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θ-CENTRALIZERS ON SEMIPRIME BANACH $*$-ALGEBRAS
θ-ЦЕНТРАЛІЗАТОРИ НА НАПІВПРОСТИХ БАНАХОВИХ $*$-АЛГЕБРАХ

By generalizing the celebrated theorem of Johnson, we prove that every left θ-centralizer on a semisimple Banach algebra with left approximate identity is continuous. We also investigate the generalized Hyers – Ulam – Rassias stability and the superstability of θ-centralizers on semiprime Banach $*$-algebras.

Шляхом узагальнення відомої теореми Джонсона доведено, що кожний лівий θ-централізатор на напівпростій банаховій алгебрі з лівою наближеною одиницею є неперервним. Також досліджено узагальнену стійкість Хайерса – Улама – Рассіаса та надстійкість θ-централізаторів на напівпростих $*$-алгебрах.

1. Introduction. The notion of centralizers has been generalized as θ-centralizer by Albas [1]. Let $\mathcal{A}$ be a $*$-algebra and θ be an algebra automorphism of $\mathcal{A}$. A mapping $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ is called a left (right) θ-centralizer on $\mathcal{A}$ if $T(xy) = T(x)\theta(y)$ $\big(T(xy) = \theta(x)T(y)\big)$ holds for all $x, y \in \mathcal{A}$. T is called a θ-centralizer if it is a left as well as a right θ-centralizer. The concept of left and right θ-centralizer covers the concept of left and right centralizer (in case $\theta = \mathrm{id}$, the identity automorphism on $\mathcal{A}$). The properties of θ-centralizers have been studied by Albas [1], Ali and Haetinger [2], Cortis and Haetinger [7], Daif [8] and Ullah and Chaudhry [22].

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation ζ must be close to an exact solution of ζ?* If the problem accepts a solution, we say that the equation ζ is stable. There are cases in which each approximate solution is actually a true solution. In such cases, we call the equation ζ superstable. The first stability problem concerning group homomorphisms was raised by Ulam [23] in 1940. Ulam problem was partially solved by Hyers [12] for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [21] has provided a lot of influence in the development of what is called the generalized Hyers – Ulam stability or the Hyers – Ulam – Rassias stability of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Gavruta [11] in 1994 by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. Badora [5] proved the generalized Hyers – Ulam stability of ring homomorphisms, which generalizes the result of D. G. Bourgin. Miura [18] proved the generalized Hyers – Ulam stability of Jordan homomorphisms. For more details about the stability of functional equations see [9 – 14].

In Section 2, by generalizing the celebrated theorem of Johnson [17], we prove that every left θ-centralizer on a semisimple Banach algebra with a left approximate identity is continuous. In Section 3, we prove the superstability of θ-centralizers on semiprime Banach $*$-algebras and we provide conditions for which a given mapping f is a left (right) θ-centralizer. In Section 4, we investigate the generalized Hyers – Ulam stability of θ-centralizers on semiprime Banach $*$-algebras. Throughout this paper, it is assumed that $\mathcal{A}$ is a semiprime Banach (complex) $*$-algebra.

2. Automatic continuity of θ-centralizers. In this section, we show that every left (right) θ-centralizer is homogenous. Also, we apply a classical theorem of B. E. Johnson to prove that every left θ-centralizer on a semisimple Banach algebra with a left approximate identity is continuous. Following [6], a Banach algebra $\mathcal{B}$ is said to have a left approximate identity (in Cohen's sense), if there exists a constant C, such that given $\epsilon > 0$, and $x_i \in \mathcal{B}$, $1 \leq i \leq m$, there exists an $e \in \mathcal{B}$, satisfying

$$\|e\| < C, \qquad \|ex_i - x_i\| < \epsilon.$$

Proposition 2.1. *Let $\mathcal{B}$ be a semiprime algebra. If $T\colon \mathcal{B} \longrightarrow \mathcal{B}$ is a left* (*right*) *θ-centralizer, then T is homogenous.*

Proof. Set $a := T(\mu x) - \mu T(x)$ for every $x \in \mathcal{B}$ and every $\mu \in \mathbb{C}$. Let $y \in \mathcal{B}$. Then there exists a $z \in \mathcal{B}$ such that $y = \theta(z)$. Therefore,

$$aya = \big(T(\mu x) - \mu T(x)\big)\theta(z)a = \big(T(\mu x)\theta(z) - \mu T(x)\theta(z)\big)a =$$

$$= \big(T(\mu xz) - T(x)\theta(\mu z)\big)a = \big(T(\mu xz) - T(x\mu z)\big)a = 0.$$

From the semiprimeness of $\mathcal{B}$ it follows that $a = 0$. Thus, T is homogenous.

Proposition 2.1 is proved.

We now generalize the result of [17] for continuity of θ-centralizers on Banach algebras.

Theorem 2.1. *Let $\mathcal{B}$ be a semisimple Banach algebra with a left approximate identity* (*in Cohen's sense*). *If $T\colon \mathcal{B} \longrightarrow \mathcal{B}$ is a left θ-centralizer, then T is linear and continuous.*

Proof. If $x_1, x_2 \in \mathcal{B}$, then by Johnson's Theorem (see [17]) one can find y_1, y_2, $z \in \mathcal{B}$ such that $x_1 = zy_1$ and $x_2 = zy_2$. Thus,

$$T(x_1 + x_2) = T\big(z(y_1 + y_2)\big) = T(z)\theta(y_1 + y_2) =$$

$$= T(z)\theta(y_1) + T(z)\theta(y_2) = T(zy_1) + T(zy_2) = T(x_1) + T(x_2).$$

Now, Proposition 2.1 implies T is linear.

If $x_m \in \mathcal{B}$ and $x_m \to 0$, then by Johnson's Theorem (see [17]) it follows that there exists a $z \in \mathcal{B}$ and a sequence y_m in $\mathcal{B}$ with $y_m \to 0$ such that $x_m = zy_m$, $m = 1, 2, \ldots$. Hence,

$$T(x_m) = T(zy_m) = T(z)\theta(y_m).$$

But a classical theorem of B. E. Johnson (see [4]) yields $\theta(y_m) \to 0$ as $m \to \infty$. Therefore, T is continuous.

Theorem 2.1 is proved.

3. Superstability. In this section, we prove the superstability of θ-centralizers on semiprime Banach $*$-algebras. Note that throughout this section $n > 4$ is a fixed integer.

We first summarize the following corollaries from [22].

***Corollary* 3.1.** *If $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ is an additive mapping such that $T(xx^*) = T(x)\theta(x^*)$ holds for all $x \in \mathcal{A}$, then T is a left θ-centralizer.*

Proof. The result follows from Theorem 2.2 of [22] and the fact that every complex $*$-algebra is a 2-torsion free ring.

***Corollary* 3.2.** *If $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ is an additive mapping such that $T(xx^*) = \theta(x^*)T(x)$ holds for all $x \in \mathcal{A}$, then T is a right θ-centralizer.*

***Corollary* 3.3.** *If $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ is an additive mapping such that $T(xx^*) = T(x)\theta(x^*) = \theta(x^*)T(x)$ holds for all $x \in \mathcal{A}$, then T is a θ-centralizer.*

We now provide conditions which imply the superstability of θ-centralizers on semiprime Banach $*$-algebras.

Theorem 3.1. *Let $p \neq 2$ and α be nonnegative real numbers and $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping such that*

$$\left\| \frac{1}{n-2} \sum_{i=1}^{n} f\left(-x_i + \sum_{j=1, j\neq i}^{n} x_j \right) - \sum_{i=1}^{n-1} f(x_i) \right\| \leq \big\| f(x_n) \big\|, \tag{3.1}$$

$$\big\| f(aa^*) - f(a)\theta(a^*) \big\| \leq \alpha \|a\|^p \tag{3.2}$$

for all $a, x_i \in \mathcal{A}$, $1 \leq i \leq n$. Then the mapping $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ is a linear left θ-centralizer. Moreover, if $\mathcal{A}$ is a semisimple Banach $*$-algebra with a left approximate identity (*in Cohen's sense*), *then f is continuous.*

Proof. Letting $x_1 = \ldots = x_n = 0$ and using $n > 4$ we conclude that $f(0) = 0$. Letting $x_1 = x$ and $x_2 = \ldots = x_n = 0$ we infer that f is odd for all $x \in \mathcal{A}$. Setting $x_3 = \ldots = x_n = 0$, we get

$$\frac{1}{n-2}\big(f(-x_1 + x_2) + f(-x_2 + x_1)\big) + f(x_1 + x_2) = f(x_1) + f(x_2)$$

for all $x_1, x_2 \in \mathcal{A}$. From the oddness of f it follows that f is additive. Assume that $p < 2$. By using the inequality (3.2), we have

$$\big\| f(aa^*) - f(a)\theta(a^*) \big\| = \frac{1}{n^2} \Big\| f\big((na)(na)^*\big) - f(na)\theta\big((na)^*\big) \Big\| \leq \frac{1}{n^2} \alpha n^p \|a\|^p$$

for all $a \in \mathcal{A}$. Thus, by letting n tend to ∞ in the last inequality, we obtain $f(aa^*) = f(a)\theta(a^*)$ for all $a \in \mathcal{A}$. Hence Corollary 3.1 implies f is a left θ-centralizer. The additivity of f together with Proposition 2.1 yield f is linear. Moreover, the continuity of f follows from Theorem 2.1. Similarly, one can obtain the result for the case $p > 2$.

Theorem 3.1 is proved.

Theorem 3.2. *Let $p \neq 2$ and α be nonnegative real numbers and $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping satisfying the inequality* (3.1) *and*

$$\big\| f(aa^*) - \theta(a^*)f(a) \big\| \leq \alpha \|a\|^p \tag{3.3}$$

for all $a \in \mathcal{A}$. Then the mapping $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a linear right θ-centralizer.

Proof. The proof is similar to the proof of Theorem 3.1 and the result follows from Corollary 3.2.

Theorem 3.3. *Let $p \neq 2$ and α be nonnegative real numbers and $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping satisfying the inequality* (3.1) *and*

$$\big\| f(aa^* + bb^*) - f(a)\theta(a^*) - \theta(b^*)f(b) \big\| \leq \alpha\big(\|a\|^p + \|b\|^p\big) \tag{3.4}$$

for all $a, b \in \mathcal{A}$. Then the mapping $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ is a linear θ-centralizer. Moreover, if $\mathcal{A}$ is a semisimple Banach $*$-algebra with a left approximate identity (*in Cohen's sense*), *then f is continuous.*

Proof. Setting $b = 0$ in (3.4) and applying Theorem 3.1, we conclude that f is a linear left θ-centralizer. Letting $a = 0$ in (3.4) and using Theorem 3.2, we deduce that f is a right θ-centralizer.

Theorem 3.3 is proved.

4. Stability. In this section we prove the generalized Hyers – Ulam stability of θ-centralizers on semiprime Banach $*$-algebras. Throughout this section $n > 3$ is a fixed integer.

The following lemma (see [19]) is needed in the rest of the paper.

Lemma 4.1. *Let X and Y be linear spaces. A mapping $f\colon X \longrightarrow Y$ satisfies*

$$\sum_{i=1}^{n} f\Bigg(-x_i + \sum_{j=1, j\neq i}^{n} x_j\Bigg) = (n-2)\sum_{i=1}^{n} f(x_i) \tag{4.1}$$

for $x_1, \ldots, x_n \in X$, if and only if f is additive.

Theorem 4.1. *Let $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping for which $f(0) = 0$ and there exists a control function $\varphi\colon \mathcal{A}^{n+1} \longrightarrow [0, \infty)$ such that*

$$\tilde{\varphi}(x) := \sum_{i=1}^{\infty} \frac{1}{2^i}\varphi\big(2^{i-1}x, 2^{i-1}x, 0, \ldots, 0\big) < \infty, \tag{4.2}$$

$$\lim_{k\to\infty} \frac{1}{2^k}\varphi\big(2^k x_1, \ldots, 2^k x_n, 2^k a\big) = 0, \tag{4.3}$$

$$\left\| \sum_{i=1}^{n} f\Bigg(-x_i + \sum_{j=1, j\neq i}^{n} x_j\Bigg) - (n-2)\sum_{i=1}^{n} f(x_i) + f(aa^*) - f(a)\theta(a^*) \right\| \leq$$

$$\leq \varphi(x_1, \ldots, x_n, a) \tag{4.4}$$

for all $a, x_1, \ldots, x_n \in \mathcal{A}$. Then there exists a unique linear left θ-centralizer $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\big\|T(x) - f(x)\big\| \leq \frac{1}{n-2}\tilde{\varphi}(x) \tag{4.5}$$

for all $x \in \mathcal{A}$.

Proof. Setting $x_1 = x_2 = x$, $a = x_3 = \ldots = x_n = 0$ in (4.4) and using $f(0) = 0$, we obtain

$$\left\| \frac{1}{2} f(2x) - f(x) \right\| \leq \frac{1}{2(n-2)}\varphi(x, x, 0, \ldots, 0) \tag{4.6}$$

for all $x \in \mathcal{A}$. Applying induction method on m, we have

$$\left\| \frac{1}{2^m} f(2^m x) - f(x) \right\| \leq \frac{1}{n-2}\sum_{i=1}^{m} \frac{1}{2^i}\varphi\big(2^{i-1}x, 2^{i-1}x, 0, \ldots, 0\big) \tag{4.7}$$

for all $x \in \mathcal{A}$. In order to show that the functions $T_m(x) = \dfrac{1}{2^m} f(2^m x)$ form a convergent sequence, we use the Cauchy convergence criterion. Replace x by $2^l x$ and divide by 2^l in (4.7), where l is an arbitrary positive integer, to find that

$$\left\| \frac{1}{2^{m+l}} f(2^{m+l}x) - \frac{1}{2^l} f(2^l x) \right\| \leq \frac{1}{n-2}\sum_{i=1+l}^{m+l} \frac{1}{2^i}\varphi\big(2^{i-1}x, 2^{i-1}x, 0, \ldots, 0\big)$$

for all positive integers $m \geq l$ and all $x \in \mathcal{A}$. Hence by the Cauchy criterion the limit $T(x) := := \lim_{m\to\infty} T_m(x)$ exists for each $x \in \mathcal{A}$. By taking the limit as $m \to \infty$ in (4.7) we see that the inequality (4.5) holds for all $x \in \mathcal{A}$. Setting $a = 0$ in (4.4), we get

$$\left\| \sum_{i=1}^{n} f\left(-x_i + \sum_{j=1, j\neq i}^{n} x_j \right) - (n-2) \sum_{i=1}^{n} f(x_i) \right\| \leq \varphi(x_1, \ldots, x_n, 0)$$

for all $x_i \in \mathcal{A}$, $1 \leq i \leq n$. Replacing x_i by $2^m x_i$, $1 \leq i \leq n$ and dividing both sides by 2^m and taking the limit as $m \to \infty$ and using (4.3) we deduce that T satisfies (4.1). Thus, it follows from Lemma 4.1 that T is additive. Setting $x_1 = \ldots = x_n = 0$ in (4.4), we get

$$\left\| f(aa^*) - f(a)\theta(a^*) \right\| \leq \varphi(0, \ldots, 0, a) \tag{4.8}$$

for all $a \in \mathcal{A}$. Replacing a by $2^m a$ in (4.8) and dividing its both sides by 2^{2m}, we obtain

$$\left\| \frac{1}{2^{2m}} f(2^{2m} aa^*) - \frac{1}{2^m} f(2^m a)\theta(a^*) \right\| \leq \frac{1}{2^{2m}} \varphi\big(0, \ldots, 0, 2^m a\big)$$

for all $a \in \mathcal{A}$. Taking the limit as $m \to \infty$ and using (4.3), we conclude that $T(aa^*) = T(a)\theta(a^*)$. So Corollary 3.1 implies T is a left θ-centralizer. Now, let $T' \colon \mathcal{A} \longrightarrow \mathcal{A}$ be another additive mapping satisfying (4.5). Consequently, we have

$$\left\| T(x) - T'(x) \right\| = \frac{1}{2^m} \left\| T(2^m x) - T'(2^m x) \right\| \leq$$

$$\leq \frac{1}{2^m} \Big(\left\| T(2^m x) - f(2^m x) \right\| + \left\| T'(2^m x) - f(2^m x) \right\| \Big) \leq \frac{2}{2^m (n-2)} \tilde{\varphi}(2^m x) =$$

$$= \frac{2}{n-2} \sum_{i=m+1}^{\infty} \frac{1}{2^i} \varphi\big(2^{i-1} x, 2^{i-1} x, 0, \ldots, 0\big)$$

for all $x \in \mathcal{A}$. The right-hand side tends to zero as $m \to \infty$. This proves the uniqueness of T. The linearity of T follows from Proposition 2.1.

Theorem 4.1 is proved.

Theorem 4.2. *Let $f \colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping for which $f(0) = 0$ and there exists a control function $\varphi \colon \mathcal{A}^{n+1} \longrightarrow [0, \infty)$ that satisfies* (4.2), (4.3) *and*

$$\left\| \sum_{i=1}^{n} f\left(-x_i + \sum_{j=1, j\neq i}^{n} x_j \right) - (n-2) \sum_{i=1}^{n} f(x_i) + f(aa^*) - \theta(a^*) f(a) \right\| \leq$$

$$\leq \varphi(x_1, \ldots, x_n, a) \tag{4.9}$$

for all $a, x_1, \ldots, x_n \in \mathcal{A}$. Then there exists a unique linear right θ-centralizer $T \colon \mathcal{A} \longrightarrow \mathcal{A}$ such that,

$$\left\| T(x) - f(x) \right\| \leq \frac{1}{n-2} \tilde{\varphi}(x) \tag{4.10}$$

for all $x \in \mathcal{A}$.

Proof. The proof is similar to the proof of Theorem 4.1.

Theorem 4.3. *Let $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping for which $f(0) = 0$ and there exists a control function $\phi\colon \mathcal{A}^{n+2} \longrightarrow [0,\infty)$ such that*

$$\tilde{\phi}(x) := \sum_{i=1}^{\infty} \frac{1}{2^i} \phi\big(2^{i-1}x, 2^{i-1}x, 0, \ldots, 0\big) < \infty, \tag{4.11}$$

$$\lim_{k\to\infty} \frac{1}{2^k} \phi\big(2^k x_1, \ldots, 2^k x_n, 2^k a, 2^k b\big) = 0, \tag{4.12}$$

$$\left\| \sum_{i=1}^{n} f\left(- x_i + \sum_{j=1, j\neq i}^{n} x_j \right) - (n-2) \sum_{i=1}^{n} f(x_i) + \right.$$

$$\left. + f(aa^* + bb^*) - f(a)\theta(a^*) - \theta(b^*) f(b) \right\| \le \phi(x_1, \ldots, x_n, a, b) \tag{4.13}$$

for all $a, b, x_1, \ldots, x_n \in \mathcal{A}$. Then there exists a unique linear θ-centralizer $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\big\|T(x) - f(x)\big\| \le \frac{1}{n-2} \tilde{\phi}(x) \tag{4.14}$$

for all $x \in \mathcal{A}$.

Proof. Setting $b = 0$ in (4.13), we obtain

$$\left\| \sum_{i=1}^{n} f\left(- x_i + \sum_{j=1, j\neq i}^{n} x_j \right) - (n-2) \sum_{i=1}^{n} f(x_i) + f(aa^*) - f(a)\theta(a^*) \right\| \le \phi(x_1, \ldots, x_n, a, 0)$$

for all $a, x_1, \ldots, x_n \in \mathcal{A}$. By taking $\varphi(x_1, \ldots, x_n, a) := \phi(x_1, \ldots, x_n, a, 0)$ for all $a, x_1, \ldots, x_n \in \mathcal{A}$ and applying the same method as in the proof of Theorem 4.1, we obtain the Cauchy sequence $\left\{ \frac{1}{2^m} f(2^m x) \right\}$ for all $x \in \mathcal{A}$. Completeness of $\mathcal{A}$ gives a unique mapping $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ which is a linear left θ-centralizer and

$$\big\|T(x) - f(x)\big\| \le \frac{1}{n-2} \tilde{\varphi}(x) = \frac{1}{n-2} \tilde{\phi}(x). \tag{4.15}$$

Setting $a = 0$ in (4.13), we obtain

$$\left\| \sum_{i=1}^{n} f\left(- x_i + \sum_{j=1, j\neq i}^{n} x_j \right) - (n-2) \sum_{i=1}^{n} f(x_i) + f(bb^*) - \theta(b^*) f(b) \right\| \le \phi(x_1, \ldots, x_n, 0, b)$$

for all $b, x_1, \ldots, x_n \in \mathcal{A}$. By taking $\varphi(x_1, \ldots, x_n, b) := \phi(x_1, \ldots, x_n, 0, b)$ for all $b, x_1, \ldots, x_n \in \mathcal{A}$ and applying the same method as in the proof of Theorem 4.2, we obtain the above Cauchy sequence which converges to the mapping $T\colon \mathcal{A} \longrightarrow \mathcal{A}$. Now, Theorem 4.2 implies the mapping T is a linear right θ-centralizer and satisfies (4.15). Therefore, T is a unique linear θ-centralizer satisfying (4.14).

Theorem 4.3 is proved.

***Corollary* 4.1.** *Let* α *and* r_j, $1 \le j \le n+2$, *be nonnegative real numbers such that* $0 < r_j < 1$. *Suppose that a mapping* $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ *with* $f(0) = 0$ *satisfies*

$$\left\| \sum_{i=1}^{n} f\left(-x_i + \sum_{j=1, j\neq i}^{n} x_j \right) - (n-2)\sum_{i=1}^{n} f(x_i) + f(x_{n+1}x_{n+1}^{*} + x_{n+2}x_{n+2}^{*}) - \right.$$

$$\left. - f(x_{n+1})\theta(x_{n+1}^{*}) - \theta(x_{n+2}^{*})f(x_{n+2}) \right\| \le \alpha \sum_{j=1}^{n+2} \|x_j\|^{r_j} \tag{4.16}$$

for all $x_1, \ldots, x_{n+2} \in \mathcal{A}$. *Then there exists a unique linear* θ*-centralizer* $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ *such that*

$$\big\|T(x) - f(x)\big\| \le \frac{\alpha}{n-2}\left(\frac{\|x\|^{r_1}}{2 - 2^{r_1}} + \frac{\|x\|^{r_2}}{2 - 2^{r_2}} \right)$$

for all $x \in \mathcal{A}$.

Proof. It is an immediate consequence of Theorem 4.3 by taking

$$\phi(x_1, \ldots, x_{n+2}) := \alpha \sum_{j=1}^{n+2} \|x_j\|^{r_j}$$

for all $x_1, \ldots, x_{n+2} \in \mathcal{A}$.

The following Corollary is Isac – Rassias type stability (see [15, 16]) for θ-centralizers on semiprime Banach $*$-algebras.

***Corollary* 4.2.** *Let* $\psi\colon \mathbb{R}^{+} \cup \{0\} \longrightarrow \mathbb{R}^{+} \cup \{0\}$ *be a function with* $\psi(0) = 0$ *such that*

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \qquad \psi(ts) \le \psi(t)\psi(s)$$

for $t, s \in \mathbb{R}^{+}$, *and* $\psi(t) < t$ *for* $t > 1$. *Suppose that* α *is a nonnegative real number and* $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ *is a mapping with* $f(0) = 0$ *satisfies*

$$\left\| \sum_{i=1}^{n} f\left(-x_i + \sum_{j=1, j\neq i}^{n} x_j \right) - (n-2)\sum_{i=1}^{n} f(x_i) + f(x_{n+1}x_{n+1}^{*} + x_{n+2}x_{n+2}^{*}) - \right.$$

$$\left. - f(x_{n+1})\theta(x_{n+1}^{*}) - \theta(x_{n+2}^{*})f(x_{n+2}) \right\| \le \alpha \sum_{j=1}^{n+2} \psi\big(\|x_j\|\big)$$

for all $x_1, \ldots, x_{n+2} \in \mathcal{A}$. *Then there exists a unique linear* θ*-centralizer* $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ *such that*

$$\big\|T(x) - f(x)\big\| \le \frac{2\alpha\psi(2)\psi(2^{-1})}{(n-2)(2 - \psi(2))}\psi\big(\|x\|\big)$$

for all $x \in \mathcal{A}$.

Proof. The result follows from Theorem 4.3 by letting

$$\phi(x_1, \ldots, x_{n+2}) := \alpha \sum_{j=1}^{n+2} \psi\big(\|x_j\|\big)$$

for all $x_1, \ldots, x_{n+2} \in \mathcal{A}$.

Theorem 4.4. *Let $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping for which there exists a control function $\varphi\colon \mathcal{A}^{n+1} \longrightarrow \longrightarrow [0,\infty)$ that satisfies* (4.4) *and*

$$\tilde{\varphi}(x) := \sum_{i=1}^{\infty} 2^i \varphi\left(\frac{1}{2^{i-1}}x, \frac{1}{2^{i-1}}x, 0, \ldots, 0\right) < \infty, \tag{4.17}$$

$$\lim_{k\to\infty} 4^k \varphi\left(\frac{x_1}{2^k}, \ldots, \frac{x_n}{2^k}, \frac{a}{2^k}\right) = 0 \tag{4.18}$$

for all $a, x_1, \ldots, x_n \in \mathcal{A}$. Then there exists a unique linear left θ-centralizer $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\big\|T(x) - f(x)\big\| \le \frac{1}{n-2}\tilde{\varphi}(x) \tag{4.19}$$

for all $x \in \mathcal{A}$.

Proof. Setting $a = x_1 = \ldots = x_n = 0$ in (4.18) we conclude that $\varphi(0,\ldots,0) = 0$. Setting $a = x_1 = \ldots = x_n = 0$ in (4.4) and using $n > 3$ we see that $f(0) = 0$. Therefore by a similar calculation as in the proof of Theorem 4.1 we can obtain (4.6). Now, replace x by $\dfrac{x}{2}$ and multiply both sides by 2 in (4.6), to get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \frac{1}{n-2}\varphi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right)$$

for all $x \in \mathcal{A}$. Using induction method on m, we have

$$\left\|f(x) - 2^m f\left(\frac{x}{2^m}\right)\right\| \le \frac{1}{n-2}\sum_{i=1}^{m} 2^{i-1}\varphi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, \ldots, 0\right) \tag{4.20}$$

for all $x \in \mathcal{A}$. Replacing x by $\dfrac{x}{2^l}$ and multiplying by 2^l in (4.20), where l is an arbitrary positive integer, we get

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{m+l} f\left(\frac{x}{2^{m+l}}\right)\right\| \le \frac{1}{n-2}\sum_{i=1+l}^{m+l} 2^{i-1}\varphi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, \ldots, 0\right) \tag{4.21}$$

for all positive integers $m \ge l$. Due to completeness of $\mathcal{A}$ the sequence $\left\{2^m f\left(\dfrac{x}{2^m}\right)\right\}$ converges for all $x \in \mathcal{A}$. Hence we can define the mapping $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ by $T(x) := \lim_{n\to\infty} 2^m f\left(\dfrac{x}{2^m}\right)$. By taking the limit as $m \to \infty$ in (4.20) we obtain the desired inequality (4.19). The rest of the proof is similar to the proof of Theorem 4.1 and we omit it.

Theorem 4.5. *Let $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping for which there exists a control function $\varphi\colon \mathcal{A}^{n+1} \longrightarrow [0,\infty)$ that satisfies* (4.9), (4.17) *and* (4.18). *Then there exists a unique linear right θ-centralizer $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ that satisfies the inequality* (4.19).

Theorem 4.6. *Let $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping for which there exists a control function $\phi\colon \mathcal{A}^{n+2} \longrightarrow [0,\infty)$ that satisfies* (4.13) *and*

$$\tilde{\phi}(x) := \sum_{i=1}^{\infty} 2^i \phi\left(\frac{1}{2^{i-1}}x, \frac{1}{2^{i-1}}x, 0, \ldots, 0\right) < \infty, \tag{4.22}$$

$$\lim_{k\to\infty} 4^k\phi\left(\frac{x_1}{2^k},\ldots,\frac{x_n}{2^k},\frac{a}{2^k},\frac{b}{2^k}\right)=0 \tag{4.23}$$

for all $a, b, x_1, \ldots, x_n \in \mathcal{A}$. Then there exists a unique linear θ-centralizer $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\big\|T(x)-f(x)\big\| \le \frac{1}{n-2}\tilde{\phi}(x) \tag{4.24}$$

for all $x \in \mathcal{A}$.

***Corollary* 4.3.** *Let α and r_j, $1 \le j \le n+2$, be nonnegative real numbers such that $r_j > 1$. Suppose that a mapping $f\colon \mathcal{A} \longrightarrow \mathcal{A}$ satisfies* (4.16). *Then there exists a unique linear θ-centralizer $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that*

$$\big\|T(x)-f(x)\big\| \le \frac{2\alpha}{n-2}\left(\frac{2^{r_1}}{2^{r_1}-2}\|x\|^{r_1}+\frac{2^{r_2}}{2^{r_2}-2}\|x\|^{r_2}\right)$$

for all $x \in \mathcal{A}$.

Proof. It is enough to define

$$\phi(x_1,\ldots,x_{n+2}) := \alpha\sum_{j=1}^{n+2}\|x_j\|^{r_j}$$

for all $x_1, \ldots, x_{n+2} \in \mathcal{A}$ and apply Theorem 4.6.

***Remark* 4.1.** In Theorems 4.3, 4.4, and 4.6 and Corollaries 4.1, 4.2, and 4.3 if $\mathcal{A}$ is replaced by a semisimple Banach $*$-algebra with a left approximate identity (in Cohen's sense), then T is continuous. Note that in this case the result follows from Theorem 2.1.

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