta_1 A_i^{(2)}(x, r, s) + A_i^{(3)}(x, r, s) - \lambda_1(x, r, s)\big(\alpha_1 s_i^1 + \beta_1 s_i^2 + s_i^3\big)\right| \leq \\ \leq\ \xi_1(x, r, s) + F_1, \end{gathered} \tag{2.5a}$$

$$\left|\alpha_2 A_i^{(1)}(x,r,s)+\beta_2 A_i^{(2)}(x,r,s)+A_i^{(3)}(x,r,s)-\lambda_2(x,r,s)\left(\alpha_2 s_i^1+\beta_2 s_i^2+s_i^3\right)\right| \le$$
$$\le \xi_2(x,r,s)+F_2, \tag{2.5b}$$
$$\left|\alpha_3 A_i^{(1)}(x,r,s)+\beta_3 A_i^{(2)}(x,r,s)+A_i^{(3)}(x,r,s)-\lambda_3(x,r,s)\left(\alpha_3 s_i^1+\beta_3 s_i^2+s_i^3\right)\right| \le$$
$$\le \xi_3(x,r,s)+F_3, \tag{2.5c}$$

$\lambda_j=\lambda_j(x,r,s)>0$, $\xi_j=\xi_j(x,r,s)>0$ are some measurable $\Omega\times\mathbb{R}^3\times\mathbb{R}^{3n}\to\mathbb{R}$ functions of x, u, v, w, u_x, v_x, w_x on which the following growth conditions are imposed:

$$\exists\,\Lambda_1,\Lambda_2>0\quad\forall\, s_i^j\in\mathbb{R}^{3n}\quad\forall\, r^j\in\mathbb{R}^3\quad\forall\, x\in\mathbb{R}^n:$$

$$0<\Lambda_1\left|\alpha_j s_i^1+\beta_j s_i^2+s_i^3\right|^{p-2}\le\lambda_j(x,r,s)\le\Lambda_2\left|\alpha_j s_i^1+\beta_j s_i^2+s_i^3\right|^{p-2}, \tag{2.6}$$

$$\xi_j(x,r,s)\le\xi_0|s|^{\nu},\qquad 0<\nu=\frac{p(p-1)(1-\kappa_1)}{n+p}; \tag{2.7}$$

ξ_0 is a positive number,

$$F_j(x,t)\in L^{\theta}(Q),\qquad \theta=\frac{p+n}{(p-1)(1-\kappa_1)},\qquad \kappa_1\in(0,1), \tag{2.8}$$

moreover

$$\alpha_1,\beta_2>1, \tag{2.9}$$

$$\alpha_2,\alpha_3,\beta_1\beta_3<1, \tag{2.10}$$

$$3\max\left[\frac{1}{p},\Lambda_2\right]\max\left[\alpha_1^{-1},\beta_2^{-1},\alpha_3,\beta_3\right]\le\frac{\Lambda_1}{2^p p}, \tag{2.11}$$

$$6\xi_0\le\frac{\Lambda_1}{2^{p+1}p}. \tag{2.12}$$

***Remark* 2.1.** It is not difficult to check by direct calculation, taking into account the fact that $F_j\in L^{(p+n)/((p-1)(1-\kappa_1))}$, that structure conditions (2.5a) – (2.5c) along with (2.6) and (2.12) imply the ellipticity condition (2.3) with $\lambda=\dfrac{\Lambda_1}{2^{p+1}p}$ and $F\equiv$ $\equiv C_1(|F_1|+|F_2|+|F_3|)^{p/(p-1)}+C_2$, $C_{1,2}$ are numbers depending only on the data.

About right-hand sides $B^j(x,r,s)$ it is assumed that they are measurable $\Omega\times$ $\times\mathbb{R}^3\times\mathbb{R}^{3n}\to\mathbb{R}$ functions satisfying:

$$\exists\,\varepsilon\in\left(0,\frac{p^2(1-\kappa_1)}{n+p}\right]\quad\Lambda_3>0\quad\forall\, s_i^j\in\mathbb{R}^{3n}\quad\forall\, r^j\in\mathbb{R}^3\quad\forall\, x\in\mathbb{R}^n:$$
$$\left|B^j(x,r,s)\right|\le\Lambda_3|s|^{\varepsilon}. \tag{2.13}$$

In what follows for the sake of conciseness we shall use the notations:

$$\tilde{u}_0=\begin{cases}u_0(x), & x\in\Omega,\quad t=0,\\ g_1(x,t), & x\in\partial\Omega,\quad t\in(0,T),\end{cases}$$

$$\tilde{v}_0 = \begin{cases} v_0(x), & x \in \Omega, \quad t = 0, \\ g_2(x,t), & x \in \partial\Omega, \quad t \in (0,T), \end{cases}$$

$$\tilde{w}_0 = \begin{cases} w_0(x), & x \in \Omega, \quad t = 0, \\ g_3(x,t), & x \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

Let us introduce in addition the following functional space:

Definition 2.3.

$$\tilde{W}(Q) = L^{p'}\Big(W^{1,p'}(0,T);\Omega\Big) \cap L^p\Big(0,T;W^{1,p}(\Omega)\Big), \qquad p' = \frac{p}{p-1},$$

i.e., the function u *belongs to* $\tilde{W}(Q)$ *if the integral*

$$\int_0^T \int_\Omega \Big(|u_t|^{p'} + |\nabla u|^p + |u|^p + |u|^{p'} \Big)$$

is finite.

On the functions $g_j(x,t)$, $(u_0, v_0, w_0)(x)$ in boundary data (2.2) we assume to be fulfilled the following assumptions:

$$\tilde{u}_0 \in \tilde{W}(\partial Q), \qquad \tilde{v}_0 \in \tilde{W}(\partial Q), \qquad \tilde{w}_0 \in \tilde{W}(\partial Q);$$

and, in addition:

$$g_j(x,t) \in L^\infty(S), \qquad (u_0, v_0, w_0)(x) \in L^\infty\big(\overline{\Omega} \times \{0\}\big).$$

3. Estimate for the sum of squares. For the ongoing considerations we need to estimate the integral of the sum of squares of the spacial derivatives of the components of solution of problem (2.1), (2.2).

Our goal in this section is to prove the following statement.

Theorem 3.1. *Let* (u, v, w) *be a solution to problem* (2.1), (2.2) *and the hypotheses* (2.5a) – (2.5c), (2.6), (2.7) – (2.12) *and* (2.13) *are satisfied, then there hold the estimates*:

$$\sup_{0<t<T} \int_\Omega |u - \tilde{u}_0|^2 + \sup_{0<t<T} \int_\Omega |v - \tilde{v}_0|^2 + \sup_{0<t<T} \int_\Omega |w - \tilde{w}_0|^2 +$$

$$+ \int_0^T \int_\Omega \Big(|\nabla(u - \tilde{u}_0)|^p + |\nabla(v - \tilde{v}_0)|^p + |\nabla(w - \tilde{w}_0)|^p \Big) \le C$$

and

$$\int_0^T \int_\Omega \Big(|\nabla u|^p + |\nabla v|^p + |\nabla w|^p \Big) \le C$$

with constant C *depending only on the data*: F_j, $\|\tilde{u}_0\|_{\tilde{W}(\partial Q)}$, $\|\tilde{v}_0\|_{\tilde{W}(\partial Q)}$, $\|\tilde{w}_0\|_{\tilde{W}(\partial Q)}$, p, n, Λ_1, Λ_2, ξ_0, κ_1, α_j, β_j, ε, $\operatorname{mes} Q$, *and independent of* u, v *and* w.

***Remark* 3.1.** By $\tilde{u}_0$, $\tilde{v}_0$ and $\tilde{w}_0$ in the formulation of the theorem and in the follow-up proof is meant any function from $\tilde{W}(Q)$ assuming the values of either $\tilde{u}_0$ or $\tilde{v}_0$, $\tilde{w}_0$ on the parabolic boundary. Therefore the final statement remains valid with the boundary norms.

Proof. Multiply the first equation of (2.1) by $(u-\tilde{u}_0)$, the second one by $(v-\tilde{v}_0)$, and the third by $(w-\tilde{w}_0)$. After adding all three together and integrating over the domain $\Omega\times(0,t)$ this results in:

$$\int\limits_{\Omega(t)}\frac{1}{2}(u-\tilde{u}_0)^2 + \int\limits_{\Omega(t)}\frac{1}{2}(v-\tilde{v}_0)^2 + \int\limits_{\Omega(t)}\frac{1}{2}(w-\tilde{w}_0)^2 +$$

$$+\int\limits_0^t\int\limits_\Omega \vec{A}^{(1)}\nabla(u-\tilde{u}_0) + \int\limits_0^t\int\limits_\Omega \vec{A}^{(2)}\nabla(v-\tilde{v}_0) + \int\limits_0^t\int\limits_\Omega \vec{A}^{(3)}\nabla(w-\tilde{w}_0) \le$$

$$\le \int\limits_0^t\int\limits_\Omega \left|B^{(1)}\right|\left|u-\tilde{u}_0\right| + \int\limits_0^t\int\limits_\Omega \left|B^{(2)}\right|\left|v-\tilde{v}_0\right| + \int\limits_0^t\int\limits_\Omega \left|B^{(3)}\right|\left|w-\tilde{w}_0\right|, \qquad (3.1)$$

where the integration by parts with respect to time variable in the first two terms and the initial condition is taken into account. On the strength of ellipticity condition (2.3) and growth conditions on $A^{(1),(2),(3)}$ (2.4), the second group of terms on the left admits the estimation:

$$\int\limits_0^t\int\limits_\Omega \left(\vec{A}^{(1)}\nabla(u-\tilde{u}_0) + \vec{A}^{(2)}\nabla(v-\tilde{v}_0) + \vec{A}^{(3)}\nabla(w-\tilde{w}_0)\right) =$$

$$= \int\limits_0^t\int\limits_\Omega \left(\vec{A}^{(1)}\nabla u + \vec{A}^{(2)}\nabla v + \vec{A}^{(3)}\nabla w - \vec{A}^{(1)}\nabla\tilde{u}_0 - \vec{A}^{(2)}\nabla\tilde{v}_0 - \vec{A}^{(3)}\nabla\tilde{w}_0\right) \ge$$

$$\ge \int\limits_0^t\int\limits_\Omega \lambda\left(|\nabla u|^p + |\nabla v|^p + |\nabla w|^p\right) - \int\limits_0^t\int\limits_\Omega \lambda\left(|\nabla u|^{p-1} + |\nabla v|^{p-1} + |\nabla w|^{p-1}\right)\times$$

$$\times\left(|\nabla u_0| + |\nabla v_0| + |\nabla w_0|\right) - \int\limits_0^t\int\limits_\Omega |F_1| + |F_2| + |F_3| \ge$$

$$\ge \int\limits_0^t\int\limits_\Omega \frac{1}{2}\lambda\left(|\nabla u|^p + |\nabla v|^p + |\nabla w|^p\right) -$$

$$- C(p,\lambda)\int\limits_0^t\int\limits_\Omega \left(|\nabla u_0|^p + |\nabla v_0|^p + |\nabla w_0|^p\right) - C \ge$$

$$\ge \int\limits_0^t\int\limits_\Omega \frac{1}{2}\lambda\left(|\nabla(u-\tilde{u}_0)|^p + |\nabla(v-\tilde{v}_0)|^p + |\nabla(w-\tilde{w}_0)|^p\right) -$$

$$- \int\limits_0^t\int\limits_\Omega \tilde{C}(p,\lambda)\left(|\nabla u_0|^p + |\nabla v_0|^p + |\nabla w_0|^p\right) - C.$$

Here the use is also made of Young's inequality and the inequality

$$\left|a+b\right|^p \le C(p)\left(|a|^p + |b|^p\right) \quad \forall\, a,b\in\mathbb{R}. \qquad (3.2)$$

The first three terms on the right of (3.1) in virtue of Young's inequality, the Sobolev inequality and growth condition (2.13) can be estimated like that:

$$\int_0^t \int_\Omega |B^{(1)}| |u-\tilde{u}_0| + \int_0^t \int_\Omega |B^{(2)}| |v-\tilde{v}_0| + \int_0^t \int_\Omega |B^{(3)}| |w-\tilde{w}_0| \le$$

$$\le \int_0^t \int_\Omega (|\nabla u| + |\nabla v| + |\nabla w|)^\varepsilon (|u-\tilde{u}_0| + |v-\tilde{v}_0| + |w-\tilde{w}_0|) \le$$

$$\le \delta_1 C_1(\varepsilon, p) \int_0^t \int_\Omega (|\nabla(u-\tilde{u}_0)| + |\nabla(v-\tilde{v}_0)| + |\nabla(w-\tilde{w}_0)|)^p +$$

$$+ \delta_2 C_2(p) \int_0^t \int_\Omega (|u-\tilde{u}_0| + |v-\tilde{v}_0| + |w-\tilde{w}_0|)^p +$$

$$+ C\big(C_{1,2}, \delta_{1,2}, \tilde{u}_0, \tilde{v}_0, \tilde{w}_0, \operatorname{mes} Q\big) \le$$

$$\le \delta_3 \int_0^t \int_\Omega (|\nabla(u-\tilde{u}_0)| + |\nabla(v-\tilde{v}_0)| + |\nabla(w-\tilde{w}_0)|)^p + C_3.$$

Here it has been taken into account that $\dfrac{\varepsilon}{p} + \dfrac{1}{p} < \dfrac{1}{2} + \dfrac{1}{p} \le 1$. Collecting the above estimates, from (3.1) we get:

$$\int_{\Omega(t)} \frac{1}{2}\Big[(u-\tilde{u}_0)^2 + (v-\tilde{v}_0)^2 + (w-\tilde{w}_0)^2\Big] +$$

$$+ \int_0^t \int_\Omega \frac{1}{2}\lambda\Big(|\nabla(u-\tilde{u}_0)|^p + |\nabla(v-\tilde{v}_0)|^p + |\nabla(w-\tilde{w}_0)|^p\Big) \le$$

$$\le \delta_5 \int_0^t \int_\Omega \Big(|\nabla(u-\tilde{u}_0)|^p + |\nabla(v-\tilde{v}_0)|^p + |\nabla(w-\tilde{w}_0)|^p\Big) +$$

$$+ C_5\big(\operatorname{mes} Q, F, \delta_5, \tilde{u}_0, \tilde{v}_0, \tilde{w}_0\big).$$

Let us choose $\delta_5 = \dfrac{1}{4}\lambda$, which yields the inequality

$$\int_{\Omega(t)} \frac{1}{2}\Big[(u-\tilde{u}_0)^2 + (v-\tilde{v}_0)^2 + (w-\tilde{w}_0)^2\Big] +$$

$$+ \int_0^t \int_\Omega \frac{1}{4}\lambda\Big(|\nabla(u-\tilde{u}_0)|^p + |\nabla(v-\tilde{v}_0)|^p + |\nabla(w-\tilde{w}_0)|^p\Big) \le$$

$$\le C_4\big(\operatorname{mes} Q, F, \delta_5, \tilde{u}_0, \tilde{v}_0, \tilde{w}_0\big). \tag{3.3}$$

On this step we take the supremum in t in the left-hand side of (3.3) and obtain the estimate

$$\sup_{0<t<T} \int_\Omega |u-\tilde{u}_0|^2 + \sup_{0<t<T} \int_\Omega |v-\tilde{v}_0|^2 + \sup_{0<t<T} \int_\Omega |w-\tilde{w}_0|^2 +$$

$$+ \int_0^T \int_\Omega \Big(|\nabla(u-\tilde{u}_0)|^p + |\nabla(v-\tilde{v}_0)|^p + |\nabla(w-\tilde{w}_0)|^p\Big) \le C_5$$

with constant C_5 depending on n, p, ε, λ, F_j, p, n, Λ_1, Λ_2, ξ_0, κ_1, α_j, β_j, ε, $\operatorname{mes} Q$, and, on the strength of Remark 2.1, the boundary norms $\|\tilde{u}_0\|_{\tilde{W}(\partial Q)}$,

$\| \tilde{v}_0 \|_{\tilde{W}(\partial Q)}$, and $\| \tilde{w}_0 \|_{\tilde{W}(\partial Q)}$ of functions in the boundary conditions only. Hence the second statement of the theorem in self-evident.

4. Estimates of L^∞-norms. Let us now turn our attention to the question of boundedness of weak solutions to a system whose coefficients satisfy assumptions (2.5a) – (2.5c). Our main result is the following theorem.

Theorem 4.1. *Let* (u, v, w) *be a solution to system* (2.1). *If there exist such numbers* α_j, β_j,

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$$

satisfying assumptions (2.5a) – (2.5c), *then for the three linearly independent functions* H_1, H_2, *and* H_3 *defined by* (1.1) *the following estimates hold:*

$$\| H_1 \|_{L_\infty(Q)} \le C_1,$$

$$\| H_2 \|_{L_\infty(Q)} \le C_2,$$

$$\| H_3 \|_{L_\infty(Q)} \le C_3.$$

Hence it is easily seen that the same estimates take place for the components of the solution themselves:

$$\| u \|_{L_\infty(Q)} \le C_1, \quad \| v \|_{L_\infty(Q)} \le C_2, \quad \| w \|_{L_\infty(Q)} \le C_3,$$

where constants $C_{1,2,3}$ *depend only on the data*: n, p, ε, λ, F^j, p, n, Λ_1, Λ_2, ξ_0, κ_1, α_j, β_j, ε, $\operatorname{mes} Q$, $\left| g_{1,2,3} \right|_{\infty,(S)}$, $| u_0, v_0, w_0 |_{\infty,(\Omega)}$; *constants in the embedding theorems and is independent of* u, v, *and* w.

To prove the theorem we need the well-known Stampacchia's lemma:

Lemma 4.1. *Let* $\psi(y)$ *be a nonnegative nondecreasing function defined on* $[l_0, \infty)$ *which satisfies*:

$$\psi(m) \le \frac{C}{(m-l)^{\vartheta}} \{ \psi(l) \}^{\delta} \quad \textit{for} \quad m > l \ge l_0,$$

with $\vartheta > 0$ *and* $\delta > 1$. *Then*

$$\psi(l_0 + d) = 0,$$

where $d = C^{1/\vartheta} \{ \psi(l_0) \}^{(\delta-1)/\vartheta} 2^{\delta/(\delta-1)}$.

For proof see [11, p. 8] (Lemma 4.1). We make also use of the following lemma (see [10, p. 7] (Proposition 3.1)).

Lemma 4.2. *If* $u \in L^\infty\big(0, T; L^2(\Omega)\big) \cap L^p\big(0, T; W_0^{1,p}(\Omega)\big)$ *then there holds the inequality*

$$\int_0^T \int_\Omega u^q \le C \left(\int_0^T \int_\Omega | \nabla u |^p \right) \left(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega | u |^2 \right)^{p/n}$$

with $q = p\dfrac{n+2}{n}$ *and constant* C *depending only on* p *and* n.

Proof of Theorem **4.1.** Let α_1, β_1 be from hypotheses (2.5a). Multiply the first equation of (2.1) by α_1, add the second one multiplied by β_1, and the third one. Choose $H_1 \equiv \operatorname{sign}(\alpha_1 u + \beta_1 v + w)(|\alpha_1 u + \beta_1 v + w| - l)_+$ as a testing function with $l \geq l_0 = \max\left[\|\alpha_1 g_1 + \beta_1 g_2 + g_3\|_{L^\infty(S)}, \|\alpha_1 u_0 + \beta_1 v_0 + w_0\|_{L^\infty(\overline{\Omega}\times\{0\})}\right]$. After integrating in t from 0 to t, $t \leq T$, and in x over the domain Ω, this results in

$$\frac{1}{2}\int_{\Omega(t)} H_1^2 + \int_0^t \int_\Omega \left\langle \alpha_1 \vec{A}^{(1)} + \beta_1 \vec{A}^{(2)} + \vec{A}^{(3)}, \nabla H_1 \right\rangle =$$

$$= \int_0^t \int_\Omega \left(\alpha_1 B^{(1)} + \beta_1 B^{(2)} + B^{(3)}\right) H_1.$$

Making use hypotheses (2.5a) – (2.5c) we have

$$\int_0^t \int_\Omega \left\langle \alpha_1 \vec{A}^{(1)} + \beta_1 \vec{A}^{(2)} + \vec{A}^{(3)}, \nabla H_1 \right\rangle =$$

$$= \int_0^t \int_\Omega \left\langle \left[\alpha_1 \vec{A}^{(1)} + \beta_1 \vec{A}^{(2)} + \vec{A}^{(3)} - \lambda_1(\alpha_1 \nabla u + \beta_1 \nabla v + \nabla w)\right] + \right.$$

$$\left. + \lambda_1(\alpha_1 \nabla u + \beta_1 \nabla v + \nabla w), \nabla H_1 \right\rangle \geq \int_0^t \int_\Omega \langle \lambda_1(\alpha_1 \nabla u + \beta_1 \nabla v + \nabla w), \nabla H_1 \rangle -$$

$$- \int_0^t \int_\Omega \left|\alpha_1 \vec{A}^{(1)} + \beta_1 \vec{A}^{(2)} + \vec{A}^{(3)} - \lambda_1(\alpha_1 \nabla u + \beta_1 \nabla v + \nabla w)\right| |\nabla H_1| \geq$$

$$\geq \int_0^t \int_\Omega \lambda_1 \langle \alpha_1 \nabla u + \beta_1 \nabla v + \nabla w, \nabla H_1 \rangle - \int_0^t \int_\Omega (\xi_1 |\nabla H_1| + |F_1| |\nabla H_1|);$$

which hence yields the inequality

$$\frac{1}{2}\int_{\Omega(t)} H_1^2 + \int_0^t \int_\Omega \lambda_1 \langle \alpha_1 \nabla u + \beta_1 \nabla v + \nabla w, \nabla H_1 \rangle \leq$$

$$\leq \int_0^t \int_\Omega (|F_1| + \xi_1)|\nabla H_1| + C\int_0^t \int_\Omega B H_1,$$

where it is denoted $B = \alpha_1 B^1 + \beta_1 B^2 + B^3$; λ and ξ are functions from (2.5a) – (2.5c); $C = C(\alpha_1, \beta_1, p, n)$ is a constant. Since $t \in (0, T]$ is arbitrary, then taking the supremum we have

$$\sup_{0<t<T} \int_\Omega H_1^2 + C_1 \int_0^T \int_\Omega |\nabla H_1|^p \leq \int_0^T \int_\Omega (|F_1| + \xi_1)|\nabla H_1| + C\int_0^T \int_\Omega B H_1, \qquad (4.1)$$

where $C = C(\Lambda_1, \alpha_1, \beta_1, p, n)$ and the use has been made of the assumption upon function λ_1 (2.6). Applying generalized Holder's inequality consequently to the terms on the right yields

$$\int_0^T \int_\Omega |F_1| |\nabla H_1| \leq \|\nabla H_1\|_{p,Q} \|F_1\|_{\theta,Q} \left(\int_0^T \int_\Omega \chi_{A(l)}\right)^{1-1/p-1/\theta}, \qquad (4.2a)$$

$$\int_0^T \int_\Omega \xi_1 |\nabla H_1| \leq \|\nabla H_1\|_{p,Q} \|\xi_1\|_{p/\nu,Q} \left(\int_0^T \int_\Omega \chi_{A(l)} \right)^{1-1/p-\nu/p}, \tag{4.2b}$$

$$\int_0^T \int_\Omega B H_1 \leq \|H_1\|_{q,Q} \|B\|_{p/\varepsilon,Q} \left(\int_0^T \int_\Omega \chi_{A(l)} \right)^{1-1/q-\varepsilon/p}, \tag{4.2c}$$

where $\chi_{A(l)}$ is a characteristic function of the set $A(l) = \{x \in \Omega \,|\, |H_1| \geq l\}(t)$. From conditions (2.8), (2.7), (2.13) and Theorem 3.1 it follows that

$$\|F_1\|_{\theta,Q} \leq C_2, \quad \|\xi_1\|_{p/\nu,Q} \leq C_3, \quad \|B\|_{p/\varepsilon,Q} \leq C_4. \tag{4.3}$$

Collecting (4.2a) – (4.2c) and taking account of (4.3) from (4.1) we obtain the inequality

$$\sup_{0<t<T} \int_\Omega H_1^2 + \int_0^T \int_\Omega |\nabla H_1|^p \leq$$

$$\leq C_1 \|\nabla H_1\|_{p,Q} \{\psi(l)\}^{1-1/p-1/\theta} + C_2 \|\nabla H_1\|_{p,Q} \{\psi(l)\}^{1-1/p-\nu/p} +$$

$$+ C_3 \|H_1\|_{q,Q} \{\psi(l)\}^{1-1/q-\varepsilon/p}, \tag{4.4}$$

here we denoted:

$$\psi(l) = \int_0^T \operatorname{mes} A\{H_1 \geq l\}(l,t)\, dt.$$

From Lemma 4.2 it follows:

$$\|H_1\|_{q,Q} \leq \left(\sup_{0<t<T} \int_\Omega H_1^2 + \int_0^T \int_\Omega |\nabla H_1|^p \right)^{(p+n)/qn}. \tag{4.5}$$

From relation (4.4) and this inequality we get

$$\sup_{0<t<T} \int_\Omega H_1^2 + \int_0^T \int_\Omega |\nabla H_1|^p \leq$$

$$\leq C_1 \left(\sup_{0<t<T} \int_\Omega H_1^2 + \int_0^T \int_\Omega |\nabla H_1|^p \right)^{1/p} \left(\{\psi(l)\}^{1-1/p-1/\theta} + \{\psi(l)\}^{1-1/p-\nu/p} \right) +$$

$$+ C_2 \left(\sup_{0<t<T} \int_\Omega H_1^2 + \int_0^T \int_\Omega |\nabla H_1|^p \right)^{(n+p)/nq} \{\psi(l)\}^{1-1/q-\varepsilon/p}. \tag{4.6}$$

Applying Young's inequality to the right-hand side of (4.6) gives

$$\sup_{0<t<T} \int_\Omega H_1^2 + \int_0^T \int_\Omega |\nabla H_1|^p \leq C_1 \{\psi(l)\}^{(1-1/p-1/\theta)(p/(p-1))} +$$

$$+ C_2 \{\psi(l)\}^{(1-1/p-\nu/p)(p/(p-1))} + C_3 \{\psi(l)\}^{(1-1/q-\varepsilon/p)(nq/(n+p))^{\#}};$$

with $C_{1,2,3} = C_{1,2,3}(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0, F, \nu, \theta, \tau, \Lambda_1, \Lambda_2, p, n)$ and $\left(\frac{nq}{n+p}\right)^{\#}$ such that $\left(\left(\frac{nq}{n+p}\right)^{\#}\right)^{-1} + \frac{nq}{n+p} = 1$. Resorting again to (4.5) implies

$$\left(\| H_1 \|_{q,Q}\right)^{nq/(n+p)} \leq C_1\{\psi(l)\}^{(1-1/p-1/\theta)(p/(p-1))} +$$

$$+ C_2\{\psi(l)\}^{(1-1/p-\nu/p)(p/(p-1))} + C_3\{\psi(l)\}^{(1-1/q-\varepsilon/p)(nq/(n+p))^{\#}}. \tag{4.7}$$

Let us estimate:

$$(m-l)\{\psi(m)\}^{1/q} = (m-l)\left(\int_0^T \int_\Omega \chi_{A(m)}\right)^{1/q} < \left(\int_0^T \int_\Omega H_1^q \chi_{A(m)}\right)^{1/q} <$$

$$< \| H_1 \|_{q,Q},$$

where $m > l \geq l_0$. Substituting this into (4.7) we come down to

$$(m-l)^q \psi(m) \leq C_1\{\psi(l)\}^{(1-1/p-1/\theta)(p(n+p)/n(p-1))} +$$

$$+ C_2\{\psi(l)\}^{(1-1/p-\nu/p)(p(n+p)/n(p-1))} + C_3\{\psi(l)\}^{(1-1/q-\varepsilon/p)(nq/(n+p))^{\#}((n+p)/n)}$$

or, succinctly

$$\psi(m) \leq \frac{C_1}{(m-l)^q}\{\psi(l)\}^{\delta_1} + \frac{C_2}{(m-l)^q}\{\psi(l)\}^{\delta_2} + \frac{C_3}{(m-l)^q}\{\psi(l)\}^{\delta_3} \tag{4.8}$$

with

$$\delta_1 = \left(1 - \frac{1}{p} - \frac{1}{\theta}\right)\left(\frac{p(n+p)}{n(p-1)}\right), \quad \delta_2 = \left(1 - \frac{1}{p} - \frac{\nu}{p}\right)\left(\frac{p(n+p)}{n(p-1)}\right),$$

and

$$\delta_3 = \left(1 - \frac{n}{p(n+2)} - \frac{\varepsilon}{p}\right) \Big/ \left(\frac{n}{n+p} - \frac{n}{p(n+2)}\right).$$

From the assumption upon F_j, (2.8), it follows that

$$1 - \frac{1}{p} - \frac{1}{\theta} > \frac{n(p-1)}{p(n+p)}; \quad \text{and thus} \quad \delta_1 > 1;$$

from the hypotheses on ξ_j (2.7)

$$0 < \nu < \frac{p(p-1)}{n+p},$$

it follows that

$$1 - \frac{1}{p} - \frac{\nu}{p} > \frac{n(p-1)}{p(n+p)}; \quad \text{and thus} \quad \delta_2 > 1;$$

from the hypotheses on B^j (2.13)

$$0 < \varepsilon < \frac{p^2}{n+p},$$

hence

$$1 - \frac{n}{p(n+2)} - \frac{\varepsilon}{p} > \frac{n}{n+p} - \frac{n}{p(n+2)}; \quad \text{and thus} \quad \delta_3 > 1.$$

Without loss of generality we may assume that $\psi(l) < 1$. In fact, from the first statement of Theorem 3.1 and (4.5) it follows that

$$(l-l_0)\{\psi(l)\}^{1/q} = (l-l_0)\left(\int_0^T\int_\Omega \chi_{A(l)}\right)^{1/q} < \left(\int_0^T\int_\Omega (|H_1|-l_0)\chi_{A(l)}\right)^{1/q} <$$

$$< \|H_1 - l_0\|_{q,Q} \le \left(\sup_{0<t<T}\int_\Omega (H_1-l_0)^2 + \int_0^T\int_\Omega |\nabla(H_1-l_0)|^p\right)^{(p+n)/qn} \le \tilde{C},$$

where $l \ge l_0$; and hence

$$\psi(l) \le \frac{\tilde{C}^q}{(l-l_0)^q}; \text{ and it's easy to see that } \psi(l) < 1 \text{ whenever } l > \tilde{C} + l_0.$$

Since $\psi(l)$ is nonincreasing function, $\psi(l) < 1$ is true for all $l > \tilde{C} + l_0$. Due to this, (4.8) yields

$$\psi(m) \le \frac{C}{(m-l)^q}\{\psi(l)\}^\delta, \tag{4.9}$$

with $\delta = \min[\delta_1, \delta_2, \delta_3]$ and $C = \max[C_1, C_2, C_3]$. On the strength of Lemma 4.1 from relation (4.9) we can conclude that

$$\psi(l_0 + d) = 0$$

for some d sufficiently large, but finite, depending only on the data: n, p, ε, λ, F^j, p, n, Λ_1, Λ_2, ξ_0, κ_1, α_j, β_j, ε, $\operatorname{mes} Q \big| g_{1,2,3}\big|_{\infty,(S)}$, $|u_0, v_0, w_0|_{\infty,(\Omega)}$; constants in the embedding theorems and is independent of u, v, and w. Analogously is done for $H_2 = \alpha_2 u + \beta_2 v + w$ and $H_3 = \alpha_3 u + \beta_3 v + w$, where $\alpha_{2,3}$ and $\beta_{2,3}$ are from (2.5b) – (2.5c).

It is not difficult to see from the previous considerations that the same estimates hold for the components (u, v, w) of solution themselves. In fact,

$$\|u\|_\infty = \frac{\|u\Delta\|_\infty}{|\Delta|} =$$

$$= \frac{\|(\alpha_1 u + \beta_1 v + w)(\beta_2 - \beta_3) - (\alpha_2 u + \beta_2 v + w)(\beta_1 - \beta_3) + (\alpha_3 u + \beta_3 v + w)(\beta_1 - \beta_2)\|_\infty}{|\Delta|} =$$

$$= \frac{\|(\beta_2 - \beta_3)H_1 - (\beta_1 - \beta_3)H_2 + (\beta_1 - \beta_2)H_3\|_\infty}{|\Delta|} \le$$

$$\le \frac{|\beta_2 - \beta_3|C_1 + |\beta_1 - \beta_3|C_2 + |\beta_1 - \beta_2|C_3}{|\Delta|},$$

$$\|v\|_\infty = \frac{\|v\Delta\|_\infty}{|\Delta|} =$$

$$= \frac{\|(\alpha_1 u + \beta_1 v + w)(\alpha_2 - \alpha_3) - (\alpha_2 u + \beta_2 v + w)(\alpha_1 - \alpha_3) + (\alpha_3 u + \beta_3 v + w)(\alpha_1 - \alpha_2)\|_\infty}{|\Delta|} =$$

$$= \frac{\|(\alpha_2-\alpha_3)H_1-(\alpha_1-\alpha_3)H_2+(\alpha_1-\alpha_2)H_3\|_\infty}{|\Delta|} \le$$

$$\le \frac{|\alpha_2-\alpha_3|C_1+|\alpha_1-\alpha_3|C_2+|\alpha_1-\alpha_2|C_3}{|\Delta|},$$

$$\|w\|_\infty = \|(\alpha_1 u+\beta_1 v+w)-\alpha_1 u-\beta_1 v\|_\infty \le$$

$$\le \|H_1-\alpha_1 u-\beta_1 v\|_\infty \le \|H_1\|_\infty + |\alpha_1|\,\|u\|_\infty + |\beta_1|\,\|v\|_\infty,$$

where $\|\cdot\|_\infty$ stands for L_∞ norm. Hence follows the statement.

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