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GENERALIZATIONS OF $\oplus$-SUPPLEMENTED MODULES

УЗАГАЛЬНЕННЯ $\oplus$-ДОПОВНЮВАНИХ МОДУЛІВ

We introduce $\oplus$*-radical supplemented* modules and *strongly* $\oplus$*-radical supplemented* modules (briefly, $srs^{\oplus}$-modules) as proper generalizations of $\oplus$-supplemented modules. We prove that (1) a semilocal ring R is left perfect if and only if every left R-module is an $\oplus$-radical supplemented module; (2) a commutative ring R is an Artinian principal ideal ring if and only if every left R-module is a $srs^{\oplus}$-module; (3) over a local Dedekind domain, every $\oplus$-radical supplemented module is a $srs^{\oplus}$-module. Moreover, we completely determine the structure of these modules over local Dedekind domains.

Введено поняття $\oplus$*-радикальних доповнюваних* модулів та *сильно* $\oplus$*-радикальних доповнюваних* модулів (скорочено $srs^{\oplus}$-модулів) як відповідних узагальнень $\oplus$-доповнюваних модулів. Доведено, що: (1) напівлокальне кільце R є досконалим зліва тоді і тільки тоді, коли кожен лівий R-модуль є $\oplus$-радикальним доповнюваним модулем; (2) комутативне кільце R є артіновим кільцем головних ідеалів тоді і тільки тоді, коли кожен лівий R-модуль є $srs^{\oplus}$-модулем; (3) над локальною дедекіндовою областю кожен $\oplus$-радикальний доповнюваний модуль є $srs^{\oplus}$-модулем. Повністю визначено структуру цих модулів над локальними дедекіндовими областями.

1. Introduction. Throughout the whole text, all rings are to be associative, unit and all modules are left unitary. Let M be such a module. We shall write $N \leq M$ ($N \ll M$) if N is a submodule of M (small in M). By $\mathrm{Rad}(M)$ we denote the radical of M. Let $U, V \leq M$. V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$. V is a supplement of U in M if and only if $M = U + V$ and $U \cap V \ll V$ (see [12]). A module M is called *supplemented* (*weakly supplemented* in [10]) if every submodule of M has a supplement in M, and it is called $\oplus$*-supplemented* if every submodule of M has a supplement that is a direct summand of M. Clearly $\oplus$-supplemented modules are supplemented.

In [13], Zöschinger introduced a notion of modules whose radical has supplements called *radical supplemented*. The author determined in the same paper and in [15] the structure of radical supplemented modules. Motivated by this, Büyükaşık and Türkmen call a module M *strongly radical suplemented* (or briefly a *srs-module*) if every submodule containing radical has a supplement [2]. So it is natural to introduce another notion that we called $\oplus$-radical supplemented. A module M is called $\oplus$*-radical supplemented* if $\mathrm{Rad}(M)$ has a supplement that is a direct summand of M. We call also a module M *strongly* $\oplus$*-radical supplemented* (or briefly $srs^{\oplus}$*-module*) provided every submodule containing radical has a supplement that is a direct summand of M.

In this paper, we obtain various properties of $\oplus$-radical supplemented and $srs^{\oplus}$-modules as a proper generalization of $\oplus$-supplemented modules. We show that the class of $srs^{\oplus}$-modules and $\oplus$-radical supplemented modules are closed under finite direct sums. A semilocal ring R is left perfect if and only if every left R-module is $\oplus$-radical supplemented, and a commutative ring R is an Artinian principal ideal ring if and only if every left R-module is a $srs^{\oplus}$-module. We prove also that a non-zero projective module M with cofinite radical is $\oplus$-supplemented if and only if it is a $srs^{\oplus}$-module if and only if it is $\oplus$-cofinitely supplemented. Over a local Dedekind domain every $\oplus$-radical supplemented module is a $srs^{\oplus}$-module, and over a local Dedekind domain the structure of these modules is completely determined.

2. Modules over any rings. Recall that a module M is called *radical* if M has no maximal submodules, that is, $\mathrm{Rad}(M) = M$. For a module M, $P(M)$ will indicate the sum of all radical submodules of M. if $P(M) = 0$, M is called *reduced*. Note that $P(M)$ is the largest radical submodule of M.

Now we have the following simple fact, which plays a key role in our working.

Lemma 2.1. *$P(M)$ is a $srs^{\oplus}$-module for every R-module M.*

Proof. Let M be any R-module. We know that $\mathrm{Rad}(P(M)) = P(M)$. So $P(M)$ has trivial supplement 0 in $P(M)$. Consequently, $P(M)$ is a $srs^{\oplus}$-module.

We begin by giving some examples of module to seperate $\oplus$-supplemented, $srs^{\oplus}$-module, $\oplus$-radical supplemented and radical supplemented.

***Example* 2.1.** Let R be a non-local Dedekind domain with quotient field K. Consider the R-module $M = K^{(\mathbb{N})}$. Since $P(M) = M$, M is a $srs^{\oplus}$-module by Lemma 2.1. If $K^{(\mathbb{N})}$ is $\oplus$-supplemented, K is supplemented as a factor module of M and so, by [14], R is a local ring. This contradicts the assumption. Hence M is not $\oplus$-supplemented.

Note that every $\oplus$-supplemented with zero radical is semisimple.

***Example* 2.2.** (1) Consider the non-Noetherian ring R which is the direct product $\prod_{i\geq 1}^{\infty} F_i$, where $F_i = F$ is any field. Clearly $\mathrm{Rad}(R) = 0$ and so the left R-module R is $\oplus$-radical supplemented. On the other hand, the left R-module R is not a $srs^{\oplus}$-module since it is not semisimple.

(2) Let $M =_{\mathbb{Z}} \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. It is well known that M is not semisimple and $\mathrm{Rad}(M) = 0$. Hence M is $\oplus$-radical supplemented, but it is not a $srs^{\oplus}$-module.

***Example* 2.3.** Let $R = \mathbb{Z}$ and I be a collection of distinct maximal ideal of $\mathbb{Z}$. Consider the left $\mathbb{Z}$-module $M = \prod_{p\in I} \left(\dfrac{\mathbb{Z}}{p^2}\right)$. Then M is radical supplemented. However, it is not $\oplus$-radical supplemented (see [13]).

Now we shall show that in general srs-modules need not be a $srs^{\oplus}$-module. To see this, we need to the following lemma.

Lemma 2.2. *Let M be a module. Suppose that* $\mathrm{Rad}(M)$ *is small in M. Then M is a $srs^{\oplus}$-module if and only if it is $\oplus$-supplemented.*

Proof. ($\Longrightarrow$) Let N be any submodule of M. Then $\mathrm{Rad}(M) \subseteq \mathrm{Rad}(M) + N \subseteq M$. Since M is a $srs^{\oplus}$-module, we have $M = \mathrm{Rad}(M) + N + L$, $(\mathrm{Rad}(M) + N) \cap L \ll L$ and $M = L \oplus L'$ for two submodules $L, L' \leq M$. Since $\mathrm{Rad}(M) \ll M$, we get $M = N + L$ and $N \cap L \ll L$. So L is a supplement of N in M such that L is a direct summand of M. Therefore M is a $\oplus$-supplemented module.

($\Longleftarrow$) Clear.

***Example* 2.4** (see [9], Corollary 2.4)**.** Let F be any field and $R = F[[X, Y]]$, the ring of formal power series over F indeterminates X, Y. Then R is a local commutative Noetherian domain. Now suppose that $M =_R \mathrm{Rad}(R)$. So $M = RX + RY$. Since R is local, by [12] (42.6), M is supplemented and so it is a srs-module. It follows from [9] (Corollary 2.4) that M is not $\oplus$-supplemented. Therefore, by Lemma 2.2, M is not a $srs^{\oplus}$-module.

Recall from [3] that a ring R is a *left Bass ring* if every non-zero left R-module has a maximal submodule. It is known that the ring R is left Bass if and only if $\mathrm{Rad}(M)$ is small in M for every non-zero left R-module M. By using Lemma 2.2, we obtain the following important corollary.

***Corollary* 2.1.** *Every $srs^{\oplus}$-module over a left Bass ring is ⊕-supplemented.*

A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. Note that coatomic modules have a small radical and so every coatomic module is ⊕-radical supplemented.

***Corollary* 2.2.** *Let M be a coatomic module. Then M is a $srs^{\oplus}$-module if and only if it is ⊕-supplemented.*

Proof. It follows from Lemma 2.2.

Now we shall prove that the class of $srs^{\oplus}$-modules and ⊕-radical supplemented modules are closed under finite direct sums.

Theorem 2.1. *Let M_i, $i = 1, 2, \ldots, n$, be any finitely collection of modules and $M = M_1 \oplus \oplus M_2 \oplus \ldots \oplus M_n$. Then*:

(1) *M is ⊕-radical supplemented if M_i is ⊕-radical supplemented for each $1 \leq i \leq n$;*

(2) *M is a $srs^{\oplus}$-module if M_i is a $srs^{\oplus}$-module for each $1 \leq i \leq n$.*

Proof. (1) The proof can be made similar to (2).

(2) Let M_i be a $srs^{\oplus}$-module for each $1 \leq i \leq n$. To prove that M is a $srs^{\oplus}$-module, it is sufficient by induction on n to prove this is the case when $n = 2$. Hence suppose $n = 2$. Let U be any submodule of M with $\mathrm{Rad}(M) \subseteq U$. Then $M = M_1 + M_2 + U$ so that $M_1 + M_2 + U$ has a supplement 0 in M. Since $M = M_1 \oplus M_2$, then $\mathrm{Rad}(M_2) \subseteq U + M_1$. It follows that $\mathrm{Rad}(M_2) \subseteq M_2 \cap (U + M_1)$ has a supplement H in M_2 such that H is a direct summand of M_2. By [5] (Lemma 1.3), H is a supplement of $M_1 + U$ in M. Moreover $\mathrm{Rad}(M_1) \subseteq U + H$. Since M_1 is a $srs^{\oplus}$-module, $M_1 \cap (U + H)$ has a supplement K in M_1 such that K is a direct summand of M_1. Again applying [5] (Lemma 1.3), we have that $H + K$ is a supplement of U in M. It is clear that $H + K$ is a direct summand of M. Therefore M is a $srs^{\oplus}$-module.

Now we shall give another example of a non-radical module which is a $srs^{\oplus}$-module but not ⊕-supplemented.

***Example* 2.5.** Consider the left $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}_p$, where p is a prime integer. Note that M has a unique maximal submodule, which means that $\mathrm{Rad}(M) \neq M$. According to Lemma 2.1, the left $\mathbb{Z}$-module $\mathbb{Q}$ is a $srs^{\oplus}$-module. By Theorem 2.1 (2), M is a $srs^{\oplus}$-module as a direct sum of two $srs^{\oplus}$-modules. On the other hand, M is not ⊕-supplemented because it is not torsion.

Proposition 2.1. *Let M be a non-radical module. If M is a ⊕-radical supplemented, then M contains a radical direct summand. In particular, if $P(M) = 0$, then $\mathrm{Rad}(M) \ll M$.*

Proof. Suppose that $\mathrm{Rad}(M) \neq M$. By the hypothesis, there exist submodules V, V' of M such that $M = \mathrm{Rad}(M) + V$, $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M) \ll V$ and $M = V \oplus V'$. It follows from [12] (21.6 (5)) that $\mathrm{Rad}(M) = \mathrm{Rad}(V) \oplus \mathrm{Rad}(V')$. So $M = \mathrm{Rad}(M) + V = \mathrm{Rad}(V') \oplus V$. Therefore by modularity, $V' = \mathrm{Rad}(V') \oplus (V \cap V') = \mathrm{Rad}(V')$, that is, V' is radical.

Suppose that $P(M) = 0$. Then $V' = 0$, which shows that $V = M$. Hence $\mathrm{Rad}(M) \ll M$.

Recall that a subset X of a ring R is called *right t-nilpotent* if, for every sequence $x_1, x_2, \ldots$ of elements in X, there exists a $k \in \mathbb{N}$ with $x_1 x_2 \ldots x_k = 0$. A ring R is called *left perfect* if R is semilocal and $\mathrm{Rad}(R)$ is right t-nilpotent [12] (43.9).

Theorem 2.2. *Let R be any ring. Then $\mathrm{Rad}(R)$ is right t-nilpotent if and only if every projective left R-module is ⊕-radical supplemented.*

Proof. ($\Longrightarrow$) Let M be any projective left R-module. By [8] (9.2.1), $\mathrm{Rad}(M) = \mathrm{Rad}(R)M$ and so, by [12] (43.5), $\mathrm{Rad}(M) \ll M$ as required.

($\Longleftarrow$) Let $M = R^{(\mathbb{N})}$. Again applying [8] (9.2.1), we have $\mathrm{Rad}(M) = \mathrm{Rad}(R)M$. Since M is $\oplus$-radical supplemented, there exist submodules V, V' of M such that $M = \mathrm{Rad}(M)+V$, $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M) \ll V$ and $V \oplus V' = M$. So V' is radical. It follows from [12] (22.3 (2)) that $V' = 0$, which means that $V = M$. Hence $\mathrm{Rad}(M)$ is small in M and, by [12] (43.5), $\mathrm{Rad}(R)$ is right t-nilpotent.

***Corollary* 2.3.** *A semilocal ring R is left perfect if and only if every left R-module is $\oplus$-radical supplemented.*

Proof. It follows from Theorem 2.2 and [12] (49.9).

Note that the condition "semilocal" in the above corollary is necessary. We see, for example, the left Bass rings which are not left perfect.

Proposition 2.2. *A non-zero projective $srs^{\oplus}$-module is $\oplus$-supplemented.*

Proof. Let M be any non-zero projective $srs^{\oplus}$-module. Therefore, it is $\oplus$-radical supplemented. Then there exist submodules V, V' of M such that $M = \mathrm{Rad}(M) + V$, $\mathrm{Rad}(V) \ll V$ and $M = V \oplus V'$. So V' is radical. By [12] (22.3(2)), $V' = 0$. It follows that $\mathrm{Rad}(M) \ll M$. Hence M is $\oplus$-supplemented by Lemma 2.2.

It is well known that a ring R is semiperfect if and only if every finitely generated free R-module is $\oplus$-supplemented. By Lemma 2.2, we know that every finitely generated $srs^{\oplus}$-module is $\oplus$-supplemented. Using these facts we obtain the following corollary.

***Corollary* 2.4.** *For any ring R with identity element, R is semiperfect if and only if every finitely generated free R-module is a $srs^{\oplus}$-module.*

Proof. Let $F = R^{(I)}$ be any free R-module for some finite set I. Since R is semiperfect, by [9] (Theorem 2.1), the left R-module R is $\oplus$-supplemented and so the module is a $srs^{\oplus}$-module. Hence F is a $srs^{\oplus}$-module by Theorem 2.1 (2). Conversely, suppose that every finitely generated free R-module is a $srs^{\oplus}$-module. Then the left R-module R is a $srs^{\oplus}$-module. By Lemma 2.2, R is semiperfect.

Let R be any ring. R is called *FGC ring* if every finitely generated R-module decomposes into a direct sum of cyclic submodules. If R is a local FGC ring, then R is an almost maximal valuation ring [1] (Theorem 4.4). It is proved [6] (Proposition 1.3) that a commutative local ring R is an almost maximal valuation ring if and only if every finitely generated R-module is $\oplus$-supplemented. Now we have the following corollary.

***Corollary* 2.5.** *For a commutative ring R, R is a finitely product of almost maximal valuation rings if and only if every finitely generated R-module is a $srs^{\oplus}$-module.*

Lemma 2.3. *Let M be an indecomposable module. If M is a $srs^{\oplus}$-module, then M is radical or M is local.*

Proof. Suppose that $\mathrm{Rad}(M) \neq M$. Then M contains a maximal submodule K. By the hypothesis, there exists a direct summand V of M such that $M = K + V$ and $K \cap V \ll V$. It follows from [12] (41.1(3)) that V is local. Since M is an indecomposable module and K is a maximal submodule of M, we get $V = M$. Thus M is local.

Theorem 2.3. *Let R be a local commutative ring and M be a uniform R-module. Then every submodule of M is a $srs^{\oplus}$-module if and only if M is uniserial.*

Proof. ($\Longrightarrow$) By [11] (Lemma 6.2), it sufficies to show that every finitely generated submodule of M is local. Let N be any finitely generated submodule of M. By assumption, N is indecomposable. So, by Lemma 2.3, N is local.

($\Longleftarrow$) Since M is uniserial, every submodule of M is hollow by [3] (2.17). Therefore every submodule of M is a $srs^{\oplus}$-module.

***Corollary* 2.6.** *Let R be a local commutative ring. Suppose that every submodule of $E\left(\dfrac{R}{\mathrm{Rad}(R)}\right)$ is a $srs^{\oplus}$-module, where $E\left(\dfrac{R}{\mathrm{Rad}(R)}\right)$ is the injective hull of the simple module $\dfrac{R}{\mathrm{Rad}(R)}$. Then R is a uniserial ring.*

Proof. Since $E\left(\dfrac{R}{\mathrm{Rad}(R)}\right)$ is uniform, the hypothesis implies that $E\left(\dfrac{R}{\mathrm{Rad}(R)}\right)$ is uniserial by Theorem 2.3. It follows from [11] (Lemma 6.2) that R is a uniserial ring.

It is shown [6] (Theorem 1.1) that a commutative ring R is an artinian principal ring if and only if every left R-module is $\oplus$-supplemented. Now we generalize this fact.

Theorem 2.4. *A commutative ring R is an artinian principal ideal ring if and only if every left R-module is a $srs^{\oplus}$-module.*

Proof. Suppose that every left R-module is a $srs^{\oplus}$-module. Then, by Lemma 2.2, the left R-module R is $\oplus$-supplemented and so R is semiperfect. By [12] (42.6), R is semilocal. It follows from Corollary 2.3 that R is left perfect. Since R is semiperfect, we can write, [12] (42.6), $R = \mathrm{Re}_1 \oplus \mathrm{Re}_2 \oplus \ldots \oplus \mathrm{Re}_n$ such that e_i is local orthogonal idempotent for $1 \leq i \leq n$ with $n \in \mathbb{N}$. For all $1 \leq i \leq n$, Re_i is commutative and it is not difficult to see that every Re_i-module is a $srs^{\oplus}$-module by assumption. Now Corollary 2.6 implies that Re_i is an uniserial ring for every $1 \leq i \leq n$. By [11] (Lemma 6.3), Re_i is a principal ideal ring, which shows that R is an artinian principal ideal ring.

Proposition 2.3. *Let R be a ring and M be a $\oplus$-radical supplemented R-module with $\mathrm{Rad}(M) \neq M$. If its ring of endomorphism is quasi local, then M is local.*

Proof. By the hypothesis, there exist submodules U, U' of M such that $M = \mathrm{Rad}(M) + U$, $\mathrm{Rad}(M) \cap U \ll U$ and $M = U \oplus U'$. By [11] (Proposition 3.11), M is an indecomposable module. So $U' = 0$, that is, $U = M$. Thus $\mathrm{Rad}(M) \ll M$. By Lemma 2.2, M is $\oplus$-supplemented. Let N be any proper submodule of M. It follows that $M = N + T$, $N \cap T \ll T$ and $M = T \oplus T'$ for some submodules $T, T' \subseteq M$. Since M is an indecomposable module, $M = T$. Then $N \ll M$. Therefore M is hollow. By [12] (41.4), M is local.

***Example* 2.6** (see [7], Example 2.3)**.** Let R be a commutative local ring which is not a valuation ring. Let x and y be elements of R, neither of them divides the other. By taking a suitable quotient ring, we may assume that $(x) \cap (y) = 0$ and $xP = yP = 0$, where P is the unique maximal ideal of R. Let F be a free module with generators a_1, a_2, a_3. Let N be the submodule generated by $xa_1 - ya_2$ and let $M = \dfrac{F}{N}$. By Theorem 2.1 (2), F is a $srs^{\oplus}$-module. Suppose that M is a $srs^{\oplus}$-module. It is clear that M is finitely generated and it follows that $\mathrm{Rad}(M) \ll M$. By Lemma 2.2, M is $\oplus$-supplemented. This is a contradiction.

Now we give some properties of factor modules of $srs^{\oplus}$-modules. Recall from [12] that a submodule U of an R-module M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. Let M be an R-module and τ be a preradical for the category of R-modules. Then $\tau(M)$ is a fully invariant submodule of M. We prove the following proposition which is a modified form of [7] (Proposition 2.5).

Proposition 2.4. *If M is a $srs^{\oplus}$-module, then $\frac{M}{U}$ is a $srs^{\oplus}$-module for every fully invariant submodule U of M.*

Proof. Let U be any fully invariant submodule of M and let $\frac{V}{U}$ be any submodule of $\frac{M}{U}$ with $\operatorname{Rad}\left(\frac{M}{U}\right) \subseteq \frac{V}{U}$. Since $\frac{\operatorname{Rad}(M)+U}{U} \subseteq \operatorname{Rad}\left(\frac{M}{U}\right)$, we have $\operatorname{Rad}(M) \subseteq V$. By the hypothesis, we have $M = V + T$, $V \cap T \ll T$ and $M = T \oplus T'$ for some submodules T, T' of M. Then by [14] (Lemma 1.2(d)), $\frac{(T+U)}{U}$ is a supplement of $\frac{V}{U}$ in $\frac{M}{U}$. Since U is a fully invariant submodule of M, we have $U = (T \cap U) + (T' \cap U)$ by [7] (Lemma 2.4). Note that

$$\frac{M}{U} = \frac{(T+U)}{U} + \frac{(T'+U)}{U}$$

and

$$\frac{(T+U)}{U} \cap \frac{(T'+U)}{U} = 0,$$

i.e., $\frac{(T+U)}{U}$ is a direct summand of $\frac{M}{U}$. Hence $\frac{M}{U}$ is a $srs^{\oplus}$-module.

Proposition 2.5. *Let M be a $\oplus$-radical supplemented module. Then $\frac{M}{P(M)}$ has a small radical.*

Proof. Since $P(M)$ is a fully invariant submodule of M, by Proposition 2.4, the factor module $\frac{M}{P(M)}$ is $\oplus$-radical supplemented. Note that $\frac{M}{P(M)}$ is reduced. It follows from Proposition 2.1 that $\frac{M}{P(M)}$ has a small radical.

Proposition 2.6. *Let M be a $srs^{\oplus}$-module. Suppose that $\frac{M}{\operatorname{Rad}(M)}$ is projective. Then $\operatorname{Rad}(M)$ is $\oplus$-supplemented if and only if M is $\oplus$-supplemented.*

Proof. ($\Longrightarrow$) Let $\operatorname{Rad}(M)$ be a $\oplus$-supplemented module. By the hypothesis, we have $M = \operatorname{Rad}(M) \oplus N$ for some submodule N of M. Since M is a $srs^{\oplus}$-module, by Proposition 2.4, $\frac{M}{\operatorname{Rad}(M)}$ is semisimple and so N is semisimple. Therefore N is $\oplus$-supplemented. By [5] (Theorem 1.4), M is $\oplus$-supplemented.

($\Longleftarrow$) Since $\operatorname{Rad}(M)$ is a fully invariant submodule of M and M is $\oplus$-supplemented, $\operatorname{Rad}(M)$ is $\oplus$-supplemented by [7] (Proposition 2.5).

A submodule N of M is said to be *cofinite* if $\frac{M}{N}$ is finitely generated.

Proposition 2.7. *Let M be a $srs^{\oplus}$-module. Suppose that a cofinite fully invariant submodule K of M is a direct summand of M. Then K is a $srs^{\oplus}$-module.*

Proof. Let U be any submodule of K with $\mathrm{Rad}(K) \subseteq N$. By the hypothesis, we have $M = K \oplus L$ for some finitely generated submodule L of M. Then $\mathrm{Rad}(L) \ll L$. Clearly $\mathrm{Rad}(M) \subseteq U + \mathrm{Rad}(L)$. And so there exist submodules V, V' of M such that $M = U + \mathrm{Rad}(L) + V$, $(U + \mathrm{Rad}(L)) \cap V \ll V$ and $M = V \oplus V'$. Since $\mathrm{Rad}(L) \ll L$, we have $M = U + V$, $U \cap V \ll V$ and $M = V \oplus V'$. It follows that $K = U + (K \cap V)$ and $U \cap (K \cap V) \ll M$. Since K is a fully invariant submodule of M, then $K = (K \cap V) \oplus (K \cap V')$. Note that $U \cap (K \cap V) \ll K \cap V$. Therefore K is a $srs^{\oplus}$-module.

***Corollary* 2.7.** *Let M be a $srs^{\oplus}$-module and let $\tau(M)$ be a cofinite direct summand of M, then $\tau(M)$ is a $srs^{\oplus}$-module.*

Lemma 2.4. *Let M be an R-module and* $\mathrm{Rad}(M) \subseteq N$. *If N is a direct summand of M, then* $\mathrm{Rad}(M) = \mathrm{Rad}(N)$. *In particular, if* $\mathrm{Rad}(M)$ *is a direct summand of* M, $\mathrm{Rad}(M) = P(M)$.

Proof. By the hypothesis, we have $M = N \oplus N'$ for some submodule N' of M. Then $\mathrm{Rad}(M) = \mathrm{Rad}(N) \oplus \mathrm{Rad}(N')$ by [8] (9.1.5). Since $\mathrm{Rad}(M) \subseteq N$, $\mathrm{Rad}(M) = \mathrm{Rad}(N) \oplus (N \cap \mathrm{Rad}(N'))$. Note that $N \cap \mathrm{Rad}(N') \subseteq N \cap N' = 0$. Hence $\mathrm{Rad}(M) = \mathrm{Rad}(N)$. Now we take $N = \mathrm{Rad}(M)$ under the similar condition. So $M = \mathrm{Rad}(M) \oplus X$ for some submodule X of M. It follows that $\mathrm{Rad}(M) = \mathrm{Rad}(\mathrm{Rad}(M)) \oplus \mathrm{Rad}(X)$. Since $\mathrm{Rad}(M) \cap X = 0$, we have $\mathrm{Rad}(X) = 0$ and so $\mathrm{Rad}(M) = \mathrm{Rad}(\mathrm{Rad}(M))$, i.e., $\mathrm{Rad}(M)$ is radical. Consequently, $\mathrm{Rad}(M) = P(M)$.

Let R be a ring and let M be an R-module. We consider the following condition.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

Proposition 2.8. *Let M be a $srs^{\oplus}$-module with (D_3) and let N be a submodule with* $\mathrm{Rad}(M) \subseteq N$. *If N is a direct summand of M, N is a $srs^{\oplus}$-module.*

Proof. Let U be a submodule of N such that $\mathrm{Rad}(N) \subseteq U$. By Lemma 2.4, $\mathrm{Rad}(M) = \mathrm{Rad}(N)$. Since M is a $srs^{\oplus}$-module, there exist submodules V, V' of M such that $M = U + V$, $U \cap V \ll V$ and $M = V \oplus V'$. Then $N = U + (N \cap V)$. Since M satisfies (D_3), $N \cap V$ is a direct summand of M. Then there exists a submodule X of M such that $M = (N \cap V) \oplus X$. It follows that $U \cap (N \cap V) \ll N \cap V$ and $N = (N \cap V) \oplus (N \cap X)$. Therefore N is a $srs^{\oplus}$-module.

***Corollary* 2.8.** *Let M be a UC-extending module. If M is a $srs^{\oplus}$-module, then every direct summand of M containing* $\mathrm{Rad}(M)$ *is a $srs^{\oplus}$-module.*

Recall that an R-module M has *summand sum property* (SSP) if the sum of two direct summands of M is again a direct summand of M. In [4], a module M is called *⊕-cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M. It is well known [4] (Theorem 2.3) that a module M with (SSP) is ⊕-cofinitely supplemented if and only if every maximal submodule of M has a supplement that is a direct summand of M. We don't know whether $srs^{\oplus}$-modules are ⊕-cofinitely supplemented, but we have the following fact.

Theorem 2.5. *Let M be a $srs^{\oplus}$-module with (SSP). Then M is ⊕-cofinitely supplemented.*

Proof. Let U be any maximal submodule of M. Then $\mathrm{Rad}(M) \subseteq U$. By the hypothesis, U has a supplement that is a direct summand of M. By [4] (Theorem 2.3), M is ⊕-cofinitely supplemented.

The following example shows that a ⊕-cofinitely supplemented module is not a $srs^{\oplus}$-module.

***Example* 2.7.** Consider that the ring $\mathbb{Z}_p$ consisting all rational numbers of the form $\dfrac{a}{b}$, where $p \nmid b$. Then $\mathbb{Z}_p$ is a local ring, which is not left perfect. So, by [4] (Theorem 2.9), every left free

$\mathbb{Z}_p$-module is $\oplus$-cofinitely supplemented. Since $\mathbb{Z}_p$ is not left perfect, there exists an infinite index set I such that $\mathbb{Z}_p^{(I)}$ is not $\oplus$-supplemented. By Proposition 2.2, $\mathbb{Z}_p^{(I)}$ is not a $srs^{\oplus}$-module.

Proposition 2.9. *Let M be a module and $\mathrm{Rad}(M)$ be cofinite. If M is $\oplus$-cofinitely supplemented, then M is a $srs^{\oplus}$-module.*

Proof. Let N be any submodule of M with $\mathrm{Rad}(M) \subseteq N$. Note that

$$\frac{\left(\dfrac{M}{\mathrm{Rad}(M)}\right)}{\left(\dfrac{N}{\mathrm{Rad}(M)}\right)} \cong \frac{M}{N}.$$

Since $\dfrac{M}{\mathrm{Rad}(M)}$ is finitely generated, N is a cofinite submodule of M. By the hypothesis, N has a supplement that is a direct summand of M. Therefore M is a $srs^{\oplus}$-module.

Theorem 2.6. *Let M be a non-zero projective module with cofinite radical. Then the following statements are equivalent*:

(1) *M is a $\oplus$-supplemented module*;

(2) *M is a $\oplus$-cofinitely supplemented module*;

(3) *M is a $srs^{\oplus}$-module.*

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) This implication follows from Proposition 2.9.

(3) $\Rightarrow$ (1) By Proposition 2.2.

3. Modules over Dedekind domains. Throughout this section R will denote a Dedekind domain unless otherwise specified.

Proposition 3.1. *Let M be an R-module. Then M is $\oplus$-radical supplemented if and only if $\dfrac{M}{P(M)}$ has a small radical.*

Proof. ($\Longrightarrow$) By Proposition 2.5.

($\Longleftarrow$) Since R is Dedekind domain, $P(M)$ is injective and so there exists a submodule N of M such that $M = P(M) \oplus N$. By the hypothesis, N is $\oplus$-radical supplemented. Thus, by Lemma 2.1 and Theorem 2.1 (1), M is $\oplus$-radical supplemented.

Note that from [14] (Lemma 2.1), over a local Dedekind domain module with small radical is coatomic. By using this fact and Proposition 3.1, we obtain the following corollary.

***Corollary* 3.1.** *Let R be a local Dedekind domain and M be a module over such a ring R. Then M is $\oplus$-radical supplemented if and only if $\dfrac{M}{P(M)}$ is coatomic.*

Proposition 3.2. *Let M be an R-module. Then M is $srs^{\oplus}$ if and only if $\dfrac{M}{P(M)}$ is a $srs^{\oplus}$-module.*

Proof. We know that $P(M)$ is a fully invariant submodule of M. So, by Proposition 2.4, $\dfrac{M}{P(M)}$ is a $srs^{\oplus}$-module. Conversely, suppose that $\dfrac{M}{P(M)}$ is a $srs^{\oplus}$-module. Since R is a Dedekind domain, we have $M = P(M) \oplus N$ for some submodule N of M. By the hypothesis, N is a $srs^{\oplus}$-module. Hence M is a $srs^{\oplus}$-module by Theorem 2.1 (2) and Lemma 2.1.

***Corollary* 3.2.** *Let R be a local Dedekind domain and M be an R-module. Then M is $\oplus$-radical supplemented if and only if it is a $srs^{\oplus}$-module.*

Proof. Suppose that M is $\oplus$-radical supplemented. By Corollary 3.1, $\dfrac{M}{P(M)}$ is coatomic and so, by [14] (Lemma 2.1) $\dfrac{M}{P(M)}$ is $\oplus$-supplemented, which shows that $\dfrac{M}{P(M)}$ is a $srs^{\oplus}$-module. By Proposition 3.2, M is a $srs^{\oplus}$-module.

Theorem 3.1. *Let R be a local Dedekind domain and M be an R-module. Then the following statements are equivalent*:

(1) *M is $\oplus$-radical supplemented*;

(2) *M is a $srs^{\oplus}$-module*;

(3) $M \cong K^{(I)} \oplus \left(\dfrac{K}{R}\right)^{(J)} \oplus R^{(n)} \oplus N$, *where K is the quotient field of R, I and J denote any index sets, n is a non-negative integer and N is a bounded R-module.*

Proof. (1) $\Longleftrightarrow$ (2) It is clear from Corollary 3.2.

(3) $\Longrightarrow$ (2) The module $K^{(I)} \oplus \left(\dfrac{K}{R}\right)^{(J)}$ is radical and so, by Lemma 2.1, $K^{(I)} \oplus \left(\dfrac{K}{R}\right)^{(J)}$ is a $srs^{\oplus}$-module. By [14] (Lemma 2.1), $R^{(n)} \oplus N$ is $\oplus$-supplemented. Hence the direct sum $K^{(I)} \oplus$ $\oplus \left(\dfrac{K}{R}\right)^{(J)} \oplus R^{(n)} \oplus N$ is a $srs^{\oplus}$-module by Theorem 2.1 (2).

(2) $\Longrightarrow$ (3) By Corollary 3.1, $\dfrac{M}{P(M)}$ is coatomic. Then by [14] (Lemma 2.1), we have $\dfrac{M}{P(M)} \cong$ $\cong R^{(n)} \oplus N$, where n is non-negative integer and N is bounded. Since $P(M)$ is radical, $P(M) \cong$ $\cong K^{(I)} \oplus \left(\dfrac{K}{R}\right)^{(J)}$ for some index sets I and J. Thus $M \cong K^{(I)} \oplus \left(\dfrac{K}{R}\right)^{(J)} \oplus R^{(n)} \oplus N$.

We know that every $\oplus$-radical supplemented module is radical supplemented. In Example 2.3, we showed that a radical supplemented module need not be $\oplus$-radical supplemented. Now we shall prove that the converse of this fact is true for torsion modules over local Dedekind domains.

Proposition 3.3. *Let R be a local Dedekind domain and M be a torsion R-module. Then M is radical supplemented if and only if it is $\oplus$-radical supplemented.*

Proof. Suppose that M is radical supplemented. By [13] (Proposition 3.1), $\dfrac{M}{P(M)}$ is bounded since M is torsion. Hence M is $\oplus$-radical supplemented by Theorem 2.1 (1).

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