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WELL-POSEDNESS OF THE RIGHT-HAND SIDE IDENTIFICATION PROBLEM FOR A PARABOLIC EQUATION

КОРЕКТНІСТЬ ПРОБЛЕМИ ПРАВОСТОРОННЬОЇ ІДЕНТИФІКАЦІЇ ПАРАБОЛІЧНОГО РІВНЯННЯ

We study the inverse problem of reconstruction of the right-hand side of a parabolic equation with nonlocal conditions. The well-posedness of this problem in Hölder spaces is established.

Досліджено обернену задачу відновлення правої частини параболічного рівняння з нелокальними умовами. Встановлено коректність цієї задачі у просторах Гьольдера.

1. Introduction. The inverse problems take an important place in various fields of science and engineering and have been studied by different authors [1–7]. The optimal overdetermination conditions are analyzed in some classical boundary conditions or and similar conditions given at a point. The literature review and various approaches for the approximate solution are given in [8–11]. Moreover, the generalized overdetermination conditions such as nonlocal, integral, and final overdetermination conditions are used [12–15].

The importance of well-posedness has been widely recognized by the researchers in the field of partial differential equations [16–22]. Moreover, the well-posedness of the right-hand side identification problem for a parabolic equation where the unknown function p is in space variable is also well investigated [23–28].

Let us give a brief summary of papers with investigations on the right-hand side identification problem for a parabolic equation where the unknown function p is in time variable and the motivation of our present paper.

In article [29], a conditional stability of Hölder type (estimate in L_p-norm) of the inverse problem of determining $p(t)$, $0<t<T$, in the heat source of the heat equation

$$\partial_t u(x,t) = \triangle u(x,t) + p(t)q(x), \qquad x\in\mathbb{R}^n, \quad t>0, \tag{1}$$

from the observation $u(x_0,t)$, $0<t<T$, at a remote point x_0 away from the support of q is established by using Hölder type inequalities.

The numerical algorithm for solving inverse problem of reconstructing a distributed right-hand side of a parabolic equation with local boundary conditions is studied in [30] and [31]. In these articles, the numerical solution of the identification problem and well-posedness of the algorithm is presented. For reconstructing the right-hand side function $f(t,x)=p(t)q(x)$, where $p(t)$ is the unknown function, the solution is observed in the form of $u(t,x)=\eta(t)q(x)+w(t,x)$, where $\eta(t)=\int_0^t p\,(s)\,ds$. Then, an approximation is given for $w(t,x)$ via fully implicit difference scheme.

The solution of system constructed by the difference scheme is searched in the form

$$w_i^{n+1} = y_i + w_k^{n+1} z_i, \quad i = 0, 1, \ldots, M,$$

where k is an interior grid point and the well-posedness of the algorithm is given by a priori estimate

$$\max_{0 \le i \le M} |z_i| \le \tau \max_{0 < i < M} \left| \frac{1}{\psi_k} (a\psi_{\overline{x}})_{x,i} \right|$$

which is based on maximum principle. Thus, in [30] $|z_i| < 1$ at small enough $\tau = O(1)$, i.e., it is necessary to use a sufficiently small time step.

In the papers [32–35], inverse problems of reconstructing a distributed right-hand side of a parabolic equation were studied by using the methods of differential equations in Banach and Hilbert spaces. The solvability of these inverse problems is dealt with by reducing it to an abstract Volterra operator of the second kind. In the paper [34], the inverse problem of the form

$$\begin{aligned} &u'(t) - Au(t) = \Phi(t)p(t) + f(t), \quad 0 \le t \le T, \\ &u(0) = u_0, \qquad \varphi\big(u(t)\big) = \psi(t), \quad 0 \le t \le T, \end{aligned} \tag{2}$$

was considered. Here u_0 is an element, A and φ are linear operators and $\Phi(\Delta)$, $f(\Delta)$, $\psi(\Delta)$ are given functions and $u(\Delta)$ and $p(\Delta)$ are the unknowns. It was assumed that A generates a strongly continuous semigroup in a Banach space X, $u_0 \in E$, $\varphi \in L(E, F)$, $\Phi(\Delta) \in C([0, T], L(F, E))$, $F(\Delta) \in C([0, T], E)$ and $\psi(\cdot) \in C^1([0, T], E)$. The author gives conditions on the coefficients that insure existence, uniqueness and continuous dependence of solutions $(u(\Delta), p(\Delta))$ on the data.

In the paper [35], a problem of determining the right-hand side of a uniformly multidimensional parabolic equation in a bounded domain with the Dirichlet boundary condition was studied. Additional information was given in integral form. An existence and uniqueness theorem for the solution of the inverse problem in the Hölder class was proven and a sufficient condition for the differentiability of the solution was given.

In the present paper, we investigate the well-posedness of the inverse problem of reconstructing the right-hand side of a one dimensional parabolic equation with nonlocal conditions

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = a(x)\frac{\partial^2 u(t,x)}{\partial x^2} - \sigma u(t,x) + p(t)q(x) + f(t,x), \\ &0 < x < l, \quad 0 < t \le T, \\ &u(t,0) = u(t,l), \quad u_x(t,0) = u_x(t,l), \quad 0 \le t \le T, \\ &u(0,x) = \varphi(x), \quad 0 \le x \le l, \\ &u\left(t, x^*\right) = \rho(t), \quad 0 \le x^* \le l, \quad 0 \le t \le T, \end{aligned} \tag{3}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $a(x) \ge \delta > 0$ and $\sigma > 0$ is a sufficiently large number with assuming that

(a) $q(x)$ is a sufficiently smooth function,

(b) $q(x)$ and $q'(x)$ are periodic with length l,

(c) $q\left(x^*\right) \neq 0$.

At the end of the section, we give a brief comparison of them with our work.

The aim of the paper is to give well-posedness theorems and their proof for the inverse problem of reconstructing the right-hand side of a parabolic equation with nonlocal conditions. In contrast to [29], we establish the coercive inequality for the solution of problem (3) in $\left(C\left([0,T], \overset{\circ}{C}{}^{2\alpha}[0,l]\right),\ C[0,T]\right)$. Comparing to stability results of [30] for the solution of difference schemes, we give the well-posedness in differential case and for the solution of problem (3) with nonlocal conditions. The application of operator tools permits to investigate inverse problems of reconstructing a distributed right-hand side of a multidimensional parabolic equation more general case than problem (3) with classical boundary conditions (see, for example, [32 – 35]).

In general, it would be interesting to establish the coercive inequality for the solution of problems of reconstructing a distributed right-hand side of a multidimensional parabolic equation more general case than problem (3). Of course, it will be possible after establishing theorem on well-posedness of abstract problem (2) in $\left(C\left([0,T], E_a\right),\ C[0,T]\right)$ and theorem on structure of interpolation spaces $E_a = E_a(E,A)$ generated by multidimensional space operator A. Unfortunately (see [36]), structure of interpolation spaces $E_a = E_a(E,A)$ generated by multidimensional space operator A with boundary local and nonlocal conditions is not well investigated.

2. Main results. To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}^{\alpha}[0,l]$, $\alpha \in (0,1)$, of all continuous functions $\phi(x)$ defined on $[0,l]$ with $\phi(0) = \phi(l)$ satisfying a Hölder condition for which the following norm is finite:

$$\|\phi\|_{\overset{\circ}{C}{}^{\alpha}[0,l]} = \max_{0<x<l} |\phi(x)| + \sup_{0<x<x+h<l} \frac{|\phi\,(x+h) - \phi(x)|}{h^{\alpha}}.$$

With the help of A we introduce the fractional spaces E_α, $0 < \alpha < 1$, consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_{E_\alpha} = \|v\|_E + \sup_{\lambda>0} \lambda^{1-\alpha} \left\|A \exp\left\{-\lambda A\right\} v\right\|_E . \tag{4}$$

It is known that under the assumption that the operator $-A$ generates an analytic semigroup $\exp\{-tA\}$, $t > 0$, with exponentially decreasing norm, when $t \to +\infty$, i.e., the following estimates hold:

$$\|\exp\{-tA\}\|_{E\to E} \le Me^{-\delta t}, \tag{5}$$

$$\|A^{\alpha} \exp\{-tA\}\|_{E\to E} \le Me^{-\delta t} t^{-\alpha}, \tag{6}$$

where $t, \delta, M > 0$ [36].

Positive constants, which can be differ in time will be indicated with an M. On the other hand $M\left(\alpha, \beta, \ldots\right)$ is used to focus on the fact that the constant depends only on $\alpha, \beta, \ldots$.

Theorem 1. *Let* $\varphi(x) \in \overset{\circ}{C}{}^{2\alpha+2}[0,l]$, $f(t,x) \in C\left([0,T], \overset{\circ}{C}{}^{2\alpha}[0,l]\right)$ *and* $\rho'(t) \in C[0,T]$. *Then for the solution of problem* (3), *the following coercive stability estimates*:

$$\|u_t\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} + \|u\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha+2}[0,l]\right)} \le M\left(x^*,q\right)\left\|\rho'\right\|_{C[0,T]} +$$

$$+M\left(a,\delta,\sigma,\alpha,x^*,q,T\right)\left(\|\varphi\|_{\overset{\circ}{C}^{2\alpha+2}[0,l]} + \|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} + \|\rho\|_{C[0,T]}\right),$$

$$\|p\|_{C[0,T]} \le M\left(x^*,q\right)\left\|\rho'\right\|_{C[0,T]} +$$

$$+M\left(a,\delta,\sigma,\alpha,x^*,q,T\right)\left[\|\varphi\|_{\overset{\circ}{C}^{2\alpha+2}[0,l]} + \|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} + \|\rho\|_{C[0,T]}\right]$$

hold.

Proof. Let us search for the solution of the inverse problem in the following form:

$$u(t,x) = \eta(t)q(x) + w\left(t,x\right), \tag{7}$$

where

$$\eta(t) = \int_0^t p\left(s\right)ds.$$

Taking derivatives from (7), we have

$$\frac{\partial u(t,x)}{\partial t} = p(t)q\left(x\right) + \frac{\partial w(t,x)}{\partial t}$$

and

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \eta(t)\frac{d^2q(x)}{dx^2} + \frac{\partial^2 w(t,x)}{\partial x^2}.$$

Moreover if we substitute $x = x^*$ in equation (7), we get

$$u\left(t,x^*\right) = \eta(t)q\left(x^*\right) + w\left(t,x^*\right) = \rho(t)$$

and

$$\eta(t) = \frac{\rho(t) - w\left(t,x^*\right)}{q\left(x^*\right)}. \tag{8}$$

Taking derivative of both sides, we obtain

$$p(t) = \frac{\rho'(t) - w_t\left(t,x^*\right)}{q\left(x^*\right)}. \tag{9}$$

Using the triangle inequality and the identity (9), we have

$$|p(t)| = \left|\frac{\rho'(t) - w_t\left(t,x^*\right)}{q\left(x^*\right)}\right| \le M(x^*,q)\left(\left|\rho'(t)\right| + \left|w_t\left(t,x^*\right)\right|\right) \le$$

$$\le M(x^*,q)\left(\max_{0\le t\le T}\left|\rho'(t)\right| + \max_{0\le t\le T}\max_{0\le x\le l}\left|w_t(t,x)\right|\right) \le$$

$$\leq M(x^*, q)\left(\max_{0\leq t\leq T}\left|\rho'(t)\right| + \max_{0\leq t\leq T}\|w_t(t)\|_{\overset{\circ}{C}^{2\alpha}[0,l]}\right) \tag{10}$$

for any $t, t \in [0,T]$. Using equations (7), (8) and under the same assumptions on $q(x)$, one can show that $w(t,x)$ is the solution of the following problem:

$$\frac{\partial w(t,x)}{\partial t} = a(x)\frac{\partial^2 w(t,x)}{\partial x^2} + a(x)\frac{\rho(t) - w(t,x^*)}{q(x^*)}\frac{d^2 q(x)}{dx^2} -$$

$$-\sigma\frac{\rho(t) - w(t,x^*)}{q(x^*)}q(x) - \sigma w(t,x) + f(t,x), \quad 0 < x < l, \quad 0 < t \leq T, \tag{11}$$

$$w(t,0) = w(t,l), \quad w_x(t,0) = w_x(t,l), \quad 0 \leq t \leq T,$$

$$w(0,x) = \varphi(x), \quad 0 \leq x \leq l.$$

So, the end of proof of Theorem 1 is based on estimate (10) and the following theorem.

Theorem 2. *For the solution of problem* (11), *the following coercive stability estimate*:

$$\|w_t\|_{\overset{\circ}{C}^{2\alpha}[0,l]} \leq M(a,\delta,\sigma,\alpha,x^*,q,T)\left(\|\varphi\|_{\overset{\circ}{C}^{2\alpha+2}[0,l]} + \|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,l]\right)} + \|\rho\|_{C[0,T]}\right)$$

holds.

Proof. We can rewrite problem (11) in the abstract form

$$w_t + Aw = \left(aq'' - \sigma q\right)\frac{\rho(t) - w(t,x^*)}{q(x^*)} + f(t), \quad 0 < t \leq T,$$

$$w(0) = \varphi$$

in a Banach space $E = \overset{\circ}{C}[0,l]$ with the positive operator A defined by

$$Au = -a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \sigma u$$

with

$$D(A) = \left\{u(x)\colon u, u', u'' \in C[0,l],\ u(0) = u(l),\ u_x(0) = u_x(l)\right\}.$$

Here, $f(t) = f(t,x)$ and $w(t) = w(t,x)$ are known and unknown abstract functions defined on $[0,T]$ with values in $E = \overset{\circ}{C}[0,l]$, $w(t,x^*)$ is unknown scalar function defined on $[0,T]$, $q = q(x)$, $q'' = q''(x)$, $\varphi = \varphi(x)$ and $a = a(x)$ are elements of $E = \overset{\circ}{C}[0,l]$ and $q(x^*)$ is a number.

By the Cauchy formula, the solution can be written as

$$w(t) = e^{-tA}\varphi - \int_0^t e^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}w(s,x^*)\,ds +$$

$$+\int_0^t e^{-(t-s)A}\frac{\rho(s)(aq'' - \sigma q)}{q^*}ds + \int_0^t e^{-(t-s)A}f(s)\,ds.$$

Taking the derivative of both sides, we obtain that

$$w_t(t) = -Ae^{-tA}\varphi + \int_0^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}w\left(s, x^*\right)ds -$$

$$-\int_0^t Ae^{-(t-s)A}\frac{\rho\left(s\right)\left(aq'' - \sigma q\right)}{q^*}ds - \int_0^t Ae^{-(t-s)A}f\left(s\right)ds.$$

Applying the formula

$$\int_0^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}w\left(s, x^*\right)ds =$$

$$= \int_0^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}\int_0^s w_z\left(z, x^*\right)dzds + \int_0^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}\varphi\left(x^*\right)ds$$

and changing the order of integration, we obtain that

$$\int_0^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}w\left(s, x^*\right)ds =$$

$$= \int_0^t \int_z^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}w_z\left(z, x^*\right)dsdz + \int_0^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}\varphi\left(x^*\right)ds.$$

Then, the following presentation of the solution of (11):

$$w_t(t) = Ae^{-tA}\varphi + \int_0^t \int_z^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}w_z\left(z, x^*\right)dsdz +$$

$$+\int_0^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}\varphi\left(x^*\right)ds - \int_0^t Ae^{-(t-s)A}\frac{\rho\left(s\right)\left(aq'' - \sigma q\right)}{q^*}ds -$$

$$-\int_0^t Ae^{-(t-s)A}f\left(s\right)ds = \sum_{k=1}^{5} G_k(t)$$

is obtained. Here,

$$G_1(t) = Ae^{-tA}\varphi,$$

$$G_2(t) = \int_0^t \int_z^t Ae^{-(t-s)A}\frac{aq'' - \sigma q}{q^*}w_z\left(z, x^*\right)dsdz,$$

$$G_3(t) = \int_0^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \varphi(x^*)\, ds,$$

$$G_4(t) = -\int_0^t Ae^{-(t-s)A} \frac{\rho(s)(aq'' - \sigma q)}{q^*} ds,$$

$$G_5(t) = -\int_0^t Ae^{-(t-s)A} f(s)\, ds.$$

It is very well known that, from the fact that the operators $R, \exp\{-\lambda A\}$ and A commute, it follows that [36]

$$\|R\|_{E_\alpha \to E_\alpha} \le \|R\|_{E \to E}\,. \tag{12}$$

Now, let us estimate $G_k(t)$ for any $k = 1, 2, 3, 4, 5$ separately. Applying the definition of norm of the spaces E_α and (12), we get

$$\|G_1(t)\|_{E_\alpha} = \left\|Ae^{-tA}\varphi\right\|_{E_\alpha} \le \left\|e^{-tA}\right\|_{E_\alpha \to E_\alpha} \|A\varphi\|_{E_\alpha} \le \left\|e^{-tA}\right\|_{E \to E} \|A\varphi\|_{E_\alpha}\,.$$

Using estimate (5), we get

$$\|G_1(t)\|_{E_\alpha} \le M_1 \|A\varphi\|_{E_\alpha} \tag{13}$$

for any t, $t \in [0, T]$. Let us estimate $G_2(t)$

$$\|G_2(t)\|_{E_\alpha} = \left\| \int_0^t \int_z^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} w_z(z, x^*)\, ds dz \right\|_{E_\alpha} \le$$

$$\le \int_0^t \int_z^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} ds\, |w_z(z, x^*)|\, dz.$$

By equation (4), we have that

$$\int_z^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} ds = \int_z^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds +$$

$$+ \sup_{\lambda > 0} \int_z^t \left\| \lambda^{1-\alpha} Ae^{-\lambda A} Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds.$$

By the definition of norm of the spaces E_α, we get

$$\int_z^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds =$$

$$= \int\limits_{z}^{t} (t-s)^{\alpha-1} \left\| (t-s)^{1-\alpha} A e^{-(t-s)A} \frac{aq''-\sigma q}{q^*} \right\|_E ds \leq$$

$$\leq \int\limits_{z}^{t} (t-s)^{\alpha-1} ds \left\| \frac{aq''-\sigma q}{q^*} \right\|_{E_\alpha} \leq \frac{T^\alpha}{\alpha} \left\| \frac{aq''-\sigma q}{q^*} \right\|_{E_\alpha} =$$

$$= M_2 \left(a, \sigma, \alpha, x^*, q, T\right).$$

Using estimate (6), we can obtain that

$$\int\limits_{z}^{t} \left\| \lambda^{1-\alpha} A e^{-\lambda A} A e^{-(t-s)A} \frac{aq''-\sigma q}{q^*} \right\|_E ds \leq$$

$$\leq \int\limits_{z}^{t} \frac{2^{2-\alpha}\lambda^{1-\alpha}}{(\lambda+t-s)^{2-\alpha}} ds \left\| \frac{\lambda+t-s}{2} A e^{-\frac{\lambda+t-s}{2}A} \right\|_{E\to E} \times$$

$$\times \left\| \left(\frac{\lambda+t-s}{2}\right)^{1-\alpha} A e^{-\frac{\lambda+t-s}{2}A} \frac{aq''-\sigma q}{q^*} \right\|_E ds \leq$$

$$\leq M_3(\alpha) \left\| \frac{aq''-\sigma q}{q^*} \right\|_{E_\alpha} \int\limits_{z}^{t} \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{2-\alpha}} ds \leq$$

$$\leq M_3(\alpha) \left\| \frac{aq''-\sigma q}{q^*} \right\|_{E_\alpha} \left(\frac{\lambda^{1-\alpha}}{(1-\alpha)(\lambda+t-z)^{1-\alpha}} \right)$$

for any $\lambda > 0$. Then,

$$\sup_{\lambda>0} \int\limits_{z}^{t} \left\| \lambda^{1-\alpha} A e^{-\lambda A} A e^{-(t-s)A} \frac{aq''-\sigma q}{q^*} \right\|_E ds \leq$$

$$\leq M_3(\alpha) \left\| \frac{aq''-\sigma q}{q^*} \right\|_{E_\alpha} \frac{1}{(1-\alpha)} = M_4 \left(a, \sigma, \alpha, x^*, q\right).$$

Then, we get

$$\int\limits_{z}^{t} \left\| A e^{-(t-s)A} \frac{aq''-\sigma q}{q^*} \right\|_{E_\alpha} ds \leq M_5 \left(a, \sigma, \alpha, x^*, q, T\right) \tag{14}$$

for any s, $0 \leq z \leq s \leq t$ and

$$\|G_2(t)\|_{E_\alpha} \leq M_6 \left(a, \sigma, \alpha, x^*, q, T\right) \int\limits_{0}^{t} \left| w_z\left(z, x^*\right) \right| dz. \tag{15}$$

$G_3(t)$ is estimated as follows:

$$\|G_3(t)\|_{E_\alpha} = \left\| \int_0^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \varphi(x^*)\, ds \right\|_{E_\alpha} \le$$

$$\le \left\| \int_0^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} ds \right\|_{E_\alpha} |\varphi(x^*)| .$$

Since

$$|\varphi(x^*)| \le \|\varphi\|_E \le \|\varphi\|_{E_\alpha} \le M \|A\varphi\|_{E_\alpha}$$

and using estimate (14) and choosing $z = 0$, we obtain

$$\|G_3(t)\|_{E_\alpha} \le M_7(a, \sigma, \alpha, x^*, q, T) \|A\varphi\|_{E_\alpha} \tag{16}$$

for any $t \in [0, T]$.

By estimate (14), the estimation of $G_4(t)$ is as follows:

$$\|G_4(t)\|_{E_\alpha} = \left\| \int_0^t Ae^{-(t-s)A} \rho(s) \frac{aq'' - \sigma q}{q^*} ds \right\|_{E_\alpha} \le$$

$$\le \int_0^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} ds\, \|\rho\|_{C[0,T]} \le$$

$$\le M_8(a, \sigma, \alpha, x^*, q, T) \|\rho\|_{C[0,T]} . \tag{17}$$

Now, let us estimate $G_5(t)$. By the definition of the norm of the spaces E_α, we get

$$\|G_5(t)\|_{E_\alpha} = \left\| \int_0^t Ae^{-(t-s)A} f(s)\, ds \right\|_{E_\alpha} =$$

$$= \left\| \int_0^t Ae^{-(t-s)A} f(s)\, ds \right\|_E + \sup_{\lambda > 0} \lambda^{1-\alpha} \left\| Ae^{-\lambda A} \int_0^t Ae^{-(t-s)A} f(s)\, ds \right\|_E .$$

Using equation (4), we have that

$$\left\| \int_0^t Ae^{-(t-s)A} f(s)\, ds \right\|_E = \int_0^t (t-s)^{\alpha-1} \left\| (t-s)^{1-\alpha} Ae^{-(t-s)A} f(s) \right\|_E ds \le$$

$$\le \int_0^t (t-s)^{\alpha-1} ds\, \|f\|_{C(E_\alpha)} = \frac{t^\alpha}{\alpha} \|f\|_{C(E_\alpha)} \le M_9(\alpha, T) \|f\|_{C(E_\alpha)} . \tag{18}$$

Now, we consider the second term. Using equation (4), we obtain

$$\lambda^{1-\alpha}\left\| Ae^{-\lambda A}\int\limits_0^t Ae^{-(t-s)A}f(s)\,ds\right\|_E \le$$

$$\le \lambda^{1-\alpha}\int\limits_0^t \left(\frac{t-s+\lambda}{2}\right)^{\alpha-1}\left(\frac{t-s+\lambda}{2}\right)^{-1}\left\|\frac{t-s+\lambda}{2}Ae^{-\frac{t-s+\lambda}{2}A}\right\|_{E\to E}\times$$

$$\times\left\|\left(\frac{t-s+\lambda}{2}\right)^{1-\alpha}Ae^{-\frac{t-s+\lambda}{2}A}f(s)\right\|_E ds \le$$

$$\le M_{10}\lambda^{1-\alpha}\int\limits_0^t \left(\frac{t-s+\lambda}{2}\right)^{\alpha-2}\|f\|_{E_\alpha}ds \le M_{10}\lambda^{1-\alpha}\int\limits_0^t \left(\frac{t-s+\lambda}{2}\right)^{\alpha-2}ds\,\|f\|_{C(E_\alpha)}$$

for any $\lambda > 0$. Then,

$$\sup_{\lambda>0}\lambda^{1-\alpha}\left\| Ae^{-\lambda A}\int\limits_0^t Ae^{-(t-s)A}f(s)\,ds\right\|_E \le \frac{M_{10}2^{1-\alpha}}{1-\alpha}\|f\|_{C(E_\alpha)} = M_{11}(\alpha)\|f\|_{C(E_\alpha)}. \quad (19)$$

By estimates (18) and (19), we get

$$\|G_5(t)\|_{E_\alpha} \le M_{12}(\alpha,T)\|f\|_{C(E_\alpha)}. \quad (20)$$

Combining estimates (13), (15), (16), (17) and (20) we have

$$\|w_t\|_{E_\alpha} \le M_1\|A\varphi\|_{E_\alpha} + M_6(a,\sigma,\alpha,x^*,q,T)\int\limits_0^t |w_z(z,x^*)|\,dz+$$

$$+M_7(a,\sigma,\alpha,x^*,q,T)\|A\varphi\|_{E_\alpha} + M_8(a,\sigma,\alpha,x^*,q,T)\|\rho\|_{C[0,T]}+$$

$$+M_{12}(\alpha,T)\|f\|_{C(E_\alpha)}.$$

Using integral inequality, we can write,

$$\|w_t\|_{E_\alpha} \le e^{M_6(a,\sigma,\alpha,x^*,q,T)}\Big[M_1\|A\varphi\|_{E_\alpha} + M_7(a,\sigma,\alpha,x^*,q,T)\|A\varphi\|_{E_\alpha}+$$

$$+M_8(a,\sigma,\alpha,x^*,q,T)\|\rho\|_{C[0,T]} + M_{12}(\alpha,T)\|f\|_{C(E_\alpha)}\Big].$$

Then, the following theorem finishes the proof of Theorem 2.

Theorem 3 [37]. *For $0 < \alpha < \dfrac{1}{2}$ the norms of the spaces $E_\alpha(C[0,l],A)$ and $C^{2\alpha}[0,l]$ are equivalent.*

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