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CAUCHY PROBLEM FOR THE ESSENTIALLY INFINITE-DIMENSIONAL HEAT EQUATION ON A SURFACE IN HILBERT SPACE*

ЗАДАЧА КОШІ ДЛЯ СУТТЄВО НЕСКІНЧЕННОВИМІРНОГО РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ НА ПОВЕРХНІ У ГІЛЬБЕРТОВОМУ ПРОСТОРІ

It is proved that the Cauchy problem for the simplest parabolic equation with essentially infinite-dimensional coefficients on bounded level surfaces of smooth functions in a Hilbert space is uniformly well-posed.

На обмежених поверхнях рівня гладких функцій у гільбертовому просторі доведена рівномірна коректність задачі Коші для найпростішого параболічного рівняння з суттєво нескінченновимірними коефіцієнтами.

This paper deals with the Cauchy problem for the parabolic equation $\partial v(x,t)/\partial t = L_S v(x,t)$ on a level surface of a smooth function in an infinite-dimensional Hilbert space. An operator L_S is supposed to be "essentially infinite-dimensional" with respect to x in the sense given below (see (2)). A similar problem on the Hilbert sphere was studied in [1].

The investigation is developed by constructing a (C_0)-semigroup on some Banach space of functions. $\overline{L}_S$ turns out to be a generator of this semigroup. The fact that L_S is essentially infinite-dimensional turns out to be a simplifying condition (Lemma 1). The solution of this problem is based on [2, 3], where an analogous problem in a linear space was investigated, and on Theorem 1 from this paper.

1. Let H be a separable infinite-dimensional real Hilbert space and let $B_s(H)$ be the Banach space of selfadjoint bounded operators on H (with operator norm). Assume that $j \in B_s(H)^*$ is a nonnegative functional which vanishes on all finite-dimensional operators from $B_s(H)$. The functional j generates the linear elliptic second-order differential operator

$$(Lu)(x) = \frac{1}{2} j(u''(x)). \tag{1}$$

Here, $u \in C^2(H)$ and, for all cylindrical functions, $Lu = 0$.

Let S be a smooth (C^2-class) surface in H of co-dimension 1. The embedding $i: S \hookrightarrow H$ induces the Riemann metric on S.

Let ∇ be the corresponding Levi–Civita connection. The embedding di_x: $T_x S \to H$ of the tangent space $T_x S$ into H at a point $x \in S$ allows us to assume that a bounded selfadjoint operator in $T_x S$ is the restriction of some bounded selfadjoint operator in H, which is determined up to a finite-dimensional operator. In this sense, we identify the second covariant derivative $\nabla^2 u(x)$ with some operator from $B_s(H)$ and the differential operator on S may be introduced by the formula

$$(L_S u)(x) = \frac{1}{2} j(\nabla^2 u(x)). \tag{2}$$

2. Let $\mathfrak{A}$ be an algebra of twice continuously Frechet-differentiable functions on H with the conditions:

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a) for any $u \in \mathfrak{A}$, there is a compact set $\mathfrak{N} = \mathfrak{N}_u \subset B_s(H)$ and a number $d = d_u > 0$ such that $u''(x) \in \mathfrak{N} + \Gamma_d$ for all $x \in H$ (where Γ_d is the set of selfadjoint Hilbert–Schmidt operators with the Hilbert–Schmidt norm $\leq d$);

b) $u''(\cdot)$ is uniformly continuous on H: $\sup_H |u(\cdot)| < \infty$.

Let X be the closure of $\mathfrak{A}$ in the norm of uniform convergence on H.

We say that $u \in C^3(H) \cap \mathfrak{A}$ belongs to $\mathfrak{A}_1$ if, for any $h \in H$, the function $(u'(\cdot), h) = v_h(\cdot) \in \mathfrak{A}$ and there exist a compact set $\mathfrak{N} \subset B_s(H)$ and $d > 0$ such that $v_h''(x) \in \mathfrak{N} + \Gamma_d$ for all $x \in H$ and $h \in \{h \mid \|h\| = 1\}$.

Obviously, $\mathfrak{A}_1$ is a subalgebra in $\mathfrak{A}$.

For an open set $G \subset H$, a functional algebra $\mathfrak{A}(G)$ is introduced by substituting G for H in conditions a) and b). Analogously, we obtain $\mathfrak{A}_1(G)$ and $X(G)$.

If the surface S may be represented as a level surface of a function $g \in \mathfrak{A}_1(S_\varepsilon)$ (S_ε is ε-neighborhood of S in H) with

$$\|g'(\cdot)\| \geq \delta > 0, \tag{3}$$

then we say that S is a surface of the class $\mathfrak{A}_1$. Assume that $S = \{x \mid g(x) = 1\}$.

Examples of such surfaces are: $\{x \mid (C_1x, x)^2 + (C_2x, x) = 1\}$, where C_1, $C_2 \in B_s(H)$; and $C_1 \geq 0$; $C_2 \geq \delta I > 0$ (see [4]).

Lemma 1. *Let $u \in C^2(S_\varepsilon)$ and S be a level surface of $g \in C^2(S_\varepsilon)$. Then, for $x \in S$,*

$$L_S u(x) = Lu(x) - \frac{(u'(x), g'(x))}{\|g'(x)\|^2} Lg(x). \tag{4}$$

Proof. Let vector fields X and Y be defined in a neighborhood $W \subset H$ of the point $x \in S$ and tangent to S on $S \cap W$. Let n be the field of a single normal vector to S.

If X'_x is the derivative of the map $y \mapsto X(y)$ in x, then

$$\nabla_Y X(y) = X'_x(Y(x)) - (X'_x(Y(x)), n(x))\, n(x) = $$
$$= X'_x(Y(x)) - (X'_x(Y(x)), g'(x)) \frac{g'(x)}{\|g'(x)\|^2}.$$

On the other hand,

$$(\nabla^2 u)(X, Y) = YXu - (\nabla_Y X)u;$$
$$(YXu - X'(Y)u)(x) = (u''(x)X(x), Y(x));$$
$$(X'(Y), g')(x) = Y(X, g')(x) - (X, g''(Y))(x) = -(g''(x)X(x), Y(x))$$

and, therefore,

$$(\nabla^2 u)(X, Y)(x) = (u''(x)X(x), Y(x)) - $$
$$- (g''(x)X(x), Y(x)) \frac{(g'(x), u'(x))}{\|g'(x)\|^2}. \tag{5}$$

This implies the assertion.

The algebra $\mathfrak{A}(S)$ on a C^2-class surface is introduced similarly to $\mathfrak{A}$ and $\mathfrak{A}(G)$ by replacing u'' by $\nabla^2 u$. Let $X(S)$ be the closure of $\mathfrak{A}(S)$ in the norm of uniform convergence.

Any function $\varphi \in \mathfrak{A}(S)$ may be extended from the surface of class $\mathfrak{A}_1$ to the function $\tilde{\varphi} \in \mathfrak{A}$ [4].

It should be observed that, for $\gamma \in C^2(\mathbb{R})$, $\varphi \in \mathfrak{A}(S)$,

$$\gamma \circ \varphi \in \mathfrak{A}(S); \quad L_S(\gamma \circ \varphi) = (\gamma' \circ \varphi) \cdot L_S \varphi.$$

Let $C_u^2(S)$ be the space of functions from $C^2(S)$ with uniformly continuous Hessians.

Lemma 2. *Suppose that S is a surface of class $\mathfrak{A}_1$, W is an open set in H; $u \in C_u^2(S)$; $L_S u > \varepsilon > 0$ in $W \cap S$. Then*

$$\sup_{W \cap S} u = \sup_{\partial(W \cap S)} u.$$

Proof. Without loss of generality, we may assume that $u(x) > 1$ on $S \cap W$. Let there be given $\beta > 0$, $C_1 > 0$ such that, in a β-neighborhood S_β of the surface S, we have $\delta / 2 < \| g'(x) \| < C_1$ (see (3)).

The flow $\Phi(t, x)$ of the vector field $g'(x)/\| g'(x) \|^2$ is defined in S_β, for $x \in S_\beta$, we have $\Phi(-g(x) + 1, x) \in S$. For $x \in S_\beta$, we set $\tilde{u}(x) = u(\Phi(-g(x) + 1, x) s(g(x)))$, where the function $s(t) = p(|t - 1|)$ of the real variable t satisfies the conditions: $p(t)$ monotonically decreases; $s \in C^2(\mathbb{R})$; $s(t) \in (0, 1)$ as $t \neq 1$, $s(1) = 1$, $s'(1) = 0$. For this reason, $(\tilde{u}', g') = 0$ on $S \cap W$ and, by virtue of Lemma 1, we have $L\tilde{u}|_{S \cap W} = L_S u$.

Let $(S \cap W)_\beta = \{\Phi(t, x) \in S_\beta \mid x \in S \cap W\}$ be the part of S_β filled with the integral curves that begin in $S \cap W$. Because of uniform continuity of $L\tilde{u}$ in S_β, decreasing β if necessary, one can consider that $L\tilde{u}(x) > \varepsilon / 2$ in $(S \cap W)_\beta$. Furthermore, there exists such $\gamma > 0$ that $\Phi(t, x) \in (S \cap W)_\beta$ for $|t| < \gamma$ ($x \in S \cap W$). For this reason, $\tilde{u}(\Phi(t, x)) \leq u(x) p(\gamma)$ ($p(\gamma) \in (0, 1)$), where $\Phi(t, x) \in \partial(S \cap W)_\beta$ and $x \in S \cap W$.

Let $C = \sup_{W \cap S} u$ and $x \in S \cap W$ be such that $u(x) > p(\gamma) C$. By virtue of the maximum principle for domains in H [3, 5], we have: $u(x) \leq \sup_{\partial((S \cap W)_\beta)} \tilde{u}$. But then there exists $y \in \partial(S \cap W)$ for which $u(y) \geq u(x)$, for otherwise there would exist $y \in \partial((S \cap W)_\beta)$ of the form $y = \Phi(t, z)$, $z \in S \cap W$, and $\tilde{u}(y) \geq u(x)$; thus $u(z) > \tilde{u}(y)/p(\gamma) > C$. The lemma is proved.

The analogous statement holds true for inf.

The next proposition proves that L_S is subtended in the space $C_u^2(S)$.

Lemma 3. *Let S be a surface of class $\mathfrak{A}_1$; $u_n \in C_u^2(S)$; $u_n \to 0$ and $L_S u_n \to v$ uniformly on S. Then $v \equiv 0$.*

Proof. Suppose the converse. Without loss of generality, we consider: $v(x_0) = \varepsilon > 0$ ($x_0 \in S$). Hence, a neighborhood $W = W_\alpha(x_0)$ of x_0 exists with radius $\alpha > 0$ such that $L_S u_n(x) > \varepsilon / 2$ ($x \in W \cap S$), beginning with some N.

Let $\alpha > 0$ be taken so that $|(x - x_0, g'(x))| \le (\delta^2/2)\left(\sup_S |Lg|\right)^{-1} \|j\|$ for any $x \in W \cap S$. This is possible as the left-hand side of the last inequality is continuous. But then, for $h(x) = c(\|x - x_0\|^2)$ $(c > 0)$, we have $(x \in S \cap W)$:

$$\left|\left(h'(x), \frac{g'(x)}{\|g'(x)\|^2}\right) Lg(x)\right| = 2c\,|(x - x_0, g'(x))|\,\frac{|Lg(x)|}{\|g'(x)\|^2} \le c\,\|j\|.$$

So $L_S h = c\,\|j\| - (h', g'/\|g'\|^2) Lg \le 2c\,\|j\|$ in $W \cap S$ and $L_S(u_n - h)(x) > \varepsilon/2 - 2c\,\|j\| > \varepsilon/4$ if $c = \varepsilon/(8\,\|j\|)$; $x \in W \cap S$. In view of Lemma 2,

$$u_n(x_0) \le \sup_{\partial(W \cap S)} u_n - \frac{\varepsilon}{8\|j\|}\alpha^2,$$

contrary to the condition: $u_n \to 0$ on S.

Lemma 4. *Let S be a surface of class $\mathfrak{A}_1$. Then, for any $u \in C_u^2(S)$, we have $\|u - L_S u\| \ge \|u\|$, where $\|u\| = \sup_S |u|$.*

Proof. We assume, without loss of generality, that $\|u\| = \sup_S u = 1$ and admit $\|u - L_S u\| = 1 - \gamma < 1$.

Take $x_0 \in S$ such that $u(x_0) = 1 - \varepsilon > 1 - \gamma$. Then $L_S u(x_0) > \gamma - \varepsilon$. By using the method of Lemma 3, we can find $\alpha > 0$ such that there exists $x \in \partial W_\alpha(x_0) \cap S$ for which $u(x) > u(x_0) + (\gamma - \varepsilon)\alpha^2/(8\,\|j\|)$.

Furthermore, we can choose α independently of ε and x_0 for $\varepsilon < \gamma/2$. It suffices to choose α under the conditions:

a) $|L_S u(x) - L_S u(y)| < \gamma/4$ for $\|x - y\| < \alpha$ (because of the uniform continuity of $L_S u$);

b) $|(x - x_0, g'(x))| \le \frac{1}{2}\delta^2\left(\sup_S |Lg|\right)^{-1} \|j\|$, $x \in W_\alpha(x_0) \cap S$,

(the left-hand side of the inequality is continuous with respect to x uniformly in x_0).

If $\varepsilon \in (0, \min(\gamma/2, \gamma\alpha^2/(16\,\|j\|)))$, we have $u(x) > 1$ at some point $x \in \partial W_\alpha \cap S$, i.e., we arrive at a contradiction.

***Remark* 1.** The proof of Lemmas 2–4 does not rely on S belonging $\mathfrak{A}_1$ and, therefore, the statements proved above hold for a considerably broader class of surfaces. Lemmas 2–4 can be generalized to the case of operator L having variable coefficients.

3. Theorem 1. *Let $B(t)$ $(t \in [0, t_0) \subset [0, \infty))$ be a one-parameter family of bounded linear operators on a Banach space X: $\|B(t)\| \le 1$; $B(0) = I$. Let $\mathcal{D}$ be a dense linear manifold in X, on which a linear operator $A: \mathcal{D} \to X$ and a nonnegative function $h: \mathcal{D} \to \mathbb{R}$ are defined. Furthermore, assume that*

1) *$B(t)\mathcal{D} \subset \mathcal{D}$ for any $t \in [0, t_0)$;*

2) *there exists $\alpha > 0$ such that $\|B(t)B(s)x - B(t+s)x\| \le t^\alpha s\,h(x)$ for any $x \in \mathcal{D}$, t, s, $t + s \in [0, t_0)$;*

3) *there exists $a > 0$ such that $h(B(t)x) \le \exp(at)h(x)$ for any $x \in \mathcal{D}$, $t \in [0, t_0)$;*

4) *$\|B(t)x - x\| \le t\,h(x)$ for any $x \in \mathcal{D}$, $t \in [0, t_0)$;*

5) *there exists a function* $q:[0,t_0)\times\mathcal{D}\to\mathbb{R}$ *such that, for any* $t\in[0,t_0)$, $x\in\mathcal{D}$, *we have* $\left\|\frac{1}{t}(B(t)x-x)-Ax\right\|\le q(t,x)$, $q(t,x)\to 0$ *as* $t\to 0+$, *and*

$$q\left(\frac{t}{n}, B\left(\frac{t}{n}\right)^n x\right)\to 0 \quad as \ n\to\infty. \tag{6}$$

Then, for any $t\in[0,\infty)$,

$$V(t) = s-\lim_{n\to\infty} B\left(\frac{t}{n}\right)^n;$$

is defined: $V(t)$ *is a contraction* (C_0)*-semigroup on* X *and* $V'(0)=\overline{A}$.

Proof. *Step* 1. We shall show that $B(t/2^n)^{2^n}x$ has a limit as $n\to\infty$ for any $t\in[0,\infty)$ and $x\in X$.

Let $x\in\mathcal{D}$. By virtue of conditions 1–3 we have

$$\left\|B\left(\frac{t}{2^{n+1}}\right)^{2^{n+1}}x - B\left(\frac{t}{2^n}\right)^{2^n}x\right\|\le$$

$$\le\sum_{k=0}^{2^n-1}\left\|B\left(\frac{t}{2^{n+1}}\right)^{2k}\left(B\left(\frac{t}{2^{n+1}}\right)^2-B\left(\frac{t}{2^n}\right)\right)B^{2^n-k-1}\left(\frac{t}{2^n}\right)x\right\|\le$$

$$\le\left(\frac{t}{2^{n+1}}\right)^{\alpha}\sum_{k=0}^{2^n-1}\exp\left(ka\frac{t}{2^n}\right)h(x)=$$

$$=\left(\frac{t}{2^{n+1}}\right)^{1+\alpha}h(x)\frac{\exp(at)-1}{\exp\left(at/2^n\right)-1}. \tag{7}$$

Therefore, the left-hand side of (7) is $O(1/2^{n\alpha})$ as $n\to\infty$. This involves the convergence

$$V(t)x = \lim_{n\to\infty} B\left(\frac{t}{2^n}\right)^{2^n}x$$

for any $x\in\mathcal{D}$ and, hence, for any $x\in X$.

Step 2. $V(t)$ is a (C_0)-semigroup on X.

We verify that $V(nt)x=V^n(t)x$ $(n\in\mathbb{N}, x\in X)$. Let $x\in\mathcal{D}$. Then

$$\|B(nt)x-B^n(t)x\|\le\sum_{k=0}^{n-2}\left\|B^k(t)(B((n-k)t)x-B(t)B((n-k-1)t)x)\right\|\le$$

$$\le h(x)\,t^{1+\alpha}\frac{(n-1)n}{2}.$$

Thus, we have, due to the identity

$$A^m-B^m=\sum_{k=0}^{m-1}A^{m-k-1}(A-B)B^k,$$

$$\left\|B^m\left(\frac{nt}{m}\right)x-B^{nm}\left(\frac{t}{m}\right)x\right\|\le h(x)\left(\frac{t}{m}\right)^{1+\alpha}\frac{n^2}{2}\sum_{k=0}^{m-1}\exp\left(\frac{knat}{m}\right)=$$

$$=h(x)\left(\frac{t}{m}\right)^{1+\alpha}\frac{n^2}{2}\frac{\exp(nat)-1}{\exp(nat/m)-1}.$$

Therefore, $V(nt)x = V^n(t)x$ when passing to the limit for $m = 2^k \to \infty$.

This implies that

$$V(rt+t)x = V(rt)V(t)x \tag{8}$$

for any $x \in X$, $r \in Q$.

Let us verify the strong right continuity of $V(t)$ for $t > 0$.

Assume that $x \in \mathcal{D}$, $\Delta t > 0$:

$$\|B(t+\Delta t)x - B(t)x\| \le \|B(t+\Delta t)x - B(t)B(\Delta t)x\| +$$

$$+ \|B(t)(B(\Delta t)x - x)\| \le h(x)(t^\alpha + 1)\Delta t;$$

$$\left\| B\left(\frac{t+\Delta t}{m}\right)^m x - B\left(\frac{t}{m}\right)^m x \right\| \le$$

$$\le \sum_{k=0}^{m-1} \left\| B^{m-k-1}\left(\frac{t+\Delta t}{m}\right)\left(B\left(\frac{t+\Delta t}{m}\right) - B\left(\frac{t}{m}\right)\right) B^k\left(\frac{t}{m}\right) x \right\| \le$$

$$\le h(x)\frac{\Delta t}{m}\left(\left(\frac{t}{m}\right)^\alpha + 1\right)\frac{\exp(at) - 1}{\exp(at/m) - 1}.$$

When passing to the limit for $m = 2^k \to \infty$, we have

$$\|V(t+\Delta t)x - V(t)x\| \le h(x)\frac{\Delta t}{at}(\exp(at) - 1).$$

Hence, condition (8) holds for any $r > 0$, $x \in X$, and, together with condition 4) of the theorem, it proves that $V(t)$ is a (C_0)-semigroup on X.

Step 3. We shall prove that $V'(0) = \overline{A}$.

Let $x \in \mathcal{D}$, $t > 0$. Due to (7),

$$\|V(t)x - B(t)x\| \le t^{1+\alpha} h(x) \sum_{k=0}^{\infty} 2^{-(k+1)(1+\alpha)}\left(\sum_{l=0}^{2^k-1} \exp\frac{alt}{2^k}\right). \tag{9}$$

Since

$$\frac{1}{m}\sum_{l=0}^{m-1} \exp\frac{alt}{m} < \exp(at),$$

the right-hand side of (9) does not exceed

$$t^{1+\alpha} h(x)\frac{1}{2^{1+\alpha}}\sum_{k=0}^{\infty}\frac{1}{2^{k\alpha}}\exp(at_0) = C t^{1+\alpha} h(x) \quad (t \in (0, t_0)).$$

Condition 5) gives now: $\|(V(t)x - x)/t - Ax\| \to 0$ as $t \to 0+$, so $V'(0)|_{\mathcal{D}} = A$ and A is subtended.

We shall verify that $V(t)x \in \mathcal{D}(\overline{A})$ for $x \in \mathcal{D} = \mathcal{D}(A)$ and $\overline{A}V(t)x = V(t)Ax$. In this case, $V(t)\mathcal{D}(\overline{A}) \subset \mathcal{D}(\overline{A})$, so $V'(0) = \overline{A}$.

In view of condition 5 for $x \in \mathcal{D}$, $t \in [0, t_0)$, we have

$$\left\| B\left(\frac{t}{n}\right)^{n+1} x - B\left(\frac{t}{n}\right)^n x - \frac{t}{n} B\left(\frac{t}{n}\right)^n Ax \right\| \le \frac{t}{n} q\left(\frac{t}{n}, x\right);$$

$$\left\| B\left(\frac{t}{n}\right)^{n+1} x - B\left(\frac{t}{n}\right)^n x - \frac{t}{n} AB\left(\frac{t}{n}\right)^n x \right\| \le \frac{t}{n} q\left(\frac{t}{n}, B\left(\frac{t}{n}\right)^n x\right),$$

whence $\left\| AB\left(\frac{t}{n}\right)^n x - B\left(\frac{t}{n}\right)^n Ax \right\| \le q\left(\frac{t}{n}, x\right) + q\left(\frac{t}{n}, B\left(\frac{t}{n}\right)^n x\right).$

Passing to the limit, we get $V(t)x \in \mathcal{D}(\overline{A})$ and $\overline{A}V(t)x = V(t)Ax$.
It remains to note that, by virtue of the Chernoff theorem [6],

$$V(t)x = \lim_{n\to\infty} B\left(\frac{t}{n}\right)^n x \quad (x \in X,\ t \in [0,\infty)).$$

***Remark* 2.** Theorem 1 is close by nature to Theorem 2.1 in Chapter VI from [7].

4. Let S be a surface of class $\mathfrak{A}_1$, $S = \{x \mid g(x) = 1\}$, $g \in \mathfrak{A}_1(S_\varepsilon)$ (see pt. 1):

$$Z(x) = -\frac{g'(x)}{\|g'(x)\|^2} Lg(x)$$

is a vector field which is defined in S_α for some $\alpha \in (0,\varepsilon)$. Let $\beta > 0$ be such that $\{x \mid |g(x) - 1| < \beta\} \subset S_\alpha$ (existence of β is guaranteed by the properties of g) and let $q \in C^\infty(\mathbb{R})$ be given by

$$q(s) = \begin{cases} 1, & s \in [1-\beta/3, 1+\beta/3], \\ 0, & s \notin (1-2\beta/3, 1+2\beta/3). \end{cases}$$

Let Y be the vector field on H, which is equal to $q(g(x))Z(x)$ on $\{x \mid |g(x) - 1| < \beta\}$ and vanishes in other points. Let $\Phi_t(x) = \Phi(t,x)$ be the flow of Y.

Basically simple but cumbersome calculation shows that $u \circ \Phi_t \in \mathfrak{A}$ for $u \in \mathfrak{A}$. In doing so, for fixed t, we have

$$(u \circ \Phi_t)'(x) = u'(\Phi_t(x)) \circ \Phi_t'(x); \tag{10}$$

$$(u \circ \Phi_t)''(x) = \left\langle u'(\Phi_t(x)), \Phi_t''(x)(\cdot,\cdot)\right\rangle + [\Phi_t'(x)]^* u''(\Phi_t(x))\,\Phi_t'(x) \tag{11}$$

(here and below, the operation of "lifting-lowering of indices" is not specified).

For fixed x, $\Phi_t'(x)$ and $\Phi_t''(x)$ satisfy the equations

$$\frac{d}{dt}\Phi_t'(x) = Y'(\Phi_t(x))\,\Phi_t'(x), \quad \Phi_0'(x) = I; \tag{12}$$

$$\frac{d}{dt}\left\langle \Phi_t''(x), h\right\rangle = Y'(\Phi_t(x))\left\langle \Phi_t''(x), h\right\rangle + \left\langle Y''(\Phi_t(x)), \Phi_t'(x)h\right\rangle \Phi_t'(x),$$
$$\Phi_0'(x) = I; \quad \Phi_0''(x) = 0 \quad (h \in H). \tag{13}$$

If $u \in \mathfrak{A}$, then

$$Yu = \lim_{t\to 0}\frac{1}{t}(u \circ \Phi_t - u) \in X \tag{14}$$

due to the uniform boundedness of

$$\frac{d^2}{dt^2}(u \circ \Phi_t)(x) \quad (x \in H,\ t \in [0, t_0]).$$

on H.

For this reason, we can speak about the well-defined operator $L + Y : \mathfrak{A} \to X$. Furthermore, $L_S u|_S(x) = Lu(x) + Yu(x)$ for $x \in S$.

We shall show that $L + Y$ is subtended and $\overline{L+Y}$ is a generator of contraction (C_0)-semigroup $V(t)$.

Let us apply Theorem 1.

Let $T(t)$ be a (C_0)-semigroup on X with a generator $\overline{L}$ [2, 3];

$$P(t)u = u \circ \Phi_t, \quad B(t) = P(t)T(t), \quad \mathcal{D} = \mathfrak{A},$$

$$h(u) = C\left(\sup_H \|u''(\cdot)\| + \sup_H \|u'(\cdot)\|\right)$$

(the constant $C > 0$ can be found from computations given below).

Condition 1 follows from the embeddings $T(t)\mathfrak{A} \subset \mathfrak{A}$; $P(t)\mathfrak{A} \subset \mathfrak{A}$:

$$\|P(t)T(t)P(s)T(s)u - P(t+s)T(t+s)u\| \le$$
$$\le \|(T(t)P(s) - P(s)T(t))T(s)u\|.$$

As follows from [7],

$$h(T(t)u) \le h(u). \tag{15}$$

Thus, to verify condition 2, it suffices to prove that $\|(T(t)P(s) - P(s)T(t))v\| \le \le t^{1/2} s h(v)$. In the notation of [2, 3],

$$T(t)(v \circ \Phi_s)(x) - T(t)v(\Phi_s(x)) =$$
$$= \lim_{n \to \infty} \int_H (v(\Phi(s, x+y)) - v(\Phi(s, x) + y))\, \mu_{tA_n}(dy).$$

Since the value of the left-hand side of the last equality depends on the value y in the ball $\|y\| \le \sqrt{t\|j\|}$ [3, 5] and

$$\|\Phi(s, x+y) - \Phi(s, x) - y\| \le \|\Phi'_s(x) - I\|\,\|y\| \le C_1 s\|y\| \le C_2 s\sqrt{t}$$

with some $C_2 > 0$ for $s \in (0, 1)$ because of (12),

$$\|T(t)P(s)v - P(s)T(t)v\| \le C_2 s\sqrt{t}\, \sup_H \|v'(\cdot)\| \le s\sqrt{t}\, h(v).$$

Standard integral inequalities yield the following estimates of solutions of equations (12), (13):

$$\left\|\Phi'_t(x)\right\| \le \exp(C_3 t); \quad \left\|\Phi''_t(x)\right\| \le C_4 t \exp(C_5 t)$$
$$(C_3, C_4, C_5 > 0;\ t \in [0, t_0] \subset [0, \infty)).$$

In view of (10), (11), we get $h(P(t)u) \le [\exp(2C_3 t) + C_4 t \exp(C_5 t)]\, h(u)$. By virtue of (15), this proves condition 3) of Theorem 1.

Let us verify condition 4).

$$\|T(t)u - u\| = \left\|\int_0^t T(t)\, Lu\, dt\right\| \le \frac{1}{2} t\|j\| \sup_H \|u''(\cdot)\|;$$

$$u(\Phi(t, x)) - u(x) = (u'(\Phi(\tau, x)), Y(\Phi(\tau, x)))t \quad (\exists\, \tau \in (0, t)),$$

so $\|(P(t) - I)u\| \le C_6 t \sup_H \|u'(\cdot)\|$ and

$$\|P(t)T(t)u - u\| \le \|(P(t) - I)T(t)u\| + \|T(t)u - u\| \le t h(u)$$

(for an appropriate constant $C > 0$);

$$\left\|\frac{1}{t}\big(P(t)T(t)u - u\big) - (L+Y)u\right\| \le \left\|\frac{1}{t}(P(t)T(t)u - T(t)u) - Y(T(t)u)\right\| +$$
$$+ \|Y(T(t)u - u)\| + \left\|\frac{1}{t}(T(t)u - u) - Lu\right\|;$$

$$\left\| \frac{1}{t}(P(t)v - v) - Yv \right\| \le \frac{1}{2} t \sup_{x \in H;\ \tau \in [0,t]} \left| \frac{d^2}{d\tau^2}\left(v \circ \Phi_\tau\right)(x) \right| \le C_7 t\, h(v);$$

$$\| Y(T(t)u - u) \| \le \sqrt{t \| j \|} \sup_H \| u'' \| \cdot \sup_H \| Y \| \le C_8 t^{1/2} h(u);$$

$$\left\| \frac{1}{t}(T(t)u - u) - Lu \right\| \le \frac{1}{t} \int_0^t \| L(T(t)u - u) \| \, dt. \tag{16}$$

Let

$$\varepsilon(\delta, u) = \sup\{ \| u''(x) - u''(y) \| \mid \| x - y \| \le \delta \} + $$
$$+ \sup\{ \| u'(x) - u'(y) \| \mid \| x - y \| \le \delta \},$$

$\varepsilon(\delta, u) \to 0$ as $\delta \to 0+$.

Since $\| (T(t)u - u)'' \| \le \varepsilon\left(\sqrt{t \| j \|}, u\right)$ and because of (16),

$$\left\| \frac{1}{t}(T(t)u - u) - Lu \right\| \le \frac{1}{2} \| j \| \, \varepsilon\left(\sqrt{t \| j \|}, u\right).$$

Since $h(T(t)u) \le h(u)$, it is natural to set

$$q(t, u) = C_7 t\, h(u) + C_8 t^{1/2} h(u) + \frac{1}{2} \| j \| \, \varepsilon\left(\sqrt{t \| j \|}, u\right).$$

To verify condition (6), it suffices to verify that

$$\varepsilon\left(\sqrt{\frac{t}{n} \| j \|}, B\left(\frac{t}{n}\right)^n u \right) \to 0 \quad \text{as} \quad n \to \infty.$$

Let

$$\alpha(\delta) = \sup\{ \| Y''(x) - Y''(y) \| \mid \| x - y \| \le \delta \} + $$
$$+ \sup\{ \| Y'(x) - Y'(y) \| \mid \| x - y \| \le \delta \}$$

(the uniform continuity of Y'' on H follows from the construction of the vector field Y). It follows from (10)–(13) by means of the integral inequalities [8] that there exist $t_0 > 0$, $c > 0$ such that

$$\varepsilon(\delta, P(t)u) \le c t\, h(u) \alpha(e^{ct}\delta) + e^{ct} \varepsilon(e^{ct}\delta, u)$$

for any $t \in (0, t_0)$, $u \in \mathfrak{A}$.

Also, we have $\varepsilon(\delta, T(t)u) \le \varepsilon(\delta, u)$. This implies that (put $c \ge a$)

$$\varepsilon\left(\delta, B\left(\frac{t}{n}\right)^n u \right) \le c t e^{ct} h(u) \alpha(e^{ct}\delta) + e^{ct} \varepsilon(e^{ct}\delta, u).$$

It only remains to note that $\alpha(\delta) \to 0$ and $\varepsilon(\delta, u) \to 0$ as $\delta \to 0$.

5. The continuation procedure of the function $v \in X(S)$ to a function on the whole H described in [4], defines a mapping $i: X(S) \to X$ for which $i(\mathfrak{A}(S)) \subset \mathfrak{A}$. The restriction of the function $u \in X$ to S gives the function $p(u) \in X(S)$. Furthermore, $p(\mathfrak{A}) = \mathfrak{A}(S)$. i and p are linear bounded operators.

We define operators $W(t)$: $X(S) \to X(S)$ $(t \ge 0)$ by the rule: $W(t) = p V(t) i$.

By virtue of (4) and (14), $L_S v \in X(S)$ for $v \in \mathfrak{A}(S)$. Put $\mathcal{D}(L_S) = \mathfrak{A}(S)$.

Theorem 2. *$W(t)$ is a contraction (C_0)-semigroup on $X(S)$, $W'(0) = \overline{L}_S$.*

Proof. To verify the semigroup rule, it suffices to verify that

$$W(t)(\operatorname{Ker} p) \subset \operatorname{Ker} p. \tag{17}$$

Lemma 5. *Let $V(t)$ be a contraction (C_0)-semigroup on a Banach space X with a generator A and let X_1 be a subspace in X invariant with respect to $(\lambda - A)^{-1}$ for $\lambda > \lambda_0 \geq 0$. Then X_1 is invariant with respect to the operators $V(t)$ $(t \geq 0)$.*

Proof. X_1 is invariant with respect to the bounded operators $A_\lambda = \lambda A(\lambda - A)^{-1}$ $(\lambda > \lambda_0)$ and, hence, it is invariant with respect to $\exp(tA_\lambda)$. It remains to note that

$$V(t) = s - \lim_{\lambda \to +\infty} \exp(tA_\lambda).$$

The lemma is proved.

In the case under consideration, $A = \overline{L+Y}$. The condition $(\lambda - A)^{-1}(\operatorname{Ker} p) \subset \subset \operatorname{Ker} p$ $(\lambda > 0)$ follows from Lemma 4. Thus, (17) is proved.

The conditions $I = s - \lim_{t \to 0+} W(t)$ and $\| W(t) \| \leq 1$ are evident.

In view of Lemma 1, $p(L+Y)i = L_S$. Hence, $L_S \subset W'(0)$. We shall show that $p(\mathcal{D}(\overline{L+Y})) = \mathcal{D}(\overline{L}_S)$. Let $u_n \in \mathfrak{A}$, $u_n \to u \in \mathcal{D}(\overline{L+Y})$, $(L+Y)u_n \to \to (\overline{L+Y})u$. Then $pu_n \to pu$, $L_S p u_n = p(L+Y)u_n \to p(\overline{L+Y})u$, so $pu \in \in \mathcal{D}(\overline{L}_S)$ and $\overline{L}_S pu = p(\overline{L+Y})u$.

For this reason, for $v \in \mathfrak{A}(S) = \mathcal{D}(L_S)$ we have $W(t)v = pV(t)iv \in \mathcal{D}(\overline{L}_S)$ and

$$\overline{L}_S W(t)v = \overline{L}_S pV(t)iv = p(\overline{L+Y})V(t)iv =$$

$$= pV(t)(L+Y)iv = pV(t)iL_S v = W(t)L_S v$$

(due to (17) for $(L+Y)iv - iL_S v \in \operatorname{Ker} p$).

Thus, $W(t)\mathcal{D}(\overline{L}_S) \subset \mathcal{D}(\overline{L}_S)$ and $W'(0) = \overline{L}_S$.

So it is proved that the Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = \overline{L}_S u, \\ u(0) \in D(\overline{L}_S) \end{cases}$$

is uniformly well-posed.

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