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OSCILLATION CRITERIA FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DAMPING

ОСЦИЛЯЦІЙНІ КРИТЕРІЇ ДЛЯ НЕЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ ІЗ ЗАТУХАННЯМ

Some new oscillation criteria are given for general nonlinear second order ordinary differential equations with damping of the form $x'' + p(t)x' + q(t)f(x) = 0$, where f is with or without monotonicity. Our results generalize and extend some earlier results of Deng.

Наведено деякі нові осциляційні критерії для загальних нелінійних звичайних диференціальних рівнянь другого порядку із затуханням вигляду $x'' + p(t)x' + q(t)f(x) = 0$, де функція f або монотонна, або немонотонна. Наведені результати узагальнюють та розширюють деякі результати, отримані раніше Денгом.

1. Introduction. Consider the second order linear differential equation with damped term

$$x'' + p(t)x' + q(t)x = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

and the more general nonlinear equation

$$x'' + p(t)x' + q(t)f(x) = 0, \quad t \geq t_0 > 0, \tag{1.2}$$

where $p \in C^1([t_0, \infty), R)$, $q \in C([t_0, \infty), R)$, $f \in C(R, R)$ is to be specified in the subsequent text, and $xf(x) > 0$ whenever $x \neq 0$.

As usual, a nontrivial solution of (1.1) or (1.2) is called oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (1.1) or (1.2) is said to be oscillatory if all of its solutions are oscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions for different classes of second order differential equations [1 – 9]. In the absence of damping, many results have been obtained for particular cases of (1.1), such as the linear equation

$$x'' + q(t)x = 0 \tag{1.3}$$

and the quasilinear equation

$$\left(|x'|^{\gamma-1}x'\right)' + q(t)|x|^{\gamma-1}x = 0, \tag{1.4}$$

where $\gamma > 0$ is a constant.

It is well known that Hille [4] studied the linear equation (1.3) and obtained that equation (1.3) is oscillatory if

$$\int_t^\infty q(s)\,ds \geq \frac{1+\delta}{4t}, \tag{1.5}$$

where δ is any small positive number. If introducing the transformation $u = x e^{-\frac{1}{2}\int^t p(s)ds}$ for the damping equation (1.1), we have

$$u'' + \left[q(t) - \frac{p^2(t)}{4} - \frac{p'(t)}{2}\right]u = 0, \tag{1.6}$$

which has the same form as equation (1.3). Hence applying condition (1.5) we can easily know that equation (1.6) is oscillatory if

$$\int_t^{\infty}\left[q(s)-\frac{p^2(s)}{4}-\frac{p'(s)}{2}\right]ds \;\geq\; \frac{1+\delta}{4t}. \tag{1.7}$$

In an early paper [5], Chunchao Huang has established interesting oscillation and nonoscillation criteria for the equation (1.3) with $q \in C[0, \infty)$ and $q(t) \geq 0$, where conditions about the integrals of $q(t)$ on every interval $[2^n t_0, 2^{n+1} t_0]$, $n = 1, 2, \ldots$, for some fixed $t_0 > 0$ are used in the results. Since that time, many authors have also investigated the oscillatory and nonoscillatory behavior of equation (1.3) by using Huang's technique, such as papers [3, 8].

Recently, by using the similar method in the proof of [6] (Lemma 3), Deng [2] presented the following result for the oscillation of equation (1.3) with $q \in L^1[t_0, \infty)$.

Theorem A. *If for large* $t \in R$,

$$\int_t^{\infty} q(s)ds \;\geq\; \frac{\alpha_0}{t}, \tag{1.8}$$

where $\alpha_0 > 1/4$, *then equation* (1.3) *is oscillatory.*

More recently, inspired by the recent work of Deng [2], Yang [9] obtain following oscillation result.

Theorem B. *If* $q \in L^1[t_0, \infty)$ *and for large* $t > t_0$,

$$t^{\gamma}\int_t^{\infty} q(s)ds \;\geq\; \alpha_0, \tag{1.9}$$

where $\alpha_0 > \dfrac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}$, *then equation* (1.4) *is oscillatory.*

It is obvious that (1.8) or (1.9) is condition on the integral of $q(s)$ in $[t, \infty)$ for arbitrarily large value of t, while the conditions of Huang [5] and Elbert [3] concern the integral of $q(s)$ in $[2^n t_0, 2^{n+1} t_0]$ for every $n \in N$, and the condition of Yang [8] concern the integral of $q(s)$ in $[t_0/\varepsilon, t_0/\varepsilon^{n+1}]$ for every $n \in N$ and $0 < \varepsilon < 1$. Therefore, they are different kinds of condition.

Motivated by the idea of Deng [2], in this paper we are study the more general equation (1.2), and obtain oscillation criteria which contain Theorems A and B as a special case, and establish oscillation criteria for equation (1.2) when $f(x) = |x|^{\lambda}\operatorname{sgn} x$ with $\lambda > 1$ or $0 < \lambda < 1$.

This paper is organized as follows. In Section 2, we shall present oscillation criteria for equation (1.2) when $q(t) \geq 0$ and the function $f(x)$ is not monotonous. Section 3 contains also oscillation criteria for equation (1.2) when $q(t)$ changes its sign and the function $f(x)$ is monotonous.

2. Oscillation results for $f(x)$ without monotonicity. In this section, we consider the oscillation of equation (1.2) when $q(t) \geq 0$ and the function $f(x)$ is not monotonous.

Theorem 2.1. *Let* $q(t) \geq 0$ *and* $\dfrac{f(x)}{x} \geq M > 0$ *for* $x \neq 0$, *where* M *is a constant. If for large* $t \in R$,

$$p(t) = O(1) \quad and \quad \int_t^{\infty} \left[Mq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \ \geq \ \frac{\alpha_0}{t}, \tag{2.1}$$

where $\alpha_0 > 1/4$, *then equation* (1.2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.2), which, without loss of generality, can be assumed to be $x(t) > 0$, $f(x(t)) > 0$ for $t \geq t_0 > 0$. By (2.1), it is easy to see that there exists $t_1 \geq t_0$ such that

$$\int_t^{\infty} \left[Mq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \ \geq \ \frac{\alpha_0}{t} \quad \text{for} \quad t \ \geq \ t_1,$$

which yields that there exists an integer $n(t) \geq t$ such that

$$\int_t^{t'} \left[Mq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \ \geq \ \frac{\alpha_1}{t} \quad \text{and} \quad \frac{\alpha_1^2}{t} - \frac{\alpha_1^2}{t'} \ \geq \ \frac{\alpha^2}{t} \quad \text{for} \quad t' \ \geq \ n(t), \tag{2.2}$$

where $\alpha_0 \geq \alpha_1 \geq \alpha > 1/4$.

Define $v(t) = x'(t)/x(t)$ for $t \geq t_0$. By equation (1.2), $v(t)$ satisfies the equation

$$v' + v^2 + pv + q\,\frac{f(x)}{x} = 0. \tag{2.3}$$

Because $\dfrac{f(x)}{x} \geq M > 0$ and $q \geq 0$, (2.3) can be rewritten as

$$v' + v^2 + pv + Mq \ \leq \ 0.$$

Let $w(t) = v(t) + \dfrac{p(t)}{2}$. Now, we have

$$w' + w^2 + Mq - \frac{p^2}{4} - \frac{p'}{2} \ \leq \ 0. \tag{2.4}$$

Integrating (2.4) from t to t', by (2.2) we obtain

$$w(t) - w(t') \ \geq \ \int_t^{t'} w^2(s)\,ds + \int_t^{t'} \left[Mq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \ \geq \ 0 \tag{2.5}$$

for $t' \geq n(t)$ and $t \geq t_1$.

If there exists $t_2 \geq t_1$ such that $w(t_2) < 0$, then from (2.5), $w(t) < 0$ for $t \geq n(t_2)$. Therefore, $w(t)$ is either eventually positive or eventually negative.

If $w(t)$ is eventually negative, then there exists $t_3 \geq t_1$ such that $w(t) < 0$ for $t \geq$ $\geq t_3$ and

$$|w(t)| \ \geq \ \int_{t_3}^{t} w^2(s)\,ds + \int_{t_3}^{t} \left[Mq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds$$

for $t \geq n(t_3)$ which, with (2.2), yields

$$|w(t)| \ \geq \ \int_{t_3}^{t} w^2(s)\,ds + \frac{\alpha_1}{t_3} \ \geq \ \frac{\alpha_1}{t_3} \ \geq \ \frac{\alpha_1}{t} \quad \text{for} \quad t \ \geq \ n(t_3). \tag{2.6}$$

Substituting this into (2.6), we obtain

$$|w(t)| \geq \alpha_1^2 \int\limits_{n(t_3)}^{t} \frac{1}{s^2}\,ds + \frac{\alpha_1}{n(t_3)} \geq \frac{\tau_0^2 + \tau_0}{t} \quad \text{for} \quad t \geq n(n(t_3)), \tag{2.7}$$

where $\tau_0 = \alpha > 1/4$.

If $w(t)$ is eventually positive, then there exists $t_4 \geq t_1$ such that $w(t) > 0$ for $t \geq t_4$ and from (2.2) and (2.5), we have

$$w(t) \geq \int\limits_{t}^{t'} w^2(s)\,ds + \int\limits_{t}^{t'} \left[Mq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \geq \frac{\alpha_1}{t} \tag{2.8}$$

for $t' \geq n(t)$ and $t \geq t_4$.

Using similar methods to those in the proof of (2.7), we get

$$|w(t)| \geq \frac{\tau_0^2 + \tau_0}{t} \quad \text{for} \quad t \geq n(t_4). \tag{2.9}$$

Setting $\tau_i = \tau_{i-1}^2 + \tau_0$, $i = 1, 2, \ldots$, and taking $t_5 = \max\{n(n(t_3)), n(t_4)\}$, we obtain

$$|w(t)| \geq \frac{\tau_1}{t} \quad \text{for} \quad t \geq t_5$$

from (2.7) and (2.9). By induction, from (2.6) and (2.8), we can prove that

$$|w(t)| \geq \frac{\tau_i}{t} \quad \text{for} \quad t \geq t_5, \quad i = 1, 2, \ldots .$$

It is easy to see that

$$|w(t)| \to \infty \quad \text{for} \quad t \geq t_5.$$

Using $p(t) = O(1)$ for large $t \in R$, we obtain $|v(t)| \to \infty$. So, this contradiction completes the proof.

Thus, $f(x)$ acts a linear function for sufficiently large x, we have the following result for the equation (1.2).

***Corollary* 2.1.** *Let $f(x)$ behaves like Mx for sufficiently large x. If for large $t \in R$,*

$$\int\limits_{t}^{\infty} \left[Mq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \geq \frac{\alpha_0}{t}, \tag{2.10}$$

where $\alpha_0 > 1/4$, then equation (1.2) *is oscillatory.*

***Remark* 2.1.** If $\alpha_0 > 1/4$, then $\alpha_0 \geq \dfrac{1+\delta}{4}$ where δ is any small positive number. Thus, condition (2.10) with $M = 1$ reduces to (1.7).

The following theorem is concerned with the oscillatory behavior of a special case of equation (1.2), namely, the equation

$$x'' + p(t)x' + q(t)|x|^{\lambda} \operatorname{sgn} x = 0, \tag{2.11}$$

where $\lambda > 0$ is a real constant.

Theorem 2.2. *Let $q(t) \geq 0$. If for large $t \in R$ and every constant $c > 0$,*

$$p(t) = O(1) \quad and \quad \int_t^\infty \left[cq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \geq \frac{\alpha_0}{t}, \tag{2.12}$$

where $\alpha_0 > 1/4$, *then*:

(i) *every unbounded solution of equation* (2.11) *with* $\lambda > 1$ *is oscillatory,*

(ii) *every bounded solution of equation* (2.11) *with* $0 < \lambda < 1$ *is oscillatory.*

Proof. Without loss of generality, we assume that $x(t)$ is a nonoscillatory solution of equation (2.11) such that $x(t) > 0$ for $t \geq t_0 > 0$. By (2.12), there exists $t_1 \geq t_0$ such that

$$\int_t^\infty \left[cq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \geq \frac{\alpha_0}{t} \quad \text{for} \quad t \geq t_1$$

so there exists an integer $n(t) \geq t$ such that

$$\int_t^{t'} \left[cq(s) - \frac{p^2(s)}{4} - \frac{p'(s)}{2} \right] ds \geq \frac{\alpha_1}{t} \quad \text{and} \quad \frac{\alpha_1^2}{t} - \frac{\alpha_1^2}{t'} \geq \frac{\alpha^2}{t} \quad \text{for} \quad t' \geq n(t),$$

where $\alpha_0 \geq \alpha_1 \geq \alpha > 1/4$.

Define $v(t) = x'(t)/x(t)$ for $t \geq t_0$. By equation (2.11), $v(t)$ satisfies the equation

$$v' + v^2 + pv + qx^{\lambda-1} = 0. \tag{2.13}$$

Next, we consider the following two cases:

(i) If $x(t)$ is an unbounded nonoscillatory solution of equation (2.11) with $\lambda > 1$ for $t \geq t_0$, then there exist a constant $k_1 > 0$ and $t_2 \geq t_1 \geq t_0$ such that $x(t) \geq k_1$ for $t \geq t_2$. Therefore

$$x^{\lambda-1}(t) \geq k_1^{\lambda-1} = c_1 \quad \text{for} \quad t \geq t_2, \tag{2.14}$$

where c_1 is a constant. Using (2.14) and $q(t) \geq 0$ in (2.13), and proceeding as in the proof of Theorem 2.1, we arrive at the desired contradiction.

(ii) If $x(t)$ is a bounded nonoscillatory solution of equation (2.11) with $0 < \lambda < 1$ for $t \geq t_0$, then there exist a constant $k_2 > 0$ and $t_2 \geq t_1 \geq t_0$ such that $x(t) \leq k_2$ for $t \geq t_2$. Therefore

$$x^{\lambda-1}(t) \geq k_2^{\lambda-1} = c_2 \quad \text{for} \quad t \geq t_2,$$

where c_2 is a constant. The rest of the proof is similar to that in the previous case and hence is omitted.

3. Oscillation results for $f(x)$ with monotonicity. In this section, we establish the oscillation of equation (1.2) under the assumption that $q(t)$ changes its sign and the function $f(x)$ is monotonous.

Theorem 3.1. *Assume that* $f \in C^1(R, R)$ *and* $f'(x) \geq K > 0$ *for all* $x \in R$, *where* K *is a constant. If for large* $t \in R$,

$$p(t) = O(1) \quad and \quad \int_t^\infty \left[q(s) - \frac{p^2(s)}{4K} - \frac{p'(s)}{2K} \right] ds \geq \frac{\alpha_0}{t}, \tag{3.1}$$

where $\alpha_0 > 1/4$, *then equation* (1.2) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1.2), say, $x(t) > 0$ when $t \geq t_0 > 0$ for some t_0 depending on the solution $x(t)$. By condition (3.1), there exists $t_1 \geq t_0$ such that

$$\int_t^\infty \left[q(s) - \frac{p^2(s)}{4K} - \frac{p'(s)}{2K} \right] ds \geq \frac{\alpha_0}{t} \quad \text{for} \quad t \geq t_1,$$

which yields that there exists an integer $n(t) \geq t$ such that

$$\int_t^{t'} \left[q(s) - \frac{p^2(s)}{4K} - \frac{p'(s)}{2K} \right] ds \geq \frac{\alpha_1}{t} \quad \text{and} \quad \frac{K\alpha_1^2}{t} - \frac{K\alpha_1^2}{t'} \geq \frac{\alpha^2}{t} \quad \text{for} \quad t' \geq n(t),$$

where $\alpha_0 \geq \alpha_1 \geq \alpha > 1/4$.

Define $v(t) = x'(t)/f(x(t))$ for $t \geq t_0$. By equation (1.2), $v(t)$ satisfies the equation

$$v' + f'(x)v^2 + pv + q = 0. \tag{3.2}$$

Because $f'(x) \geq K > 0$, and setting $w(t) = v(t) + \dfrac{p(t)}{2K}$, (3.2) can be rewritten as

$$w' + Kw^2 + q - \frac{p^2}{4K} - \frac{p'}{2K} \leq 0.$$

The rest of the proof is similar to that of Theorem 2.1, and is omitted.

Now, by combining some ingredients of the proofs of Theorem 2.2 and of Theorem 3.1, we give the following theorem, whose proof is similar to that of Theorem 2.2, for the equation (2.11).

Theorem 3.2. *If for large $t \in R$ and every constant $\beta > 0$,*

$$p(t) = O(1) \quad \textit{and} \quad \int_t^\infty \left[q(s) - \frac{p^2(s)}{4\beta} - \frac{p'(s)}{2\beta} \right] ds \geq \frac{\alpha_0}{t},$$

where $\alpha_0 > 1/4$, then:

(i) *every unbounded solution of equation* (2.11) *with $\lambda > 1$ is oscillatory,*

(ii) *every bounded solution of equation* (2.11) *with $0 < \lambda < 1$ is oscillatory.*

***Remark* 3.1.** Theorems 2.1 and 3.1 extend and improve Theorem A for the nonlinear equation (1.2). In addition to this, they are true for the linear equation (1.1). Also note that when $f(x) \equiv x$, it is not necessary to assume $q(t) \geq 0$ in Theorem 2.1. So, if $p(t) \equiv 0$ then Theorems 2.1 and 3.1 reduce to Theorem A, Theorem B with $\gamma =$ $= 1$ and the well known Hille's result.

***Remark* 3.2.** If we compare Theorem 2.2 or Theorem 3.2 with Theorem 2.2 given in [1], it is easy to see that these results include different types of sufficient conditions for the oscillation.

Finally, we give an example to illustrate the efficiency of our results. The example is not covered by any of the results of Deng [2] and Yang [9].

Example. Consider the equations

$$x'' + \frac{2}{t}x' + \frac{\theta}{t^2}[x + x^3] = 0, \tag{3.3}$$

$$x'' + \frac{2}{t}x' + \frac{\theta}{t^2}|x|^\lambda \operatorname{sgn} x = 0, \tag{3.4}$$

and

$$x'' + \frac{1}{t}x' + \frac{\mu}{t^2}x\left(1 + \frac{1}{1+x^2}\right) = 0, \tag{3.5}$$

where $t > 0$, $\theta > \frac{1}{4}$, $\lambda > 0$ and $\mu > 0$. Note that, for the equation (3.3) $\frac{f(x)}{x} = 1 + x^2 \geq 1 = M$ or $f'(x) = 1 + 3x^2 \geq 1 = K$ for all x, and $\frac{p^2(t)}{2} + p'(t) = 0$ when $p(t) = \frac{2}{t}$. Applying Theorem 2.1 or Theorem 3.1 for the equation (3.3), and Theorem 3.2 for the equation (3.4), it is easy to verify that $\int_t^\infty \frac{\theta}{s^2}\,ds = \frac{\theta}{t} > \frac{1}{4t}$. Hence, equation (3.3), every unbounded solution of equation (3.4) with $\lambda > 1$ and every bounded solution of equation (3.4) with $0 < \lambda < 1$ are oscillatory for $\theta > \frac{1}{4}$. Note that $f(x) = x\left(1 + \frac{1}{1+x^2}\right)$ and $f'(x) = 1 + \frac{1-x^2}{(1+x^2)^2}$ for the equation (3.5). It is clear that Theorem 3.1 cannot be applied to equation (3.5). Nevertheless, we can prove the oscillatory character of equation (3.5) by using Theorem 2.1 or Corollary 2.1. Taking into account that $\frac{f(x)}{x} = 1 + \frac{1}{1+x^2} \geq 1 = M$ for $x \neq 0$, we get

$$\int_t^\infty \left[\frac{\mu}{s^2} - \frac{1}{4s^2} + \frac{1}{2s^2}\right] ds = \left(\mu + \frac{1}{4}\right)\frac{1}{t} > \frac{1}{4t}.$$

Hence, equation (3.5) is oscillatory for $\mu > 0$.

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