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A NEW METHOD OF GENERATING OF TRAVELING WAVE SOLUTIONS FOR COUPLED NONLINEAR EQUATIONS*

НОВИЙ МЕТОД ГЕНЕРУВАННЯ РОЗВ'ЯЗКІВ ТИПУ БІЖУЧИХ ХВИЛЬ ДЛЯ ЗЧЕПЛЕНИХ НЕЛІНІЙНИХ РІВНЯНЬ

A new algebraic transformation method is constructed for finding traveling-wave solutions of complicated nonlinear wave equations on the basis of simpler ones. The generalized Dullin – Gottwald – Holm (DGH) equation and mKdV equations are chosen to illustrate our method. The solutions of the DGH equation can be obtained directly from solutions of the mKdV equation. Conditions under which different solutions appear are also given. Abundant traveling-wave solutions of the generalized DGH equation are obtained, including periodic solutions, smooth solutions with decay, solitary solutions, and kink solutions.

Побудовано новий метод алгебраїчних перетворень для знаходження розв'язків типу біжучих хвиль для складних нелінійних хвильових рівнянь на основі більш простих. Для ілюстрації методу використано узагальнене рівняння Далліна – Готвальда – Холма та модифіковане рівняння Кортевега – де Фріза. Розв'язки рівняння Далліна – Готвальда – Холма можна отримати безпосередньо із розв'язків модифікованого рівняння Кортевега – де Фріза. Наведено також умови для отримання різних розв'язків. Отримано чисельні розв'язки типу біжучих хвиль для узагальненого рівняння Далліна – Готвальда – Холма, серед яких періодичні розв'язки, гладкі розв'язки з запізненням, солітонні розв'язки та кінк-розв'язки.

1. Introduction. Nonlinear wave phenomena appear in a wide variety of scientific applications, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid state physics and optical fibers. These nonlinear phenomena are often related to nonlinear wave equations. Investigation of the traveling wave solutions can make a better understanding of those phenomena and their application in real life. Although many methods have been developed to construct traveling wave solutions, it is still a difficult task to find traveling wave solutions of complicated equations with nonlinear terms.

The main purpose of this paper is to devise a new method to get traveling wave solutions of the complicated wave equations from solutions of simpler equations. This method is different from the classic Miura transformation [1], that is, the classic Miura transformation is between two equations with a linear dispersive term, and our method has more advantages in that one can obtain abundant solutions of the aimed equation with a nonlinear dispersive term.

The generalized DGH equation

$$u_t + 2\omega u_x - \alpha^2 u_{xxt} + au^m u_x + \gamma u_{xxx} = \alpha^2 (2u_x u_{xx} + uu_{xxx}) \qquad (1.1)$$

includes two separately integrable soliton equations for water waves. If taking $m = 1$ and $a = 3$, Eq. (1.1) becomes the DGH equation [2]

$$u_t + 2\omega u_x - \alpha^2 u_{xxt} + 3uu_x + \gamma u_{xxx} = \alpha^2 (2u_x u_{xx} + uu_{xxx}), \qquad (1.2)$$

which arises as a model for the unidirectional shallow water waves over a flat bottom. Here α^2

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and γ are squares of length scales, and the constant ω is the critical shallow water speed for undisturbed water at rest at spatial infinity. Eq. (1.2) has been studied by many researchers. The existence of several special solutions was proved in [3 – 6]. Those solutions include smooth solitary solutions, peakons, global conservative solutions and low regularity solutions with continuously periodic initial data. In [7 – 11], the authors studied the solution properties of the DGH equation and its relative equations. In this present letter, we will consider all possible integral constants.

Letting $\omega = 0$, $m = 2$ and $\alpha^2 \to 0$, Eq. (1.1) becomes the famous mKdV equation

$$u_t + au^2 u_x + \gamma u_{xxx} = 0, \tag{1.3}$$

which plays an important role in many nonlinear scientific fields [13 – 16]. Eq. (1.3) has been used to describe acoustic waves in certain anharmonic lattices and Alfven waves in a collisionless plasma. It also appears in the models of Schottky barriers transmission lines and traffic congestion. Many solutions of (1.3) in integral form have been given for some special integral constants. However, for any integral constants, all the possible solutions have not been determined.

In view of the close relationship between Eq. (1.1) and Eq. (1.3), it is possibble to establish an explicit connection between solutions of these two equations. If the connection does exist, one can easily obtain traveling wave solutions of Eq. (1.1) from the already known solutions of Eq. (1.3). This is one motivation of our work. Another motivation is that, whether the generalized DGH equation still preserves the integrability, Hamiltonian structure and some important conservative laws, like the two integral equations (1.2) and (1.3).

The remainder of the paper is organized as follows. In Section 2, a kind of the generalized DGH equation is firstly proved integrable. Meanwhile, the Hamiltonian structure and some important conservative laws of Eq. (1.1) are given. In Section 3, under different parameter conditions, the classification of traveling wave solutions of the mKdV equation is given by a qualitative method in which all possible integral constants are considered. In Section 4, motivated by the Fan subequation method [17], we verify directly an explicit connection between the mKdV equation and the generalized DGH equation. Furthermore, abundant traveling wave solutions of the generalized DGH equation are determined from the known solutions of the mKdV equation, and some examples of explicit solutions are also given. The last section is conclusion.

2. Painleve property and conservative laws. An equation is called Painleve integrable when it has Painleve property which means its solutions are single valued about an arbitrary singular manifold. In this section, we will study the Painleve integrability of the generalized DGH equation.

According to the Kruskal method [18], we expand u in Eq. (1.1) by a local Laurent expansion in the neighborhood of the singular manifold $\phi(x,t) = 0$ as

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+\alpha}. \tag{2.1}$$

Substituting (2.1) into (1.1) leads to conditions on α and recursion relation for the functions u_j. If α is a negative integer and the recursion relation is consistent, then we say the system (1.1) is integrable.

Substituting $u \sim u_j \phi^{\alpha}$ into Eq. (1.1), the leading order analysis implies $\alpha = -\dfrac{2}{m-1}$, so we can deduce $m = 2$ or $m = 3$.

For $m = 2$, we obtain the recursion relation of the expansion coefficients u_j as

$$(j+1)(j-6)(j-8)u_j = F_j(\phi_x, \phi_t, \ldots, u_0, u_1, \ldots, u_{j-1}), \tag{2.2}$$

where F_j is a function of $u_0, u_1, \ldots, u_{j-1}$ and the derivatives of ϕ. Therefore the resonances occur at $j = -1, 6, 8$. The resonance at $j = -1$ represents the arbitrariness of the singular manifold $\phi(x,t) = 0$. For the integability of Eq. (1.1), we only prove the existence of arbitrary functions at the cases $j = 6, 8$ with $m = 2$. According to the Kruskal's method, one can take $\phi = x + \psi(t)$, where $\psi(t)$ is an arbitrary function of t. After a lengthy computation, we obtain

$$u_0 = \frac{24}{a}, \qquad u_1 = u_3 = u_5 = 0, \qquad u_2 = -\frac{1}{3}\psi_t - \frac{1}{3}\gamma,$$

$$u_4 = \frac{1}{20}\psi_t - \frac{w}{10} - \frac{a}{180}\psi_t^2 - \frac{a\gamma}{90}\psi_t - \frac{a\gamma^2}{180}, \tag{2.3}$$

$$u_7 = \frac{13a}{5760}\psi_{tt} - \frac{a^2}{8640}\psi_t\psi_{tt} - \frac{a^2\gamma}{8640}\psi_{tt}.$$

Substituting (2.3) into the recursion relations (2.2), one can find that (2.2) are satisfied identically. Hence the generalized DGH equation (1.1) with $m = 2$ is integrable.

Similarly, for $m = 3$, we obtain the recursion relation of the expansion coefficients u_j as

$$(j+1)(j-4)(j-5)u_j = F_j(\phi_x, \phi_t, \ldots, u_0, u_1, \ldots, u_{j-1}).$$

When the resonance occurs at $j = 5$, we obtain

$$F_5 = \psi_{tt}(-32 + 6a\gamma\psi_t + 3a\psi_t^2 + 3a\gamma^2) \neq 0.$$

Therefore the generalized DGH equation (1.1) with $m = 3$ is not integrable.

The generalized DGH equation, combining the DGH equation and the mKdV equation, still preserve the Hamiltonian structure and some important conservative laws. Indeed, the generalized DGH equation, analogous to the case of the DGH equation, has the following conservative laws

$$M(u) = \int_{\mathbb{R}} u\,dx, \qquad Q(u) = \frac{1}{2}\int_{\mathbb{R}} (u^2 + {u_x}^2)\,dx,$$

$$H(u) = -\frac{1}{2}\int_{\mathbb{R}} \left(\frac{2a}{(m+1)(m+2)} u^{m+2} + uu_x^2 + 2\omega u^2 - \gamma {u_x}^2 \right) dx$$

and the Hamiltonian structure

$$u_t = -\partial_x \frac{\delta H}{\delta u} = \{u, H\},$$

where the Poisson bracket structure is defined as $\{u(x), u(y)\} = \partial_x \delta(x - y)$.

3. Classification of traveling wave solutions of the mKdV equation. In this section, we will classify traveling wave solutions of the mKdV equation in different regions of parametric space by a qualitative method.

For a traveling wave $u(x,t) = \phi(\xi) = \phi(bx - ct)$, Eq. (1.3) takes the form

$$-c\phi_\xi + ab\phi^2\phi_\xi + \gamma b^3\phi_{\xi\xi\xi} = 0, \tag{3.1}$$

where b and c are constants to be determined. Integrated twice with respect to x, Eq. (3.1) turns to be the equivalent integrated form

$$\phi_\xi^2 = h_1(\phi^4 - h_2\phi^2 + d_1\phi) + d_2, \tag{3.2}$$

where $h_1 = -\dfrac{\alpha}{6b^2\gamma}$, $h_2 = \dfrac{6c}{\alpha b}$, d_1 and d_2 are arbitrary constants of integration. For some special values of d_1 and d_2, it is not difficult to obtain traveling wave solutions of Eq. (3.2). However, for any integral constants, it is a hard task to determine the type of the solutions. Fortunately, we can use a qualitative analysis method to deal with this problem, in which we consider Eq. (3.2) for all possible constants of integration. Let

$$\phi_x^2 = F(\phi) = h_1(\phi^4 - h_2\phi^2 + d_1\phi) + d_2 \tag{3.3}$$

for determining the solutions of Eq. (3.2). Let the polynomial $P(\phi) = h_1\phi G(\phi)$ with a simple root at $\phi = 0$, where $G(\phi) = \phi^3 - h_2\phi + d_1$. The solutions of Eq. (3.2) correspond to different behaviors of this polynomial. Once the integral constant d_1 is fixed, a change in d_2 will shift the graph vertically up or down, accordingly change the zero points.

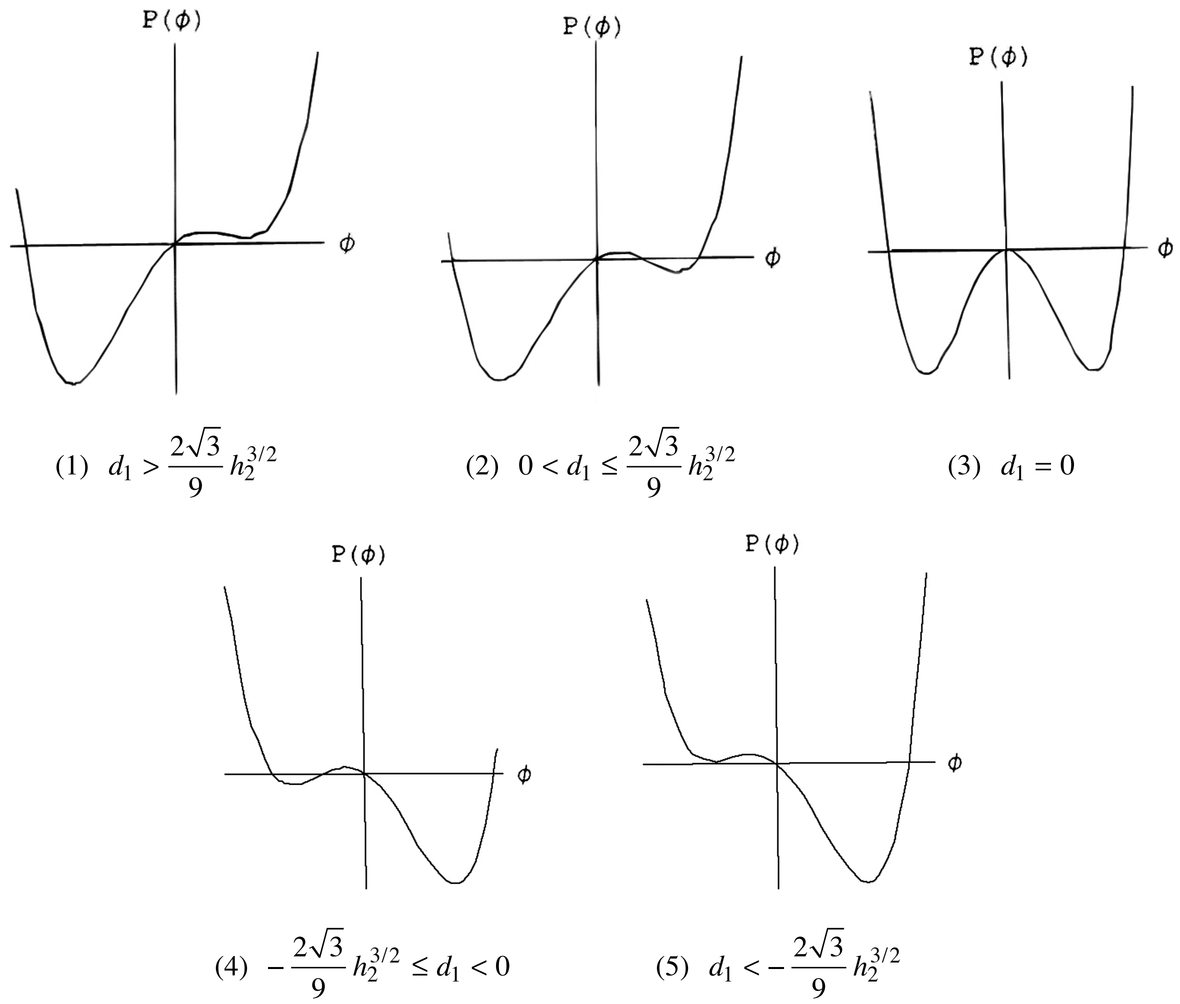
Similar to the method in [12], we can obtain five qualitatively different cases of $P(\phi)$ when $h_1 > 0$ (see Figure), and we can also establish the following structure of traveling wave solutions of Eq. (3.2):

***Case* 1.** When $h_1 > 0$, any traveling wave solution falls into one of the following categories:

(1) If $d_1 > \dfrac{2\sqrt{3}}{9} h_2^{3/2}$, for some negative d_2, there are smooth solutions with decay and periodic solutions.

(2) If $0 < d_1 \le \dfrac{2\sqrt{3}}{9} h_2^{3/2}$, there are smooth solutions with decay and periodic solutions for some positive d_2. For some negative d_2, there are periodic solutions.

(3) If $d_1 = 0$, there are kink solutions and periodic solutions for some positive d_2.



Five cases of $P(\phi)$ as $h_2 > 0$.

(4) If $-\frac{2\sqrt{3}}{9} h_2^{3/2} \leq d_1 < 0$, there are periodic solutions for some negative d_2. For some positive d_2, there exists smooth solutions with decay and periodic solutions.

(5) If $d_1 < -\frac{2\sqrt{3}}{9} h_2^{3/2}$, there are smooth solutions with decay and periodic solutions for some negative d_2.

For $h_1 < 0$, we can obtain the following results.

***Case* 2.** When $h_1 < 0$, any traveling wave solution falls into one of the following categories:

(1) If $d_1 \neq 0$, for some negative d_2, there are periodic solutions. For some positive d_2, there are smooth solutions with decay and periodic solutions.

(2) If $d_1 = 0$, there are solitary solutions for $d_2 = 0$ and periodic solutions for some positive or negative d_2.

Remark. If $h_2 \leq 0$, there are no bounded solutions of Eq. (3.2).

4. The new transformation method and traveling wave solutions. For a traveling wave $u(x,t)=u(\xi)$, $\xi=bx-ct$, Eq. (1.1) is reduced to

$$(2\omega b-c)\,u_\xi+(\gamma b+\alpha^2 c)\,u_{\xi\xi\xi}+abu^m u_\xi \;=\; \alpha^2 b^3(2u_\xi u_{\xi\xi}+uu_{\xi\xi\xi}), \tag{4.1}$$

where b and c are constants to be determined. The purpose of this section is to find a transformation between Eq. (4.1) and Eq. (3.2). Furthermore, by this transformation, we can obtain solutions of Eq. (4.1) from those to Eq. (3.2).

The main idea of our method is to expand a solution of Eq. (4.1) in the form

$$u \;=\; \sum_{i=0}^{n} a_i\phi^i, \tag{4.2}$$

where ϕ is a solution of Eq. (3.2). Once the parameters n and a_i are determined, one can obtain the solutions of Eq. (4.1) from already known solutions of Eq. (3.2) easily.

Firstly, to determine the parameter n, we give the following results by (3.2):

$$\frac{d\phi}{d\xi} \;=\; \varepsilon\sqrt{h_1(\phi^4-h_2\phi^2+d_1\phi)+d_2}, \tag{4.3}$$

$$\frac{d}{d\xi} \;\to\; \varepsilon\sqrt{h_1(\phi^4-h_2\phi^2+d_1\phi)+d_2}\,\frac{d}{d\phi}, \tag{4.4}$$

$$\frac{d^2}{d\xi^2} \;\to\; \varepsilon^2\left[\frac{1}{2}h_1(4\phi^3-2h_2\phi+d_1)\frac{d}{d\phi}+(h_1(\phi^4-h_2\phi^2+d_1\phi)+d_2)\frac{d^2}{d\phi^2}\right], \tag{4.5}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where $\varepsilon=\pm 1$. Substituting (4.2) together with (4.3) – (4.5) into Eq. (4.1) and balancing the highest derivative term with the nonlinear convection term, we can obtain $n=\dfrac{2}{m-1}$. Noting that the positive integer n implies $m=2$ or 3, the parameter n must be taken 2 or 1 correspondingly.

Secondly, to determine the parameters a_i, substituting (4.2) into Eq. (4.1) and setting coefficients of all powers of ϕ^i and $\phi^i\sqrt{h_1(\phi^4-h_2\phi^2+d_1\phi)+d_2}$ to zero, we can obtain a system of algebraic equations from which the parameters a_i can be found explicitly.

Based on the above method, we obtain the following results with the aid of *Maple*.

Theorem 1. *For* $m=2$, *corresponding to the traveling wave solution* ϕ *of the mKdV equation* (1.3),

$$u \;=\; 2c\alpha^2-\frac{c}{3b}-\frac{c}{3\alpha^2}-\alpha^2\phi^2 \tag{4.6}$$

is a solution of the generalized DGH equation (1.1), *where* b, c *are arbitrary constants*, $d_1=0$ *and*

$$d_2 = \frac{9bc\alpha^4 - 16ab^2 - 8abc\alpha^2 - ac^2\alpha^4 - 18b^2c^2\alpha^8 - 18b^2\omega\alpha^4}{36b^4\alpha^8}.$$

Theorem 2. *For* $m = 3$ *and* $d_1 \neq 0$, *corresponding to the traveling wave solution* ϕ *of the mKdV equation* (1.3),

$$u = -\frac{2}{\alpha^2} - \frac{c}{4b} + \frac{24c - 24abc - 48b\omega - 6ac^2\alpha^2}{ab\alpha^2 d_1}\phi \tag{4.7}$$

is a solution of the generalized DGH equation (1.1) *with* $m = 3$, *where* d_1 *and* b *are arbitrary*,

$$d_2^2 = \frac{1728(8abc^2 + 16bc\omega + 2ac^3\alpha^2 - 4c^2 - 16b^2\omega^2 - 16ac\omega b^2 - 4ab\omega c^2\alpha^2 - a^2b^2c^3\alpha^6)}{5a^2b^2\alpha^6}$$

and c *satisfies*

$$\alpha^2c^2 + (8b\alpha^2 - 4b^2\alpha^6)c + 16b^2 = 0. \tag{4.8}$$

Theorem 3. *For* $m = 3$ *and* $d_1 = 0$, *corresponding to the traveling wave solution* ϕ *of the mKdV equation* (1.3),

$$u = -\frac{2}{\alpha^2} - \frac{c}{4b} + a_1\phi, \tag{4.9}$$

is a solution of the generalized DGH equation (1.1), *where* $b = \dfrac{4c - a\alpha^2c^2}{4ac + 8\omega}$ *and* c *satisfies* (4.8).

Based on the above facts, combining with the results in Section 3, we can obtain that there exist periodic solutions, kink solutions and smooth solutions with decay to Eq. (1.1) under some parameter conditions.

In the following we will give some examples of explicit solutions.

***Example* 1.** If $d_1 = d_2 = 0$, we can easily get the explicit solution of (3.2). Then substituting the solution into (4.6), we obtain a smooth solution of Eq. (1.1) with $m = 3$

$$u_1 = 2b^3\gamma\left(\alpha^2 - \frac{1}{3b} - \frac{1}{3\alpha^2}\right) - \frac{6\gamma\alpha^2b^2}{a}\operatorname{sech}^2(bx - b^3\gamma t). \tag{4.10}$$

Furthermore, if $b = \dfrac{\alpha^2}{3\alpha^4 - 1}$, (4.10) becomes the solitary wave solution. According to Theorem 1, we find that the parameters of Eq. (1.1) should satisfy

$$9\gamma b^2\alpha^4 - 16a - 8a\gamma b^2\alpha^2 - ab^4\gamma^2\alpha^4 - 18b^6\gamma^2\alpha^8 - 18\omega\alpha^4 = 0.$$

***Example* 2.** If $d_1 = \dfrac{-32b^3\gamma}{a}\sqrt{\dfrac{2\gamma}{a}}$ and $d_2 = 0$, solving (3.2) and using (4.7), we obtain a periodic solution of Eq. (1.1) with $m = 3$

$$u_3 = -\frac{2}{\alpha^2} - b^2\gamma + \frac{8a\gamma b^3 + 6\omega + 48a\alpha^2\gamma^2 b^5 - 31\gamma b^2}{\alpha^2 b^2 \gamma} \frac{\cos^2(bx - 4b^3\gamma t)}{3 - 2\cos^2(bx - 4b^3\gamma t)},$$

where the parameters of Eq. (1.1) satisfy

$$27a(4ab^5\gamma^2 + 2\gamma\omega b^2 + 4ab^7\gamma^3\alpha^2 - 13b^4\gamma^2 - 2ab^3\gamma\,\omega -$$

$$- \omega^2 - 2a\omega b^5\gamma^2\alpha^2 - 2a^2\alpha^6 b^9\gamma^3) - 5b^6\gamma^3\alpha^6 = 0,$$

and b is determined by $\alpha^2\gamma^2 b^4 - \gamma\alpha^6 b^3 + 2\gamma\alpha^2 b^2 + 1 = 0$.

***Example* 3.** If $d_1 = 0$ and $d_2 = -\frac{3\gamma b^2}{2a}$, solving (3.2) and using (4.9), we obtain a kink solution of Eq. (1.1) with $m = 3$

$$u_2 = -\frac{2}{\alpha^2} - \frac{c}{4b} + a_1 \tanh\left[\, bx + 2b^3\gamma t \,\right],$$

where $b = \frac{4c - a\alpha^2 c^2}{4ac + 8\omega}$ and c satisfies (4.8).

5. Conclusions. A new method was devised to construct traveling wave solutions of the complicated nonlinear wave equations from solutions of the simpler equations. Abundant traveling wave solutions can be obtained by this method easily. As an example, we obtained periodic solutions, smooth solutions with decay, solitary solutions and kink solutions of the generalized DGH equation. This new method is direct and efficient. It can be widely applied to other nonlinear wave equations.

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