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PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PULSE INFLUENCE IN A BANACH SPACE*

ПЕРІОДИЧНІ РОЗВ'ЯЗКИ НЕЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ІМПУЛЬСНОЮ ДІЄЮ У БАНАХОВОМУ ПРОСТОРІ*

Rank conditions for control of linear pulse systems are established. An example of control synthesis in a problem for linear pulse systems is given.

Встановлено рангові ознаки керування для лінійних імпульсних систем. Наведено приклад синтезу керування в задачі для лінійних імпульсних систем.

In finite-dimensional spaces, periodic, almost periodic, and bounded solutions of differential equations with pulse influence were thoroughly investigated (see, e.g., [1–8]). Much less attention has been paid up to now to the equations in infinite-dimensional spaces [9–12].

The theory of discontinuous dynamical systems is quickly developing now and its applications [13–17] demonstrate the necessity of the study of differential equations with pulse influence in the infinite-dimensional spaces as well.

This paper deals with the study of one of the simplest problems, namely, with the problem of periodic solutions which is quite important for applications. By using the methods of functional analysis, we establish conditions under which nonlinear differential equations in a Banach space subjected to the pulse influence (both at fixed and nonfixed moments of time) possess periodic solutions.

1. The Object of Investigation. Assume that E is an arbitrary Banach space with norm $\|\cdot\|_E$, $L(E,E)$ is an algebra of all linear continuous operators $A: E \to E$, $\mathbb{R}$ is a set of all real numbers, and $\mathbb{Z}$ is a set of all integers.

Consider a system of equations

$$\begin{cases} \dfrac{dx}{dt} = A(t)x + f(t,x), & t \neq \tau_i + \varepsilon\tau_i(x), \\ \Delta x|_{t=\tau_i+\varepsilon\tau_i(x)} = (B_i x + J_i(x))|_{t=\tau_i+\varepsilon\tau_i(x)-0}, & i \in \mathbb{Z}, \end{cases} \tag{1}$$

where $\Delta x|_{t=\tau} = x(\tau+0) - x(\tau-0)$, $A(t)$ is an ω-periodic $L(E,E)$-valued function continuous on $\mathbb{R}$; $f(t,x)$ is an E-valued function continuous on $\mathbb{R}\times E$ and ω-periodic in t for all $x \in E$; $B_i \in L(E,E)$ $\forall i \in \mathbb{Z}$, $J_i(x)$ is an E-valued function continuous in x on E and having values in $\mathbb{R}$ for every $i \in \mathbb{Z}$; $\varepsilon \geq 0$, and

$$B_{i+p} = B_i \quad \forall i \in \mathbb{Z}, \quad J_{i+p}(x) = J_i(x) \quad \forall i \in \mathbb{Z}, \quad x \in E,$$

$$\tau_{i+p} = \tau_i + \omega \quad \forall i \in \mathbb{Z}, \quad \tau_{i+p}(x) = \tau_i(x) + \omega \quad \forall i \in \mathbb{Z}, \quad x \in E$$

for some positive integer p.

In this paper, we investigate the problem of ω-periodic solutions of the system of equations (1) under some additional restrictions on $f(t,x)$, $\tau_i(x)$, and $J_i(x)$ presented below. Note that this problem was studied in the papers [1, 7] for the case where $E = R^n$, and the functions $f(t,x)$, $J_i(x)$, and $\tau_i(x)$ satisfy the Lipschitz conditions.

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2. Periodic Solutions of System (1) for $\varepsilon = 0$. Let $T = \{\tau_i : i \in \mathbb{Z}\}$. Denote by $\mathcal{P}^{(0)}$ a Banach space of all continuous ω-periodic E-valued functions $x = x(t)$, defined on $\mathbb{R} \setminus T$, for which there exist $x(\tau_i + 0)$ and $x(\tau_i - 0)$ (for all $i \in \mathbb{Z}$) having the norm $\| x \|_{\mathcal{P}^{(0)}} = \sup_{t \in \mathbb{R} \setminus T} \| x(t) \|_E$. By $\mathcal{P}^{(1)}$ we denote the Banach space of all functions $x = x(t) \in \mathcal{P}^{(0)}$, for which $dx(t)/dt \in \mathcal{P}^{(0)}$ with a norm

$$\| x \|_{\mathcal{P}^{(1)}} = \| x \|_{\mathcal{P}^{(0)}} + \left\| \frac{dx(t)}{dt} \right\|_{\mathcal{P}^{(0)}}.$$

Let $\mathcal{Y}$ be a Banach space of bilateral p-periodic sequences $g = g_i \in E$ $(i \in \mathbb{Z})$ with the norm $\| g \|_{\mathcal{Y}} = \max_{i \in \mathbb{Z}} \| g_i \|_E$.

We define operators $\mathfrak{U} : \mathcal{P}^{(1)} \to \mathcal{P}^{(0)} \times \mathcal{Y}$ and $\mathfrak{N} : \mathcal{P}^{(0)} \to \mathcal{P}^{(0)} \times \mathcal{Y}$ by $\mathfrak{U}x = (\alpha, \beta)$, $\mathfrak{N}x = -(\gamma, \delta)$, where

$$\alpha(t) = \frac{dx(t)}{dt} - A(t)x(t) \in \mathcal{P}^{(0)}, \quad \beta_i = \Delta x(t)|_{t=\tau_i} - B_i x(\tau_i - 0) \in \mathcal{Y},$$

$\gamma(t) = f(t, x(t)) \in \mathcal{P}^{(0)}$, and $\delta_i = J_i(x(\tau_i - 0)) \in \mathcal{Y}$. These operators enable us to reduce the problem of ω-periodic solutions of the system of equations (1) with $\varepsilon = 0$ to the problem of existence of solutions of the equation

$$(\mathfrak{U} + \mathfrak{N})x = 0 \tag{2}$$

in the space $\mathcal{P}^{(1)}$.

Let $R(\mathfrak{U}) = \{ \mathfrak{U}x : x \in \mathcal{P}^{(1)} \}$, $\operatorname{Ker} \mathfrak{U} = \{ x \in \mathcal{P}^{(1)} : \mathfrak{U}x = 0 \}$ and $\|(a,b)\|_{\mathcal{P}^{(0)} \times \mathcal{Y}} = \| a \|_{\mathcal{P}^{(0)}} + \| b \|_{\mathcal{Y}}$.

Theorem 1. *Assume that*

(*i*) $R(\mathfrak{U})$ *is closed*;

(*ii*) $\operatorname{Ker} \mathfrak{U}$ *possesses a closed complementary space*;

(*iii*) $R(\mathfrak{N}) \subset R(\mathfrak{U})$;

(*iv*) $\| \mathfrak{N}x - \mathfrak{N}y \|_{\mathcal{P}^{(0)} \times \mathcal{Y}} \le M \| x - y \|_{\mathcal{P}^{(0)}}$ *for all* $x; y \in \mathcal{P}^{(0)}$ *and some* $M > 0$.

Then, for all sufficiently small M, *a set of solutions of equation* (2) *is nonempty.*

Proof. We write $\mathcal{P}^{(1)}$ as a direct sum $\mathcal{P}^{(1)} = \operatorname{Ker} \mathfrak{U} \oplus \mathcal{P}$, where $\mathcal{P}$ is a closed space complementary to $\operatorname{Ker} \mathfrak{U}$. Denote by $\mathfrak{U}_1$ the restriction of the operator $\mathfrak{U}$ to the subspace $\mathcal{P}$. We represent each element $x \in \mathcal{P}^{(1)}$ in the form $x = u + v$, where $u \in \operatorname{Ker} \mathfrak{U}$ and $v \in \mathcal{P}$. Then equation (2) is equivalent to the equation

$$\mathfrak{U}_1 v + \mathfrak{N}(u + v) = 0. \tag{3}$$

By virtue of the Banach theorem on an inverse operator [18, p. 225], the operator $\mathfrak{U}_1 : \mathcal{P} \to R(\mathfrak{U})$ possesses the inverse operator $\mathfrak{U}_1^{-1}$, since $(\operatorname{Ker} \mathfrak{U}_1) \cap \mathcal{P} = \{0\}$ and $R(\mathfrak{U}_1) = R(\mathfrak{U}) = \overline{R(\mathfrak{U})}$. Therefore, equation (3) is equivalent to the equation

$$v + \mathfrak{U}_1^{-1} \mathfrak{N}(u + v) = 0. \tag{4}$$

If $M \| \mathfrak{U}_1^{-1} \| < 1$, then we can apply to equation (1) the principle of contracting mappings [18, p. 73]. This guarantees the existence of a unique solution $v \in \mathcal{P}$ for each fixed $u \in \operatorname{Ker} \mathfrak{U}_1$. Thus, equation (2) possesses at least one solution $x \in \mathcal{P}^{(1)}$. Theorem 1 is proved.

***Remarks.* 1.** In Theorem 1, condition (iv) can be replaced by the following one:

$$\max_{i\in[0,\omega)} \| f(t,x) - f(t,y) \|_E + \max_{i=\overline{1,p}} \| J_i(x) - J_i(y) \|_E \le M \| x - y \|_E \qquad (5)$$

$$\forall x, y \in E \quad (M > 0).$$

2. If the second condition in Theorem 1 is replaced by the condition $\operatorname{Ker} \mathfrak{U} = \{0\}$, then equation (2) possesses a unique solution for sufficiently small M.

Consider now the case where $f(t,x)$ and $J_i(x)$ $(i \in \mathbb{Z})$ are not Lipschitz functions $\mathfrak{N}$.

Theorem 2. *Assume that*

(*i*) $R(\mathfrak{U})$ *is closed*;

(*ii*) $\operatorname{Ker} \mathfrak{U}$ *possesses a closed complementary subspace*;

(*iii*) $R(\mathfrak{N}) \subset R(\mathfrak{U})$;

(*iv*) $\mathfrak{N} : \mathcal{P}^{(0)} \times \mathcal{Y}$ *is a completely continuous operator*;

(*v*) $$\overline{\lim}_{\|x\|_{\mathcal{P}^{(0)}} \to +\infty} \frac{\| H x \|_{\mathcal{P}^{(0)}, \mathcal{Y}}}{\| x \|_{\mathcal{P}^{(0)}}} < \infty.$$

Then, for all sufficiently small M, *equation* (2) *has at least one solution* $x \in \mathcal{P}^{(1)}$.

Proof. We proceed just as in the proof of the previous theorem and employ its proof up and including deriving equation (4). Further, by virtue of the boundedness of the operator $\mathfrak{U}_1^{-1}$ and the complete continuity of $\mathfrak{N}$, the operator $C: \mathcal{P}^{(0)} \in \mathcal{P}^{(1)}$ defined by the equality $Cx = \mathfrak{U}_1^{-1} \mathfrak{N}(u + x)$ is completely continuous. If $M \| \mathfrak{U}_1^{-1} \| <$ < 1, then one can find a closed ball $S_r = \{ x \in \mathcal{P} : \| x \|_{\mathcal{P}^{(1)}} = r \}$ for which $CS_r \subset S_r$. By the Schauder theorem on a fixed point [19], there exists $z \in S_r$ such that $z +$ $+ \mathfrak{U}_1^{-1} \mathfrak{N}(u + z) = 0$, i.e., equation (2) possesses at least one solution $x \in \mathcal{P}^{(1)}$ for each $M \in [0, \| \mathfrak{U}_1^{-1} \|^{-1})$. Theorem 2 is proved.

***Remarks.* 3.** In Theorem 2, condition (iv) can be replaced by the following condition: The sets

$$M_1(r) = \overline{\{ f(t,x) \in E : t \in [0,w), \| x \|_E \le r \}}$$

and

$$M_2(r) = \overline{\{ J_i(x) \in E : i = \overline{1,p}, \| x \|_E \le r \}}$$

are compact for every $r > 0$. Condition (v) can be replaced by the condition

$$M = \left(\overline{\lim}_{\|x\|_E \to \infty} \max_{t\in[0,w)} \frac{\| f(t,x) \|_E}{\| x \|_E} + \overline{\lim}_{\|x\|_E \to \infty} \max_{i=\overline{1,p}} \frac{\| J_i(x) \|}{\| x \|_E} \right) < \infty. \qquad (6)$$

4. When $\dim E < \infty$, the operator $\mathfrak{U}$ is a Fredholm operator [20]. In this case, $\dim \operatorname{Ker} \mathfrak{U} = \operatorname{def} R(\mathfrak{U}) \le \dim E$. Therefore, conditions (i) and (ii) in Theorems 1 and 2 are satisfied. Taking this into account, we obtain the following statements:

***Corollary* 1.** *Assume that* $\dim E < \infty$, *relation* (5) *holds, and* $R(\mathfrak{N}) \subset R(\mathfrak{U})$. *Then, for sufficiently small* M, *the system of equations* (1) *with* $\varepsilon = 0$ *possesses at least one solution.*

***Corollary* 2.** *Let* $\dim E < \infty$ *and* $R(\mathfrak{N}) \subset R(\mathfrak{U})$. *Assume that the sets* $M_1(r)$ *and* $M_2(r)$ *are compact for every* $r > 0$, *and relation* (6) *holds. Then, for sufficiently small* M, *the system of equations* (1) *with* $\varepsilon = 0$ *possesses at least one solution.*

***Remark* 5.** In the previous statement, the condition $R(\mathfrak{N}) = R(\mathfrak{U})$ is rather restrictive, and it is sometimes difficult to check it. For example, this conditions holds if $R(\mathfrak{U}) = \mathcal{P}^{(0)} \times \mathcal{Y}$, and, in particular, when the operator $\mathfrak{U}$ has a continuous inverse operator (this takes place if, e.g., $\dim E < \infty$ and $\operatorname{Ker} \mathfrak{U} = \{0\}$; see also Theorem 4).

3. Periodic Solutions of System (1) for Sufficiently Small $\varepsilon \neq 0$. Assume that the following conditions hold for system (1):

$$0 < \tau_1 < \tau_2 < \ldots < \tau_p < \omega, \tag{7}$$

for some $C \in (0, +\infty)$,

$$\sup_{x \in E, i = \overline{1,p}} \| \tau_i(x) \|_E \leq C < \infty, \tag{8}$$

$$\max_{i = \overline{1,p}} \| \tau_i(x) - \tau_i(y) \|_E \leq \| x - y \|_E \quad \forall x, y \in E, \tag{9}$$

$$\tau_i(x) \geq \tau_i(x + B_i x + J_i(x)) \quad \forall x \in E, \quad i = \overline{1,p}, \tag{10}$$

$J_i \colon E \to E$ $(i = \overline{1,p})$ is a completely continuous mapping, uniformly continuous on every bounded closed set, and for some $M \in [0, \infty)$, we have

$$\| f(t,x) - f(t,y) \|_E \leq M \| x - y \|_E \quad \forall x, y \in E, \quad t \in [0, \omega), \tag{11}$$

$$\varlimsup_{\|x\|_E \to +\infty} \max_{i = \overline{1,p}} \frac{\| J_i(x) \|_E}{\| x \|_E} \leq M. \tag{12}$$

Note that by (7) and (8), there exists a number $\varepsilon_0 > 0$ such that

$$0 < \tau_1 + \varepsilon \tau_1(x_1) < \tau_2 + \varepsilon \tau_2(x_2) < \ldots < \tau_p + \varepsilon \tau_p(x_p) < \omega$$

for all $\varepsilon \in [0, \varepsilon_0]$ and $x_1, x_2, \ldots, x_p \in E$. In addition, according to [7, p. 22–27] and the restrictions imposed on $f(t, x)$, B_i, and $J_i(x)$, relations (9) and (11), for every $\varepsilon \in [0, \varepsilon_1]$ $(\varepsilon_1 \in [0, \varepsilon_0])$, guarantee the absence of pulsation of solutions of system (1) on the hypersurface $t = \tau_i + \varepsilon \tau_i(x)$ $(i = \overline{1,p})$.

Under the restrictions presented above, the following theorem holds:

Theorem 3. *Assume that the operator* $\mathfrak{U}$ *has a continuous inverse operator. Then, for all sufficiently small* M *and* ε, *system* (1) *has at least one* ω*-periodic solution* x.

***Remark* 6.** In this theorem, unlike the analogous theorem in [7, p. 149], we do not assume that the Lipschitz condition holds for $J_i(x)$ and that the space E is finite-dimensional.

The statement of Theorem 3 follows from the results of the next two sections.

4. Conditions of Invertibility of the Operator $\mathfrak{U}$ and Stability of the Invertibility of This Operator under Small Perturbations of the Set T. Consider system (1) for $\varepsilon = 0$ in the case where $f(t, x) \equiv f(t) \in \mathcal{P}^{(0)}$ and $J_i(x) \equiv a_i \in \mathcal{Y}$. The system of equations takes the form

$$\begin{cases} \dfrac{dx}{dt} = A(t)x + f(t), & t \neq \tau_i, \\ \Delta x|_{t=\tau_i} = B_i x(\tau_i - 0) + a_i, & i \in \mathbb{Z}. \end{cases} \tag{13}$$

The corresponding uniform system of equations has the form

$$\begin{cases} \dfrac{dx}{dt} = A(t)x, & t \neq \tau_i, \\ \Delta x|_{t=\tau_i} = B_i x(\tau_i - 0), & i \in \mathbb{Z}. \end{cases} \tag{14}$$

It follows from the restrictions imposed on $A(t)$ and B_i that, for arbitrary $s \in \mathbb{R} \setminus T$ and $z \in E$, system (14) possesses a unique solution $y(t)$ $(t \geq s)$ satisfying the condition $y(s) = z$. We define the operator $X(t, s): E \to E$ by the equality $X(t, s)z = y(t)$. This operator is called an evolutionary (solving) operator of system (14).

Assume that $U(t, \tau)$ is an evolutionary operator of the differential equation $dx(t)/dt = A(t)x(t)$ [21].

Consider arbitrary $s \geq 0$ $(s \notin T)$ and $t \geq s$ $(t \notin T)$. Let

$$[s, t] \cap T = \{\tau_k, \tau_{k+1}, \dots, \tau_m\} \quad (\tau_k < \tau_{k+1} < \dots < \tau_m).$$

Denote $C_l = (I + B_l)U(\tau_l, \tau_{l+1})$ $(l \geq k+1)$, where I is a unique operator. Analyzing system (14), we find that

$$X(t, s) = U(t, \tau_m) C_m C_{m-1} \dots C_{k+1}(I + B_k) U(\tau_k, s).$$

If $[s, t] \cap T$ consists of a single element $\tau_i \in T$, then $X(t, s) = U(t, \tau_i)(I + B_i)U(\tau_i, s)$. But if $[s, t] \cap T = \varnothing$, then $X(t, s) = U(t, s)$.

The operator $X(t, s)$ enables us to represent each solution $x(t)$ of system (13) in the form

$$x(t) = X(t, 0)x(0) + \int_0^t X(t, s) f(s)\, ds + \sum_{0 < \tau_i < t} X(t, \tau_i + 0)\, a_i, \quad t \notin T, \tag{15}$$

provided that $x(0)$ is known (see also [7, 11]). Here and below, we regard $X(t, s)$, $s \in T$, in the integrand as a unilateral limit

$$X(t, s + 0) = \lim_{s_1 \to s + 0} X(t, s_1),$$

which exists for all $s \in T$ and $t \notin T$ by the continuity of $U(t, s)$ on $\mathbb{R} \times \mathbb{R}$. The properties of solutions to the system of equations (13) depend on the spectrum of the operator $X(\omega, 0)$.

Theorem 4. *The operator* $\mathfrak{U}$ *has a continuous inverse operator if and only if* $1 \notin \sigma(X(\omega, 0))$.

Proof. Let $1 \notin \sigma(X(\omega, 0))$. Then the operator $I - X(\omega, 0)$ has a continuous inverse operator $(I - X(\omega, 0))^{-1}$. Consider a function

$$y(t) = X(t, 0)(I - X(\omega, 0))^{-1} \left(\int_0^\omega X(\omega, \tau) f(\tau) d\tau + \sum_{i=1}^{p} X(\omega, \tau_i + 0) a_i \right) +$$

$$+ \int_0^t X(t, \tau) f(\tau) d\tau + \sum_{0 < \tau_i < t} X(t, \tau_i + 0) a_i, \tag{16}$$

where $f(t)$ and a_i are arbitrary elements of the spaces $\mathcal{P}^{(0)}$ and $\mathcal{Y}$, respectively. According to (15), this function satisfies system (13) for $t \geq 0$; furthermore,

$$y(0) = y(\omega) = (I - X(\omega, 0))^{-1} \left(\int_0^\omega X(\omega, \tau) f(\tau) d\tau + \sum_{i=1}^{p} X(\omega, \tau_i + 0) a_i \right).$$

Consequently, for arbitrary $f(t) \in \mathcal{P}^{(0)}$ and $a_i \in \mathcal{Y}$, system (13) has a solution $y(t) \in \mathcal{P}^{(1)}$. This solution is unique. Indeed, according to (15), an ω-periodic solution $x(t)$ of system (13) should satisfy the relation

$$(I - X(\omega, 0))\, x(0) = \int_0^{\omega} X(\omega, s) f(s) ds + \sum_{i=1}^{p} X(\omega, \tau_i + 0) a_i. \tag{17}$$

If the solution of system (13) is not unique, then $\operatorname{Ker}(I - X(\omega, 0)) \neq \{0\}$, but this contradicts the assumption that $1 \notin \sigma(X(\omega, 0))$.

Thus, $\operatorname{Ker} \mathcal{U} = \{0\}$ and $R(\mathcal{U}) = \mathcal{P}^{(0)} \times \mathcal{Y}$. Therefore, by the Banach theorem on inverse operator, the operator $\mathcal{U}$ has a continuous inverse operator.

Hence, the relation $1 \notin \sigma(X(\omega, 0))$ guarantees the invertibility of the operator $\mathcal{U}$.

Assume now that the operator $\mathfrak{U}$ has a continuous inverse operator. Then, for arbitrary $f(t) \in \mathcal{P}^{(0)}$ and $a_i \in \mathcal{Y}$, relation (17), regarded as an equation with respect to $x(0)$, possesses a unique solution, i.e.,

$$\operatorname{Ker}(I - X(\omega, 0)) = \{0\}. \tag{18}$$

Let us show that $R(I - X(\omega, 0)) = E$.

Let $a_i \equiv 0$ and let $f(t)$ be an element of the space $\mathcal{P}^{(0)}$ such that

$$f(t) = \begin{cases} U^{-1}(\omega, t) \dfrac{\varphi(t)}{\int_{\tau_p}^{\omega} \varphi(s) ds} b, & \text{if } t \in [\tau_p, \omega], \\ 0, & \text{if } t \in [0, \tau_p), \end{cases}$$

for $t \in [0, \omega]$, where $f(t)$ is a function continuous on $[\tau_p, \omega]$, for which $\varphi(\tau_p) = \varphi(\omega) = 0$ and $\varphi(t) > 0$ if $t \in (\tau_p, \omega)$; b is an arbitrary element of the space E. Then relation (17) takes the form $(I - X(\omega, 0))\, x(0) = b$.

Since b is an arbitrary element of the space E, we have $R(I - X(\omega, 0)) = E$.

This and (18) imply that the operator $I - X(\omega, 0)$ has a continuous inverse operator, i.e., $1 \notin \sigma(X(\omega, 0))$. Theorem 4 is proved.

***Remarks.* 7.** When studying ω-periodic solution of system (13), we did not demand that the operators $I + B_i$ $(i = \overline{1, p})$ should satisfy the condition of invertibility which was used in [7]. In this paper, this condition is not necessary.

8. It follows from the proof of Theorem 4 and formula (16) that ω-periodic solutions of system (13) can be represented in the form

$$x(t) = \int_0^{\omega} G(t, s) f(s) ds + \sum_{i=1}^{p} G(t, \tau_i + 0) a_i \quad (t \notin T), \tag{19}$$

where

$$G(t, \tau) = \begin{cases} X(t, 0)(I - X(\omega, 0))^{-1} X(\omega, \tau) + X(t, \tau), & \text{if } 0 \le \tau \le t, \\ X(t, 0)(I - X(\omega, 0))^{-1} X(\omega, \tau), & \text{if } t < \tau \le \omega \end{cases}$$

(see also [7, p. 144]). The function $G(t, \tau)$ is called Green's function.

We take arbitrary sufficiently small numbers $\gamma_1, \gamma_2, \ldots, \gamma_p$ such that the relation

$$0 < \tau_1 + \gamma_1 < \tau_2 + \gamma_2 < \ldots < \tau_p + \gamma_p < \omega$$

(similar to (7)) holds and consider a system of equations

$$\begin{cases} \dfrac{dx}{dt} = A(t) x(t), & t \neq \tau_i + \gamma_i, \\ \Delta x|_{t = \tau_i + \gamma_i} = B_i x(\tau_i + \gamma_i - 0), & i \in \mathbb{Z}, \end{cases} \tag{20}$$

similar to (14). Denote the evolutionary operator of this system by $X_{\gamma_1,\ldots,\gamma_p}(t, s)$; the corresponding Green's function and the operator establishing the correspondence between each function $y(t) \in \mathcal{P}^{(1)}_{\gamma_1,\ldots,\gamma_p}$ and the pair $(\alpha, \beta) \in \mathcal{P}^{(0)}_{\gamma_1,\ldots,\gamma_p} \times \mathcal{Y}$, where

$$\alpha(t) = \frac{dy(t)}{dt} - A(t)y(t) \in \mathcal{P}^{(0)}_{\gamma_1,\ldots,\gamma_p} \quad \text{and} \quad B_i = \Delta y|_{t=\tau_i+\gamma_i} - B_i y(\tau_i + \gamma_i - 0) \in \mathcal{Y},$$

are denoted by $G_{\gamma_1,\ldots,\gamma_p}(t, s)$ and $\mathfrak{U}_{\gamma_1,\ldots,\gamma_p}$ respectively, where the spaces $\mathcal{P}^{(k)}_{\gamma_1,\ldots,\gamma_p}$ $(k = \overline{1,p})$ have the meaning of respective spaces $\mathcal{P}^{(k)}$ $(k = \overline{0,1})$, where the set T is considered instead of $\{\tau_i + \gamma_i + k\omega : i = \overline{1,p},\ k \in \mathbb{Z}\}$.

The definition of an evolutionary operator and the restrictions on $A(t)$ imply that one can find numbers $\gamma_0 > 0$ and $Q > 0$ such that

$$\| X(t_1, s_1) - X_{\gamma_1,\ldots,\gamma_p}(t_2, s_2) \| \le Q(|t_1 - t_2| + |s_1 - s_2| + |\gamma_1| + \ldots + |\gamma_p|) \quad (21)$$

$$\text{for all} \quad t_1, t_2, s_1, s_2 \in [0, \omega] \setminus \bigcup_{i=1}^{p} [a_i, b_i] \quad (t_1 \ge s_1,\ t_2 \ge s_2),$$

and $\gamma_i \in [-\gamma_0, \gamma_0]$, $i = \overline{1,p}$, where $a_i = \min\{\tau_i, \tau_i + \gamma_i\}$, $b_i = \max\{\tau_i, \tau_i + \gamma_i\}$.

This and the property of semicontinuity of the spectrum of a bounded operator [22] imply that if γ_0 is a sufficiently small number, then

$$1 \notin \sigma(X_{\gamma_1,\ldots,\gamma_p}(\omega, 0)), \qquad (22)$$

when $|\gamma_i| \le \gamma_0 \ \forall i = \overline{1,p}$. Therefore, the following theorem holds:

Theorem 5. *Assume that the operator* $\mathfrak{U}$ *has a continuous inverse operator and that* γ_0 *is a sufficiently small number. Then the operator* $\mathfrak{U}_{\gamma_1,\ldots,\gamma_p}$ *possesses a continuous inverse operator provided that* $|\gamma_i| \le \gamma_0 \ \forall i = \overline{1,p}$.

This theorem shows that the property of invertibility of the operator $\mathfrak{U}$ is stable under small perturbations of the set T.

The invertibility of the operator $\mathfrak{U}_{\gamma_1,\ldots,\gamma_p}$ is equivalent to the invertibility of the operator $I - X_{\gamma_1,\ldots,\gamma_p}(\omega, 0)$ and, consequently, to the existence of Green's function $G_{\gamma_1,\ldots,\gamma_p}(t, s)$. Therefore, by virtue of the definition of an evolutionary operator, the restrictions imposed on $A(t)$, and relations (21) and (22), we conclude that, for some positive number Q_1 and sufficiently small number $\gamma_0 \in (0, a)$, where

$$a < \min\left\{ \frac{\tau_1}{2}, \frac{\tau_2 - \tau_1}{2}, \ldots, \frac{\tau_p - \tau_{p-1}}{2}, \frac{\omega - \tau_p}{2} \right\},$$

the relation

$$\begin{aligned} &\left\| G_{\gamma_1',\ldots,\gamma_p'}(t_1, s_1) - G_{\gamma_1'',\ldots,\gamma_p''}(t_2, s_2) \right\| \le \\ &\le Q_1\left(|t_1 - t_2| + |s_1 - s_2| + |\gamma_1' - \gamma_1''| + \ldots + |\gamma_p' - \gamma_p''| \right) \end{aligned} \qquad (23)$$

holds for all $t_1, t_2, s_1, s_2 \in \bigcup_{i=1}^{p} [\tau_i + m_i, \tau_i + M_i]$ and $\gamma_i', \gamma_i'' \in [-\gamma_0, \gamma_0]$, where $m_i = \min\{\gamma_i', \gamma_i''\}$ and $M_i = \max\{\gamma_i', \gamma_i''\}$.

This property of Green's function will be used below.

5. An Auxiliary Mapping Acting in E^p. Assume that the conditions given in Section 3 are satisfied. Consider an arbitrary element

$$(y_1, \ldots, y_p) \in E^p = \underbrace{E \times \ldots \times E}_{p \text{ times}}$$

and the system of equations

$$\begin{cases} \dfrac{dx}{dt} = A(t)x + f(t,x), & t \neq \tau_i + \varepsilon\tau_i(y_i), \\ \Delta x|_{t=\tau_i+\varepsilon\tau_i(\gamma_i)} = B_i x(\tau_i + \varepsilon\tau_i(y_i) - 0) + J_i(y_i), & i \in \mathbb{Z}, \end{cases} \tag{24}$$

where $\varepsilon \in [0, \min\{\varepsilon_1, \gamma_0/C\}]$ (here, γ_0 is a number satisfying relation (22)).

Then the problem of existence of a solution of system (24) in the space $\mathcal{P}^{(1)}_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}$ is equivalent to the problem of existence of the solution of

$$x(t) = \int_0^\omega G_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(t,s) f(s, x(s))\, ds +$$

$$+ \sum_{i=1}^p G_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(t, \tau_i + \varepsilon\tau_i(y_i) + 0) J_i(y_i) \tag{25}$$

in the same space. By the results given in Section 4 and the restrictions imposed on ε, the equation (25) has a unique solution $x(t) \in \mathcal{P}^{(1)}_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}$ for sufficiently small M; this follows from the principle of contracting mappings. This solution depends on $y_1, \ldots, y_p$. We denote it by $x(t, y_1, \ldots, y_p)$.

Consider a mapping $S: E^p \to E^p$ that associates each vector $(y_1, \ldots, y_p) \in E^p$ with the vector

$$(x(\tau_1 + \varepsilon\tau_1(y_1) - 0, y_1, \ldots, y_p), \ldots, x(\tau_p + \varepsilon\tau_p(y_p) - 0, y_1, \ldots, y_p)) \in E^p.$$

Theorem 6. *If* $0 \le \varepsilon \le \min\{\varepsilon_1, \gamma_0/C\}$ *and* M *is a sufficiently small number, then the mapping* S *has at least one fixed point in* E^p.

Proof. It follows from the restrictions imposed on ε and $\tau_i(x)$ and relation (23) that there exists a number G for which

$$\sup\{\| G_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(t,s)\| : t, s \in \mathbb{R}\setminus T, \ (y_1, \ldots, y_p) \in E^p,$$
$$0 \le \varepsilon \le \min\{\varepsilon_1, \gamma_0/C\}\} \le G. \tag{26}$$

Consider the functions

$$h(r) = \sup_{\|y\|_E \le r,\, i=\overline{1,p}} \|J_i(y)\|_E \quad (r > 0),$$

$$x_0 = x_0(t, y_1, \ldots, y_p) = \sum_{i=1}^p G_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(t, \tau_i + \varepsilon\tau_i(y_i) + 0) J_i(y_i), \tag{27}$$

$$x_n = x_n(t, y_1, \ldots, y_p) = \int_0^\omega G_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(t,s) f(s, x_{n-1}(s, y_1, \ldots, y_p))\, ds +$$
$$+ x_0(t, y_1, \ldots, y_p) \quad (n \ge 1) \tag{28}$$

and the sets

$$W = \{(t, y_1, \ldots, y_p) \in \mathbb{R} \times E^p : t \neq \tau_i + \varepsilon\tau_i(y_i), \quad i = \overline{1,p}\},$$

$$W_r = \{(t, y_1, \ldots, y_p) \in W\colon \|y_k\|_E \le r, \quad k = \overline{1,p}\}.$$

It is clear that

$$\sup_{(t, y_1, \ldots, y_p) \in W_r} \|x_0(t, y_1, \ldots, y_p)\|_E \le p\,G h(r),$$

$$\sup_{(t, y_1, \ldots, y_p) \in W_r} \|x_1(t, y_1, \ldots, y_p) - x_0(t, y_1, \ldots, y_p)\|_E \le p\,\omega G^2 M h(r) + \omega G \Gamma, \tag{29}$$

where $\Gamma = \max_{t \in [0,\omega]} \|f(t, 0)\|_E$ and

$$\sup_{(t, y_1, \ldots, y_p) \in W_r} \|x_{n+1}(t, y_1, \ldots, y_p) - x_n(t, y_1, \ldots, y_p)\|_E \le$$

$$\le (\omega G M)^n (\omega G^2 M h(r) + \omega G \Gamma) \quad \forall n \ge 0. \tag{30}$$

Let

$$\omega G M < 1. \tag{31}$$

Then equation (25) has a unique solution $x(t, y_1, \ldots, y_p) \in \mathcal{P}^{(1)}_{\varepsilon\tau_1(y_1), \ldots, \varepsilon\tau_p(y_p)}$ which can be represented in the form

$$x(t, y_1, \ldots, y_p) = x_0 + (x_1 - x_0) + \ldots + (x_{n+1} - x_n) + \ldots \tag{32}$$

due to (29), (30), and (31). This solution is continuous with respect to the collection of variables $(t, y_1, \ldots, y_p)$ on W. This follows from the fact that series (32) can be majorized by the convergent geometric progression with the denominator $\omega G M < 1$ (see (29) and (30)), because of continuity of the functions $x_n(t, y_1, \ldots, y_p)$ $(n \ge 0)$ on W, (27), (28), the restrictions imposed on $f(t, x)$, $\tau_i(x)$, and $J_i(x)$ $(i = \overline{1,p})$, and relations (23) and (26).

The solution $x(t, y_1, \ldots, y_p)$ is also uniformly continuous with respect to the collection of variables on every bounded closed set $K \subset W$. Indeed, the functions $J_i(y)$ are continuous on $\{x \in E\colon \|x\|_E \le r\}$, and the set $\{J_i(x) \in E\colon i = \overline{1,p}, \|x\|_E \le r\}$ is compact for every $r > 0$. Therefore, the functions $J_i(y)$ $(i = \overline{1,p})$ are uniformly continuous in y on $\{x \in E\colon \|x\|_E \le r\}$ for every $r > 0$. The functions $\tau_i(y)$ $(i = \overline{1,p})$ are also uniformly continuous on E since they are Lipschitz functions (see (9)). Therefore, according to (23), the function $x_0(t, y_1, \ldots, y_p)$ defined by the relation (27) is also uniformly continuous with respect to the collection of its variables on every bounded closed set $K \subset W$. By (28), (11), and (23), this is also true for the functions $x_n(t, y_1, \ldots, y_p)$ $(n \ge 1)$. Then, due to the fact that series (32) can be majorized by a convergent geometric progression, we can conclude that the function $x(t, y_1, \ldots, y_p)$ is uniformly continuous on every bounded closed set $K \subset W$.

The uniform continuity of the function $x(t, y_1, \ldots, y_p)$ guarantees the continuity of the limits

$$\lim_{\delta \to -0} x(\tau_i + \varepsilon\tau_i(y_i) + \delta, y_1, \ldots, y_p) = x(\tau_i + \varepsilon\tau_i(y_i) - 0, y_1, \ldots, y_p) \quad (i = \overline{1,p})$$

on E^p.

Indeed, passing in (25) to the limit as $t \to \tau_k + \varepsilon\tau_k(y_k) - 0$, we obtain the equality

$$x(\tau_k + \varepsilon\tau_k(y_k) - 0, y_1, \ldots, y_p) = x_0(\tau_k + \varepsilon\tau_k(y_k) - 0, y_1, \ldots, y_p) +$$

$$+\int_0^{\omega} G_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(\tau_k+\varepsilon\tau_k(y_k)-0,s)f(s,x(s,y_1,\ldots,y_p))\,ds, \tag{33}$$

$k=\overline{1,p}$. In this equality, the unilateral limits $x(\tau_k+\varepsilon\tau_k(y_k)-0,y_1,\ldots,y_p)$ and $x_0(\tau_k+\varepsilon\tau_k(y_k)-0,y_1,\ldots,y_p)$ exist because $x(t,y_1,\ldots,y_p)$, $x_0(t,y_1,\ldots,y_p)\in \in \mathcal{P}^{(1)}_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}$. The limit transition under the integral sign is possible due to relation (23), the continuity of the function $f(t,x(t,y_1,\ldots,y_p))$ and its boundedness on $[0,\omega]\setminus\{\tau_k+\varepsilon\tau_k(y_k)\colon k=\overline{1,p}\}$, and due to the existence of the limit $\lim\limits_{t\to\tau_k+\varepsilon\tau_k(y_k)-0} G_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(t,s)$ for all $s\in[0,\omega]\setminus\{\tau_k+\varepsilon\tau_k(y_k)\colon k=\overline{1,p}\}$ because analogous limits exist for $X_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(t,s)$. The function $f(t,s)$ continuous on $[0,\omega]\times E$ satisfies the Lipschitz condition with respect to the variable x (see (11)), and the function $x(t,y_1,\ldots,y_p)$ is uniformly continuous on every bounded closed set $K\subset W$; therefore, the similar property takes place for the function $f(t,x(t,y_1,\ldots,y_p))$. This function is also bounded. It follows from (23) that the operator-function $G_{\varepsilon\tau_1(y_1),\ldots,\varepsilon\tau_p(y_p)}(\tau_k+\varepsilon\tau_k(y_k)-0,s)$ is also continuous at every point $(y_1,\ldots,y_p)\in E^p$ uniformly in $s\in[0,\omega]\setminus\bigcup_{k=1}^{p}[\tau_k+\varepsilon\tau_k(y_k)-\sigma,\ \tau_k+\varepsilon\tau_k(y_k)+\sigma]$ for any sufficiently small $\delta>0$. Therefore, if we also take into account (26), then the last term in (33) is continuous in $(y_1,\ldots,y_p)\in E^p$. The function $x_0(\tau_k+\varepsilon\tau_k(y_k)-0,y_1,\ldots,y_p)$ is also continuous on E^p by (27) and (23). By virtue of (33), this ensures the continuity of the functions $x(\tau_k+\varepsilon\tau_k(y_k)-0,\ y_1,\ldots,y_p)$ $(k=\overline{1,p})$ on E^p. Hence, $S\colon E^p\to E^p$ is a continuous mapping.

Taking into account (27), (28), the fact that series (32) can be majorized by a convergent geometric progression, and the complete continuity of the mappings $J_i\colon E\to E$ $(i=\overline{1,p})$, we conclude that the mapping $S\colon E^p\to E^p$ is also completely continuous.

Let us show that there exists a number $r>0$ such that

$$SK_r\subset K_r \tag{34}$$

for the set $K_r=\{(y_1,\ldots,y_p)\in E^p\colon \|y_k\|_E\le r,\ k=\overline{1,p}\}$. Indeed, it follows from (25) that

$$\sup_{(t,y_1,\ldots,y_p)\in W_r}\|x(t,y_1,\ldots,y_p)\|_E\le \omega GM\sup_{(t,y_1,\ldots,y_p)\in W_r}\|x(t,y_1,\ldots,y_p)\|_E+$$
$$+\ \omega G\Gamma+pG\sup_{\|y\|_E\le r,\ i=\overline{1,p}}\|J_i(y)\|_E.$$

Therefore,

$$\sup_{(t,y_1,\ldots,y_p)\in W_r}\|x(t,y_1,\ldots,y_p)\|_E\le\frac{G}{1-\omega GM}\left(p\sup_{\|y\|_E\le r,\ i=\overline{1,p}}\|J_i(y)\|_E+\omega\Gamma\right). \tag{35}$$

Assume that the number M is sufficiently small so that not only relation (31) holds but also the relation $(p+\omega)GM<1$ is valid. Then, according to (35) and (12), relation (34) holds for sufficiently large $r>0$.

Thus, the mapping $S\colon E^p\to E^p$ is completely continuous and satisfies relation (34) for sufficiently large $r>0$ (note that K_r is a bounded closed convex set).

Therefore, by the Schauder theorem on a fixed point, the mapping S has at least one fixed point in $K_r \subset E^p$. Theorem 6 is proved.

6. Proof of Theorem 3. Let $0 \le \varepsilon \le \min\{\varepsilon_1, \gamma_0/C\}$ and $(p+\omega)GM < 1$. According to Theorem 6 and its proof, one can find a vector $(y_1^*,\dots,y_p^*) \in E^p$ which is a fixed point for S, i.e.,

$$y_k^* = x(\tau_k + \varepsilon\tau_k(y_k^*) - 0,\ y_1^*,\dots,y_p^*) \quad (k = \overline{1,p}). \tag{36}$$

Then the function $x(t,\ y_1^*,\dots,y_p^*)$ is a solution of system (24) if $y_i = y_i^*$ $(i = \overline{1,p})$. Clearly, by virtue of (36) and the absence of pulsations of solutions of system (1) on hypersurfaces $t = \tau_i + \varepsilon\tau_i(x)$ $(i = \overline{1,p})$, the same solution is a solution of system (1).

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