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Yan Yang, He-guo Liu (Hubei Univ., Wuhan; Hubei Univ. Arts and Sci., Xiangyang, China)

ON SPLIT METACYCLIC GROUPS *

ПРО РОЗЩЕПЛЮВАНІ МЕТАЦИКЛІЧНІ ГРУПИ

We consider sufficient conditions for metacyclic groups to split. Speciifically, we show that a finite metacyclic group of odd order G is split on its cyclic normal subgroup K if K is such that G/K is cyclic and $|K| = \exp G$.

Розглянуто достатні умови для розщеплення метациклічних груп, а саме, показано, що скінченна метациклічна група G непарного порядку розщеплюється на своїй циклічній нормальній підгрупі K, якщо K є такою, що G/K є циклічною та $|K| = \exp G$.

1. Introduction. G is a metacyclic group if and only if there exists a cyclic normal subgroup K of G such that G/K is cyclic, and $G = SK$ is called a metacyclic factorization, when S is cyclic. In particular, if G has a split metacyclic factorization $G = SK$ such that $S \cap K = 1$, then G is called split metacyclic group, otherwise is called nonsplit. C. E. Hempel and Hyo-Seob Sim have given the classifications of metacyclic groups in their papers [1, 2].

Resently, the structure of the automorphism group of metacyclic group have been given much attention. The automorphism groups of the split metacyclic p-groups have been given in [3] for p is odd and in [4] for $p = 2$. And the automorphism groups of the finite split metacyclic groups have been given in [5]. But the case of nonsplit groups is much more complicated than the case of split group, and only the automorphism groups of nonsplit metacyclic p-group of odd order have been given in [6]. Thus it is necessary to determine wether the metacyclic group is split or not, before we study its automorphism. And in this paper, of particular interest are some sufficient coditions to show a special type of finite metacyclic groups are split.

2. Notation and preliminaries. In this section, we present some general facts that will be useful in this paper. We first define some notation which will be kept throughout.

$\pi(G)$: the set of all prime divisors of the order of a finite group G.

$r(p)$: the largest integer i such that p^i divides the positive integer r.

G is a metacyclic group if and only if there exists a cyclic normal subgroup K of G such that G/K is cyclic. And such a subgroup K is called a kernel of G.

Lemma 1. *A p-group P of odd order is metacyclic with a kernel of order p^γ and of index p^α if and only if it has a presentation*

$$P = \langle a, b \,|\, a^{p^\alpha} = b^{p^\beta}, b^{p^\gamma} = 1, b^a = b^{p^\delta + 1} \rangle,$$

where α, β, γ, δ are positive integers such that $\gamma \le \min(\alpha + \delta, \beta + \delta)$.

Lemma 2 ([1], Lemma 2.1)**.** *A group G is metacyclic with a kernel of order γ and of index α if and only if it has a presentation*

$$G = \langle a, b \,|\, a^\alpha = b^\beta, b^\gamma = 1, b^a = b^\delta \rangle,$$

where α, β, γ, δ are positive integers such that $\gamma | \delta^\alpha - 1$ and $\gamma | \beta(\delta - 1)$.

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Lemma 3 ([2], Lemma 3.4). *Let $P = SK$ be a metacyclic factorization of a p-group P of odd order. P is present by the presentation*

$$P = \langle x, y \mid x^{p^\alpha} = y^{p^\beta}, y^{p^\gamma} = 1, y^x = y^{p^\delta+1}\rangle,$$

where

$$p^\alpha = |S\colon S \cap K|, \quad p^\beta = |K\colon S \cap K|, \quad p^\gamma = |K|, \quad p^\delta = |K\colon P'|.$$

Lemma 4 ([5], Corollary 4.1). *Let p be a odd prime, k, s, t be non-negative integers. Then*

(i) $(1 + p^k)^{p^s t} \equiv 1 \pmod{p^{k+s}}$;

(ii) *if $k > 0$, then*

$$1 + r^s + \ldots + r^{s(p^t-1)} \equiv p^t \pmod{p^{t+k}},$$

for $r = 1 + p^k$.

Lemma 5 ([2], Lemma 5.3). *Let G be a metacyclic group with a metacyclic factorization $G = SK$. To each set π of prime numbers, the subgroup $H = S_\pi K_\pi$ is the unique Hall π-subgroup of G such that $S_\pi = S \cap H$ and $K_\pi = K \cap H$, so $H = (S \cap H)(K \cap H)$.*

Definition 1 ([2], Definition 5.4). *Let G be a group with a metacyclic factorization $G = SK$ and let π denote the set $\{p \in \pi(G)\colon G$ has a normal Hall p'-subgroup$\}$. Let H denote the Hall π-subgroup $S_\pi K_\pi$ and let N denote the the Hall π'-subgroup $S_{\pi'} K_{\pi'}$. Then the semidirect decomposition $G = HN$ is called the standard Hall-decomposition for the metacyclic factorization $G = SK$.*

Lemma 6 ([2], Lemma 5.6). *Let $G = HN$ be a Hall-decomposition for the metacyclic factorization $G = SK$ and π=$\{p \in \pi(G)\colon G$ has a normal Hall p'-subgroup$\}$. Then*

(i) $H = S_\pi K_\pi = (S \cap H)(K \cap H)$, *and H is nilpotent;*

(ii) $N = S_{\pi'} K_{\pi'} = (S \cap N)(K \cap N)$, *and $K_{\pi'} = G' \cap N$, $S_{\pi'} \cap K_{\pi'} = 1$;*

(iii) $S_{\pi'} \leq C_N(H)$.

3. Split metacyclic group. ***3.1. Split metacyclic p-group.***

Lemma 7. *If $P = SK$ be a metacyclic factorization of a p-group P of odd order, then* $\exp P = \max(|S|, |K|)$.

Proof. By Lemma 3, P can be represented as

$$P = \langle a, b | a^{p^\alpha} = b^{p^\beta}, b^{p^\gamma} = 1, b^a = b^{p^\delta+1}\rangle,$$

where $|a| = |S|$ and $|b| = |K|$. For any $a^s b^t \in P$, by Lemma 4, we have

$$(a^s b^t)^{p^m} = a^{sp^m} b^{t(1+(p^\delta+1)^s+\ldots+(p^\delta+1)^{s(p^m-1)})} = a^{sp^m} b^{t(p^m+kp^{\delta+m})} = 1,$$

where $p^m = \max(|S|, |K|)$ and k is an integer.

Lemma 8. *P is a metacyclic p-group of odd order. If K is a normal cyclic subgroup with $|K| = \exp P$, then P/K is cyclic.*

Proof. Let

$$P = \langle a, b | a^{p^\alpha} = b^{p^\beta}, b^{p^\gamma} = 1, b^a = b^{p^\delta+1}\rangle,$$

be a presentation of P. Suppose $K = \langle a^m b^n\rangle$, and we observe that $|K| = \exp P = \max(|a|, |b|)$ from Lemma 7. Then by Lemma 4 we have

$$(a^pb^p)^{p^{\gamma-1}} = a^{p^\gamma} b^{p(1+(1+p^\delta)^p+\ldots+(1+p^\delta)^{p(p^{\gamma-1}-1)})} = 1,$$

which shows $(m,p) = 1$ or $(n,p) = 1$. If $(m,p) = 1$, we know that P/K is cyclic since $P = \langle b\rangle\langle a^m b^n\rangle$. If $(n,p) = 1$, P/K is also cyclic since $P = \langle a\rangle\langle a^m b^n\rangle$.

Theorem 1. *Let P be a metacyclic p-group of odd order. If P has a normal cyclic subgroup K of order* $\exp P$, *then P is split.*

Proof. P/K is cyclic by Lemma 8. Let $P = HK$ be a metacyclic factorization of P. Thus by Lemma 3, choosing the generators x and y for H and K, respectively, we have the presentation

$$P = \langle x, y \mid x^{p^\alpha} = y^{p^\beta}, y^{p^\gamma} = 1, y^x = y^{p^\delta+1}\rangle,$$

where

$$p^\alpha = |H\colon H\cap K| = |P|/\exp P, \quad p^\beta = |K\colon H\cap K|,$$

$$p^\gamma = |K| = \exp P, \quad p^\delta = |K\colon P'| = |P/P'|/p^\alpha.$$

If $\beta = \gamma$, then P is split. Now we assume that $\beta < \gamma$. Denote $a = xy^{fp^{\beta-\alpha}}$, where $f = p^{\gamma-\beta} - 1$, then we will show $\langle a\rangle \cap \langle y\rangle = 1$. From Lemma 1 we can obtain

$$a^{p^\alpha} = (xy^{fp^{\beta-\alpha}})^{p^\alpha} = x^{p^\alpha} y^{fp^{\beta-\alpha}(1+(p^\delta+1)+\ldots+(p^\delta+1)^{p^\alpha-1})} = x^{p^\alpha} y^{fp^{\beta-\alpha}p^\alpha} =$$

$$= y^{p^\beta+fp^\beta} = y^{p^\gamma} = 1,$$

and

$$a^{p^{\alpha-1}} = (xy^{fp^{\beta-\alpha}})^{p^{\alpha-1}} = x^{p^{\alpha-1}} y^{fp^{\beta-\alpha}p^{\alpha-1}} \notin K,$$

which implies $\langle a\rangle \cap \langle y\rangle = 1$. $P = \langle a\rangle\langle y\rangle$ is obvious. Thus P is split. Finally, because of

$$y^a = y^{xy^{fp^{\beta-\alpha}}} = y^x = y^{1+p^\delta},$$

we can get the presentation of P

$$P = \langle a, y \mid a^{p^\alpha} = y^{p^\gamma} = 1, y^a = y^{p^\delta+1}\rangle,$$

where $p^\alpha = |P|/\exp P$, $p^\gamma = \exp P$, $p^{\alpha+\delta} = |P/P'|$.

***Example* 1.** The above theorem may not hold for a metacyclic 2-group

$$Q_8 = \langle x, a \mid x^2 = a^2, a^4 = 1, a^x = a^{-1}\rangle,$$

where $|a| = 4 = \exp Q_8$ and $\langle a\rangle \lhd Q_8$, but Q_8 is not a split group.

Applying Theorem 1 and Lemma 1, we can deduce the following corollary.

***Corollary* 1.** *A metacyclic p-group P of odd order is non-split if and only if P has the presentation of the form in Lemma* 1

$$P = \langle a, b \mid a^{p^\alpha} = b^{p^\beta}, b^{p^\gamma} = 1, b^a = b^{p^\delta+1}\rangle,$$

where $\alpha > \beta > \delta \geq 1$ and $\beta < \gamma \leq \beta + \delta$.

3.2. Split metacyclic group. In the remaining part of this section, we will proof that the conclusion above is also hold for a metacyclic group of odd order.

Lemma 9. *Let P be a non-abelian metacyclic p-group of odd order. Suppose $P = SK$ and $P = S_1K_1$ both are the split metacyclic factorizations. Then*

(i) $|S| = |S_1|$ *and* $|K| = |K_1|$;

(ii) $P = SK_1$ *and* $P = S_1K$ *also are the split metacyclic factorizations. Here,* $P = SK$ *is a split metacyclic factorization means* $P = SK$ *is a metacyclic factorization and* $S \cap K = 1$.

Proof. (i) Suppose

$$P = SK = \langle a, b \,|\, a^{p^\alpha} = b^{p^\gamma} = 1, b^a = b^{p^\delta+1}\rangle,$$

$$P = S_1K_1 = \langle a_1, b_1 \,|\, a_1^{p^{\alpha_1}} = b_1^{p^{\gamma_1}} = 1, b_1^{a_1} = b_1^{p^{\delta_1}+1}\rangle.$$

It is obvious that $|P| = p^{\alpha+\gamma} = p^{\alpha_1+\gamma_1}$ and $\exp P = \max(\alpha, \gamma) = \max(\alpha_1, \gamma_1)$. Thus if $\alpha = \alpha_1$, (i) holds by Lemma 7. If $\alpha \neq \alpha_1$, then $\alpha = \gamma_1$ and $\alpha_1 = \gamma$. And we observe that $P/P' \cong$ $\cong Z_{p^\alpha} \times Z_{p^\delta} \cong Z_{p^{\alpha_1}} \times Z_{p^{\delta_1}}$. Consequently,

$$\gamma_1 = \alpha = \delta_1, \quad \gamma = \alpha_1 = \delta,$$

which contradicts that P is non-abelian. Thus $\alpha = \alpha_1$, $\gamma = \gamma_1$, $\delta = \delta_1$.

(ii) Let $b_1 = a^m b^n$, where m, n are non-negative integers. We will show that $(n, p) = 1$. Suppose $p|n$. Then

$$b_1^{p^\delta} = (a^m b^n)^{p^\delta} = b^{p(1+(1+p^\delta)^m+\ldots+(1+p^\delta)^{m(p^\delta-1)})} \in \langle b^{p^{\delta+1}}\rangle,$$

which contradicts that $P' = \langle b^{p^\delta}\rangle = \langle b_1^{p^\delta}\rangle = \langle (a^m b^n)^{p^\delta}\rangle$, since $|K_1/P'| = |K/P'| = p^\delta$. Therefore, $P = \langle a, b\rangle = \langle a, b_1\rangle = SK_1$ and $S \cap K_1 = 1$.

Theorem 2. *Let G be a metacyclic group of odd order and Y is a normal cyclic subgroup of G with $|Y| = \exp G$. Then G/Y is cyclic and G is split .*

Proof. Let $G = HN$ be a Hall-decomposition for a metacyclic factorization $G = SK$, $\pi =$ $= \pi(H)=\{p \in \pi(G)\colon G$ has a normal Hall p'-subgroup$\}$ and $\pi' = \pi(N)$. Thus $S_\pi = S \cap H$, $K_\pi = K \cap H$, $S_{\pi'} = S \cap N$, $K_{\pi'} = K \cap N$, by Lemma 5.

For $p \in \pi$, denote H_p and $Y_p = \langle y_p\rangle$ as the sylow p-subgroup of H and Y, respectively. Thus $Y_p \lhd H_p$ and $|Y_p| = \exp H_p$. Then from Lemma 9, we know H_p is split. Applying Lemma 9, we can find a $x_p \in H_p$, such that $H_p = \langle x_p\rangle\langle y_p\rangle$ is a split metacyclic factorization. Let $x_1 = \prod_{p\in\pi} x_p$, $y_1 = \prod_{p\in\pi} y_p$. Then $H = \langle x_1\rangle\langle y_1\rangle$ is a split metacyclic factorization, since H is nilpotent.

If $N = 1$, then $G = H$ is split. If N is non-trivial, then from Lemma 7, we know $N = S_{\pi'}K_{\pi'}$ is a split metacyclic factorazition, since $S_{\pi'} \cap K_{\pi'} = 1$. Next we will show $N = S_{\pi'}Y_{\pi'}$ is a split metacyclic factorization, where $Y_{\pi'}$ is the Hall π'-subgroup of Y.

For any $q \in \pi'$, denote Y_q and $N_q = S_qK_q$ as the the sylow q-subgroup of Y and N, respectively. Note that $Y_q \lhd N_q$ and $|Y_q| = \exp N_q$. We know Y_q is a kernel of N_q by Theorem 1. This implies that $N_q = S_qY_q$ is a split metacyclic factorization by $S_q \cap K_q = 1$ and Lemma 9. Thus

$$N = \prod_{q\in\pi(N)} N_q = \prod_{q\in\pi(N)} S_qK_q = \prod_{q\in\pi(N)} S_qY_q = \prod_{q\in\pi(N)} S_q \prod_{q\in\pi(N)} Y_q = S_{\pi'}Y_{\pi'}.$$

Futher, $S_{\pi'} \cap Y_{\pi'} = 1$, since $S_q \cap Y_q = 1$.

Let $S_{\pi'} = \langle x_2 \rangle$ and $Y_{\pi'} = \langle y_2 \rangle$, then $S_{\pi'} \leq C_N(H)$ and $Y \lhd G$ yields that

$$G = HN = \langle x_1 \rangle \langle y_1 \rangle \langle x_2 \rangle \langle y_2 \rangle = \langle x_1 \rangle \langle x_2 \rangle \langle y_1 \rangle \langle y_2 \rangle = \langle x_1 x_2 \rangle \langle y_1 y_2 \rangle = XY,$$

where $X = \langle x_1 x_2 \rangle$ and $X \cap Y = 1$.

The above theorem may not hold for a group of even order.

***Example* 2.** Let

$$G \cong Q_8 \times Z_3 = \langle x, b \,|\, x^2 = b^6, b^{12} = 1, b^x = b^7 \rangle,$$

where $|b| = 12 = \exp G$ and $\langle b \rangle \lhd G$, but G is not a split group.

In Theorem 2, if Y is not normal, G may not be a split group.

***Example* 3.** Let

$$G = \langle a, b \,|\, a^{3^4} = b^{3^3}, b^{3^5} = 1, b^a = b^{1+3^2} \rangle.$$

It is obviously that $|a| = \exp G$, but

$$[b, a] = b^{-1} b^a = b^{3^2} \overline{\in} \langle a \rangle$$

shows $\langle a \rangle$ is not normal. And G is non-split.

Lemma 10. *If $G = SK$ is a metacyclic factorization of a metacyclic group G of odd order. Then* $\exp G = \mathrm{lcm}\,(|S|, |K|)$.

Proof. Let $S = \langle x \rangle, K = \langle y \rangle$, we will show that

$$|x^m y^n| \leq \mathrm{lcm}\,(|S|, |K|) \quad \forall x^m y^n \in G.$$

Let $r = \mathrm{lcm}(|S|, |K|)$ and $t = |K|$. Suppose $y^x = y^\theta$, and it is obvious that $\theta^r \equiv 1 (\mathrm{mod}\, |K|)$. Then for $p \in \pi(G)$ and $kp^{r(p)} = r$

$$\theta^r = \theta^{kp^{r(p)}} \equiv 1 \ (\mathrm{mod}\, p^{t(p)}),$$

which implies $\theta^{mkp^{r(p)}} \equiv 1 (\mathrm{mod}\, p^{t(p)})$. Then from Lemma 4 we have

$$1 + \theta^{mk} + \ldots + \theta^{mk(p^{r(p)}-1)} \equiv p^{r(p)} (\mathrm{mod}\, p^{t(p)}).$$

Thus

$$1 + \theta^m + \ldots \theta^{m(r-1)} = \sum_{i=0}^{k-1} \theta^{mi} (1 + \theta^{mk} + \ldots + \theta^{mk(p^{r(p)}-1)}) \Longrightarrow$$

$$\Longrightarrow p^{t(p)} | 1 + \theta^m + \ldots + \theta^{m(r-1)} \Longrightarrow t | 1 + \theta^m + \ldots + \theta^{m(r-1)} \Longrightarrow$$

$$\Longrightarrow (x^m y^n)^r = x^{mr} y^{n(1+\theta^m + \ldots + \theta^{m(r-1)})} = 1.$$

***Corollary* 2.** *Let $G = SK$ be a metacyclic factorization of a metacyclic group G of odd order, and $|K| \big| |S|$. Then G is split.*

***Example* 4.** Let $G = SK = \langle a\rangle\langle b\rangle$ be a metacyclic factorization.

(i) Suppose $G = \langle a, b \mid a^{3^2} = b^{3^3}, b^{3^5} = 1, b^a = b^{1+3^3}\rangle$. Then $|K| = 3^5 = \exp G$ yields that G is split. Let $x = ab^{24}$, and we can get a presentation of G

$$G = \langle x, b \mid x^{3^2} = 1, b^{3^5} = 1, b^a = b^{1+3^3}\rangle.$$

(ii) Suppose $G = \langle a, b \mid a^{3^4} = b^{3^3}, b^{3^5} = 1, b^a = b^{1+3^3}\rangle$. Then $S \lhd G$ and $|S| = 3^6 = \exp G$ yields that G is split, for $P' = \langle b^{3^3}\rangle \lhd S$. Let $x = b^{-1}a^6$, and we can get a presentation of G

$$G = \langle x, a \mid x^{3^3} = 1, a^{3^6} = 1, a^x = b^{1+3^4}\rangle.$$

(iii) Suppose $G = \langle a, b \mid a^{3^3} = b^{3^3\times 7}, b^{3^4\times 7} = 1, b^a = b^{415}\rangle$. Then $|K| = 3^4 \times 7 = \exp G$ yields that G is split. Let $x = ab^{14}$, and we can get a presentation of G

$$G = \langle x, b \mid x^{3^3} = 1, b^{3^4\times 7} = 1, b^x = b^{415}\rangle.$$

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