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GROUPS WITH FEW NONMODULAR SUBGROUPS*

ГРУПИ З НЕБАГАТЬМА НЕМОДУЛЯРНИМИ ПІДГРУПАМИ

Let G be a Tarski-free group such that the join of all nonmodular subgroups of G is a proper subgroup in G. It is proved that G contains a finite normal subgroup N such that the factor group G/N has modular subgroup lattice.

Нехай G — група, що не має секцій, ізоморфних групам Тарського, всі немодулярні підгрупи якої породжують в G деяку власну підгрупу. Доведено, що G містить скінченну нормальну підгрупу N, фактор-група G/N якої є групою з модулярною решіткою підгруп.

1. Introduction. Let G be a group. A subgroup of G is said to be *modular* if it is a modular element of the lattice $\mathfrak{L}(G)$ of all subgroups of G, and G is called *modular group* if $\mathfrak{L}(G)$ is a modular lattice. Groups in which every subgroup is normal and any Tarski group (i.e., an infinite simple group in which any proper nontrivial subgroup has prime order) are clearly modular groups; and it is known that a projective image (i.e., an isomorphism between subgroup lattices) of a normal subgroup is a modular subgroup, thus modular subgroups may be a considered as a lattice generalization of normal subgroups. Groups with modular subgroup lattice have been completely described by K. Iwasawa and R. Schmidt; for a detailed account on this subject, we refer the reader to [1].

An important role among modular groups is played by permutable subgroups. A subgroup H of a group G is called *permutable* (or even *quasinormal*), when $HK = KH$ for every subgroup K of G. Obviously, a permutable subgroup is a modular subgroup and a result of Stonehewer (see [2]) states that a modular subgroup is permutable if and only if it is ascendant. Groups in which every subgroup is permutable are called *quasihamiltonian*, thus they are locally nilpotent groups with modular subgroup lattice.

Generalizations of modular groups are the object of many papers (see, for instance, [3 – 7]). In particular, Mainardis [7] consider groups G in which the join $Q(G)$ of all nonpermutable subgroups is a proper subgroup; it is simple to see that such a group is generated by the set $G \setminus Q(G)$ and thus G is locally nilpotent by the above quoted result of Stonehewer. It was proved in [7] that a group G such that $G \neq Q(G)$ is either quasihamiltonian or the direct product $G = H \times K$, where H is p-group (p prime) such that $\{1\} \neq Q(H) \neq H$, and K is a periodic quasihamiltonian group that does not contain elements of order p. If G is a group, we define, similary, $M(G)$ to be the join of all nonmodular subgroups of G. Clearly, $M(G)$ is a characteristic subgroup of G and G can be generated by the set $G \setminus M(G)$; moreover, every

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Tarski-free, the index $|L_{i+1} : L_i|$ is finite for every i (see [9], Theorem B); therefore, M has finite index in $\langle M, x\rangle$ and, hence, $\langle M, x\rangle$ is locally finite. On the other hand, since M and $\langle x\rangle$ are modular subgroups of G, also $\langle M, x\rangle$ is a modular subgroup of G contradicting the maximality of M in $\mathfrak{F}$. Therefore, $G \setminus M(G)$ is contained in M and so $G = M$ is locally finite.

A normal subgroup N of a group G is said to be *hypercyclically embedded* in G if it has an ascending G-invariant series with cyclic factors. It was proved by Schmidt that in a finite group the normal closure of any cyclic modular subgroup is hypercyclically embedded in the group (see [1], 5.2.5).

Proposition 1. *Let G be a Tarski-free group such that $G \neq M(G)$. Then G is locally supersoluble.*

Proof. Since any nonperiodic modular group is locally nilpotent, we may assume by Theorem A that G is periodic. By lemma 1, we may also suppose that G is finitely generated and so even finite by Lemma 4. Let x be an element of $G \setminus M(G)$. By the above quoted result of Schmidt, the subgroup $\langle x\rangle^G$ has a finite G-invariant series with cyclic factors. On the other hand, the factor group $G/\langle x\rangle^G$ has modular subgroup lattice and so, since G is Tarski-free, it is supersoluble. Therefore, G is supersoluble.

The proposition is proved.

Recall here that a group G is called *P-group* if it is either abelian of prime exponent or it is the semidirect product of a normal abelian subgroup A of prime exponent p and a cyclic subgroup of prime order $q \neq p$ which induce a nontrivial power automorphism on A. In particular, it is simple to see that any P-group has modular subgroup lattice (see [1], 2.4.1).

A subgroup H of a group G is said to be *P-embedded* in G if G/H_G is periodic and the following conditions are satisfied:

$G/H_G = \left(Dr_{i\in I}(S_i/H_G)\right) \times L/H_G$ where each S_i/H_G is a nonabelian P-group;

$\pi(S_i/H_G) \cap \pi(S_j/H_G) = \varnothing = \pi(S_i/H_G) \cap \pi(L/H_G)$ for every $i \neq j$;

$H/H_G = \left(Dr_{i\in I}(P_i/H_G)\right) \times (H \cap L)/H_G$ where each P_i/H_G is a nonnormal Sylow subgroup of S_i/H_G;

$H \cap L$ is a permutable subgroup of G.

Clearly, every P-embedded subgroup is modular, and Stonehewer has proved that a modular subgroup of a locally finite group is either permutable or P-embedded (see [9], Theorem 3.2 and Theorem E).

***Proof of Theorem* B.** The group G is locally supersoluble by Proposition 1. Clearly, assume that $M(G)$ is not trivial, thus G is periodic by Theorem A. Let $x \in G \setminus M(G)$ and, without loss of generality, assume that x has prime-power order; moreover, let X be any subgroup of G. Thus, $H = \langle X, x\rangle$ is a modular subgroup of G. We now will prove that X has finite index in H, thus, without loss of generality, we may suppose that $\langle x\rangle$ is a nonpermutable core-free subgroup of H and that $x \notin X$. Since $\langle x\rangle$ is a modular subgroup of G, $\langle x\rangle$ is P-embedded in H and so, as x has prime-power order, we have that

$$H = S \times K,$$

where S is a nonabelian P-group, $\pi(S) \cap \pi(K) = \varnothing$, and $\langle x\rangle$ is a nonnormal Sylow subgroup of S; in particular, x must have prime order (see[1], 2.2.2). Therefore,

$$S = S \cap H = S \cap \langle X, x\rangle = \langle x, S \cap X\rangle$$

and

$$[S/S \cap X] \simeq [\langle x\rangle/\langle x\rangle \cap X],$$

thus the index $|S : S \cap X|$ is finite; thus also the index

$$|H : X| = |SX : X| = |S : S \cap X|$$

is finite.

It follows that every subgroup of G has a finite index in a modular subgroup of G, and hence G contains a finite normal subgroup N such that G/N is a modular group (see [5]). Since G is Tarski-free, G/N is metabelian and hence G'' is finite.

The theorem is proved.

Let G be a periodic group such that $G \neq Q(G)$. Thus, as quoted in the introduction, $G = H \times K$, where H is a p-group (p prime) such that $\{1\} \neq Q(H) \neq H$, and K is a periodic quasihamiltonian group that does not contains elements of order p. Mainardis has also proved that G is metabelian if its 2-Sylow subgroup is quasihamiltonian (see [7], Theorem E), and that the subgroup H is finite-by-quasihamiltonian if either H has finite exponent and p is odd or H has infinite exponent (see [7], Theorem C and Theorem D; [3], Theorem 1). As consequence of Theorem B, we have the following statement:

Corollary. *Let G be a periodic group such that $Q(G) \neq G$. Then G is finite-by-quasihamiltonian. Moreover, G is hypercentral and G'' is finite.*

Proof. By Theorem B, there exists a finite normal subgroup N of G such that G/N is a modular group and G'' is finite. Since G is locally nilpotent, G/N is quasihamiltonian. Moreover, as any Sylow subgroup of G is hypercentral (see [4], Lemma 2.4), the group G is hypercentral.

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