UDC 517.9

E. Ö. Canfes, A. Özdeğer (Istanbul Techn. Univ., Kadir Has Univ., Istanbul, Turkey)

A CHARACTERIZATION OF TOTALLY UMBILICAL HYPERSURFACES OF A SPACE FORM BY GEODESIC MAPPING

ХАРАКТЕРИСТИКА ТОТАЛЬНО ОМБІЛІЧНИХ ГІПЕРПОВЕРХОНЬ ПРОСТОРОВОЇ ФОРМИ ЗА ДОПОМОГОЮ ГЕОДЕЗИЧНИХ ВІДОБРАЖЕНЬ

The idea of considering the second fundamental form of a hypersurface as the first fundamental form of another hypersurface has found very useful applications in Riemannian and semi-Riemannian geometry, specially when trying to characterize extrinsic hyperspheres and ovaloids. Recently, T. Adachi and S. Maeda gave a characterization of totally umbilical hypersurfaces in a space form by circles. In this paper, we give a characterization of totally umbilical hypersurfaces of a space form by means of geodesic mapping.

Ідея використання другої фундаментальної форми гіперповерхні як першої фундаментальної форми іншої гіперповерхні знайшла дуже важливі застосування у рімановій та напіврімановій геометрії, зокрема при описі зовнішніх гіперсфер та овалоїдів. Нещодавно T. Adachi та S. Maeda навели характеристику тотально омбілічних гіперповерхонь у просторовій формі за допомогою кіл. У цій роботі ми наводимо характеристику тотально омбілічних гіперповерхонь просторової форми за допомогою геодезичних відображень.

1. Introduction. Let M_n and M'_n be two hypersurfaces of the space form $\bar{M}_{n+1}$ [3–5] and let g, g' and $\bar{g}$ be the respective positive definite metric tensors. Denote by ∇, ∇' and $\bar{\nabla}$ the corresponding connections induced by g, g' and $\bar{g}$.

In this paper, we choose the first fundamental form of M'_n as

$$g' = e^{2\sigma}\omega, \tag{1.1}$$

where ω is the second fundamental form of M_n which is supposed to be positive definite and σ is a differentiable function defined on M_n.

Let $\{x^i\}$, $\{x'^i\}$ and $\{y^\alpha\}$ be the respective coordinate systems in M_n, M'_n and $\bar{M}_{n+1}$ and let f be a one-to-one differentiable mapping of M_n upon M'_n defined by

$$x'^i = f^i(x^1, x^2, \ldots, x^n), \quad i = 1, 2, \ldots, n, \tag{1.2}$$

in which f^i are smooth functions defined on M_n and have a non-vanishing Jacobian. Then, it is clear that the corresponding points of M_n and M'_n are represented by the same set of coordinates and that the coordinate vectors correspond.

Let $\bar{R}$, R and R' be the covariant curvature tensors of $\bar{M}_{n+1}$, M_n and M'_n respectively and let $\bar{K}$ be the Riemannian curvature of $\bar{M}_{n+1}$.

We then have[1]

$$\bar{R}_{\beta\gamma\delta\epsilon} = \bar{K}(\bar{g}_{\beta\delta}\bar{g}_{\gamma\epsilon} - \bar{g}_{\beta\epsilon}\bar{g}_{\gamma\delta}). \tag{1.3}$$

On the other hand, under the condition (1.3) the Codazzi equations

[1] In the sequel, Latin indices $i, j, k, \ldots$ run from 1 to n, while the Greek indices α, β, γ will run from 1 to $n+1$.

$$\nabla_k \omega_{ij} - \nabla_j \omega_{ik} + \bar{R}_{\beta\gamma\delta\epsilon} N^\beta \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\delta}{\partial x^j} \frac{\partial y^\epsilon}{\partial x^k} = 0$$

and the Gauss equation

$$R_{ijkl} = \bar{R}_{\beta\gamma\delta\epsilon} \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} \frac{\partial y^\delta}{\partial x^k} \frac{\partial y^\epsilon}{\partial x^l} + (\omega_{ik}\omega_{jl} - \omega_{il}\omega_{jk})$$

transform, respectively, into

$$\nabla_k \omega_{ij} - \nabla_j \omega_{ik} = 0 \tag{1.4}$$

and

$$R_{ijkl} = \bar{K}(g_{ik}g_{jl} - g_{il}g_{jk}) + (\omega_{ik}\omega_{jl} - \omega_{il}\omega_{jk}) \tag{1.5}$$

in which N^β are the components of the unit normal vector field of M_n [4].

2. Relation between the connections ∇ and ∇'. It is well-known that the connection coefficients of a Riemannian space whose metric tensor is g are given by [5]

$$\Gamma_{ij}^l = \frac{1}{2} g^{lh} (\partial_i g_{jh} + \partial_j g_{ih} - \partial_h g_{ij}), \qquad \partial_k = \frac{\partial}{\partial x^k}. \tag{2.1}$$

Replacing g in (2.1) by the metric tensor g' of M'_n given by (1.1) and doing the necessary calculations we first find the connection coefficients Γ'^l_{ij} of M'_n as

$$\Gamma'^l_{ij} = \frac{1}{2} e^{2\sigma} g'^{lk} (\partial_j \omega_{ik} + \partial_i \omega_{jk} - \partial_k \omega_{ij}) + (\partial_j \sigma)\delta_i^l + (\partial_i \sigma)\delta_j^l - (\partial_k \sigma) g'^{lk} g'_{ij}. \tag{2.2}$$

On the other hand, for the covariant derivative of the second fundamental tensor ω of M_n we have [3, 4]

$$\nabla_i \omega_{jk} = \partial_i \omega_{jk} - \Gamma_{ij}^h \omega_{hk} - \Gamma_{ik}^h \omega_{jh}. \tag{2.3}$$

Changing the indices i, j and k cyclically we obtain two more equations:

$$\nabla_j \omega_{ki} = \partial_j \omega_{ki} - \Gamma_{ij}^h \omega_{hk} - \Gamma_{kj}^h \omega_{ih}, \tag{2.4}$$

$$\nabla_k \omega_{ij} = \partial_k \omega_{ij} - \Gamma_{ki}^h \omega_{hj} - \Gamma_{kj}^h \omega_{ih}. \tag{2.5}$$

Subtracting (2.5) from the sum of (2.3) and (2.4) and using the Codazzi equations (1.4), we obtain

$$\nabla_i \omega_{jk} = \partial_i \omega_{jk} + \partial_j \omega_{ik} - \partial_k \omega_{ij} - 2\omega_{hk} \Gamma_{ij}^h. \tag{2.6}$$

In view of (2.6), (2.2) becomes

$$\Gamma'^l_{ij} = \Gamma_{ij}^l + \delta_i^l \partial_j \sigma + \delta_j^l \partial_i \sigma - g'^{lk} g'_{ij} \partial_k \sigma + \frac{1}{2} e^{2\sigma} g'^{lk} \nabla_i \omega_{jk}. \tag{2.7}$$

(2.7) is the desired relation connecting the connection coefficients of M_n and M'_n.

3. Geodesic mappings of M_n upon M'_n. If the map f defined by (1.2) transforms every geodesic in M_n into a geodesic in M'_n, f is called a geodesic mapping of M_n into M'_n.

M_n and M'_n will be in geodesic correspondence if and only if the respective connection coefficients Γ_{ij}^h and Γ'^h_{ij} of M_n and M'_n are related by [3]

$$\Gamma'^i_{jk} = \Gamma_{jk}^i + \delta_j^i \psi_k + \delta_k^i \psi_j, \tag{3.1}$$

where ψ_k are the components of some 1-form which is known to be a gradient.

We first prove the following lemma which will be needed in our subsequent work.

Lemma 3.1. *Let M_n and M'_n be hypersurfaces of the space form $\bar{M}_{n+1}$ and let the metric tensor of M'_n be defined by* (1.1). *If M_n and M'_n are in geodesic correspondence, then the* 1*-form ψ_k is the gradient of* 2σ.

Proof. Since ∇' is a metric connection we have

$$0 = \nabla'_k g'_{ij} = \partial_k g'_{ij} - g'_{lj}\Gamma'^{l}_{ik} - g'_{li}\Gamma'^{l}_{jk}$$

so that with the help of (1.1) and (3.1) we obtain

$$0 = 2\omega_{ij}\partial_k\sigma + \nabla_k\omega_{ij} - 2\psi_k\omega_{ij} - \psi_i\omega_{kj} - \psi_j\omega_{ki}. \tag{3.2}$$

Interchanging the indices j and k in (3.2) we find

$$0 = 2\omega_{ik}\partial_j\sigma + \nabla_j\omega_{ik} - 2\psi_j\omega_{ik} - \psi_i\omega_{kj} - \psi_k\omega_{ji}. \tag{3.3}$$

Subtracting (3.3) from (3.2) and putting

$$\phi_k = \psi_k - 2\partial_k\sigma \tag{3.4}$$

in (3.3) we conclude that

$$\omega_{ij}\phi_k - \omega_{ik}\phi_j = 0 \tag{3.5}$$

in which the Codazzi equations (1.4) have been used.

We note that, since ψ_k is a gradient, it follows from (3.4) that ϕ_k is also a gradient. Multiplying (3.5) by $e^{2\sigma}$ and using (1.1) we obtain

$$\phi_k g'_{ij} - \phi_j g'_{ik} = 0 \tag{3.6}$$

or, multiplying (3.6) by g'^{ij} and summing with respect to i and j we find for $n > 1$ that

$$\phi_k = 0. \tag{3.7}$$

Combination of (3.4) and (3.7) yields $\psi_k = 2\partial_k\sigma$.

We next prove the following theorem.

Theorem 3.1. *The hypersurface M_n of a space form $\bar{M}_{n+1}$ will be totally umbilical if and only if M_n can be geodesically mapped upon M'_n.*

Proof. *Sufficiency.* Let γ be a geodesic through the point $p \in M_n$ which is defined by $x^i = x^i(s)$, s being the arc length of γ. Then, the normal curvature, say κ_n, of M_n in the direction of γ, i.e., in the direction of $\dfrac{dx^i}{ds}$, is [4]

$$\kappa_n = \omega_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}. \tag{3.8}$$

Multiplying (3.2) by $\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}$ and summing with respect to $i,\ j,\ k$ and using (3.8) we obtain

$$2\kappa_n(\partial_k\sigma)\frac{dx^k}{ds} + (\nabla_k\omega_{ij})\frac{dx^k}{ds}\frac{dx^i}{ds}\frac{dx^j}{ds} - 2\left(\psi_k\frac{dx^k}{ds}\right)\kappa_n - \left(\psi_i\frac{dx^i}{ds}\right)\kappa_n - \left(\psi_j\frac{dx^j}{ds}\right)\kappa_n = 0. \tag{3.9}$$

Since ψ_k is a gradient, there exists a differentiable function ψ such that $\psi_k = \partial_k\psi$. On the other hand, differentiating (3.8) covariantly in the direction of γ and using the Frenet's formula [3]

$$\left(\nabla_k \frac{dx^i}{ds}\right)\frac{dx^k}{ds} = \kappa_g \underset{1}{\eta}^i,$$

where κ_g is the geodesic curvature and $\underset{1}{\eta}$ is the unit principal normal vector field of γ relative to M_n, we find that

$$(\nabla_k\omega_{ij})\frac{dx^k}{ds}\frac{dx^i}{ds}\frac{dx^j}{ds} = \frac{d\kappa_n}{ds} - 2\kappa_g\omega_{ij}\,\underset{1}{\eta}^i\,\frac{dx^j}{ds}. \tag{3.10}$$

Using (3.10) in (3.9) and remembering that γ is a geodesic $(\kappa_g = 0)$ in M_n, we get

$$\left[\frac{\partial\kappa_n}{\partial x^i} + \left(2\frac{\partial\sigma}{\partial x^i} - 4\frac{\partial\psi}{\partial x^i}\right)\kappa_n\right]\frac{dx^i}{ds} = 0,$$

or

$$\left[\frac{\partial}{\partial x^i}\left(\ln|\kappa_n| + 2\sigma - 4\psi\right)\right]\frac{dx^i}{ds} = 0 \tag{3.11}$$

along γ.

On the other hand, by (1.1) and (3.11), we find

$$ds'^2 = g'_{ij}dx^i dx^j = e^{2\sigma}\omega_{ij}dx^i dx^j = e^{2\sigma}\omega_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}ds^2 = e^{2\sigma}\kappa_n ds^2,$$

from which it follows that $\kappa_n > 0$. From (3.1) it follows that,

$$\ln\kappa_n + 2\sigma - 4\psi = \text{const} = \mathrm{C}_1 \tag{3.12}$$

along γ.

By Lemma 3.1, $\psi = 2\sigma + C_2$, $C_2 = \text{Const}$ and therefore (3.12) gives

$$\kappa_n = ce^{6\sigma}, \tag{3.13}$$

where c is an arbitrary positive constant.

From (3.13) it follows that the lines of curvature of M_n are indeterminate at all points of M_n. Consequently, M_n is totally umbilical.

Necessity. Assume that M_n is a totally umbilical hypersurface of $\bar{M}_{n+1}$ which means that $\omega_{ij} = \frac{H}{n}g_{ij}$ where H is the mean curvature of M_n. In this case, (1.1) becomes

$$g'_{ij} = \rho^2 g_{ij} \qquad \left(\rho^2 = e^{2\sigma}\frac{H}{n}\right), \tag{3.14}$$

so that M_n and M'_n are conformal.

From (1.5) it follows that

$$R_{ijkl} = \left(\bar{K} + \frac{H^2}{n^2}\right)(g_{ik}g_{jl} - g_{il}g_{jk})$$

showing that M_n has the constant curvature $\bar{K} + \frac{H^2}{n^2}$. So H is constant.

We will show that M_n can also be geodesically mapped upon M'_n. Since M_n is conformal to M'_n, their connection coefficients are related by [6]

$$\Gamma'^h_{ij} = \Gamma^h_{ij} + \delta^h_j \rho_i + \delta^h_i \rho_j - g_{ij}\rho^h \qquad \Big(\rho_i = \nabla_i \rho, \;\; \rho^h = g^{th}\rho_t\Big). \tag{3.15}$$

To show that this conformal mapping between M_n and M'_n is also a geodesic mapping, according to (3.15) and (3.1) we have to find a 1-form ψ_k such that

$$\Gamma^h_{ij} + \delta^h_j \psi_i + \delta^h_i \psi_j = \Gamma^h_{ij} + \delta^h_j \rho_i + \delta^h_i \rho_j - g_{ij}\rho^h$$

or

$$\delta^h_j(\psi_i - \rho_i) + \delta^h_i(\psi_j - \rho_j) + g_{ij}\rho^h = 0. \tag{3.16}$$

Transvecting (3.16) by g^{ij} we get

$$g^{ih}(\psi_i - \rho_i) + g^{jh}(\psi_j - \rho_j) + n\rho^h = 0$$

or

$$2g^{ih}(\psi_i - \rho_i) + n\rho^h = 0. \tag{3.17}$$

Multiplying (3.17) by g_{hj} and summing for h we obtain

$$2\psi_j + (n-2)\rho_j = 0.$$

Then, by (3.14) we find that

$$\psi_j = \left(\frac{2-n}{2\sqrt{n}}\sqrt{H}\right)\partial_j e^{\sigma}, \quad H > 0.$$

With this choice of ψ_j the conformal mapping mentioned above becomes also a geodesic mapping.

Theorem 1.1 is proved.

In the special case where $\sigma = 0$ throughout M_n, i.e., when $g' = \omega$, we may mention below some properties of M_n which is in geodesic correspondence with M'_n.

1. From Lemma 3.1 and the relation (3.1) we conclude that any geodesic mapping of M_n upon M'_n is connection preserving.

2. By (3.13) it follows that M_n has constant normal curvature along each geodesic through a point $p \in M_n$.

3. The underlying geodesic mapping is a homothety.

1. *Verpoort S.* The geometry of the second fundamental form: curvature properties and variational aspects: Ph. D Thesis. – Katholieke Univ. Leuven, 2008.
2. *Adachi T., Maeda S.* Characterization of totally umbilic hypersurfaces in a space form by circles // Czechoslovak Math. J. – 2005. – **55**, № 1. – P. 203 – 207.
3. *Gerretsen J.* Lectures on tensor calculus and differential geometry. – Groningen: P. Noordhoff, 1962.
4. *Weatherburn C. E.* An Introduction to Riemannian geometry and the tensor calculus. – Cambridge Univ. Press, 1966.
5. *Yano K., Kon M.* Structures on manifolds. – Worldsci., 1984.
6. *Chen B. Y.* Geometry of submanifolds. – New York: Marcel Dekker Inc., 1973.

Received 02.02.11