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PERIODIC MOVING WAVES ON 2 D LATTICES WITH NEAREST NEIGHBOR INTERACTIONS*

ПЕРІОДИЧНІ РУХОМІ ХВИЛІ НА ДВОВИМІРНИХ ҐРАТКАХ ІЗ ВЗАЄМОДІЯМИ НАЙБЛИЖЧИХ СУСІДІВ

We study the existence of periodic moving waves on two-dimensional periodically forced lattices with linear coupling between nearest particles and with periodic nonlinear substrate potentials. Such discrete systems can model molecules adsorbed on a substrate crystal surface.

Вивчено питання існування періодичних рухомих хвиль на двовимірних періодично збурених ґратках із лінійним зчепленням між найближчими частинками та з періодичними нелінійними потенціалами підкладинки. Такі дискретні системи можуть моделювати молекули, що адсорбуються на кристалічну поверхню підкладинки.

1. Introduction. Recently, several papers have been devoted to the dynamics of structures on two-dimensional (2d) lattice systems. For instance [1, 2], 2d Frenkel – Kontorova type models are used to study either coherent localized and extended defects such as dislocations, domain walls, vortices, grain boundaries, etc., which play an important role in the dynamical properties of materials with applications to the problem of adsorbates deposited on crystal surfaces; or in superlattices of ultrathin layers; or in large-area Josephson junctions. On the other hand [3], the existence of longitudinal solitary waves is shown for 2d cubic Hamiltonian lattices of particles interacting via harmonic springs between nearest and next nearest neighborhoods which appear in elastostatic investigation modeling a particle interaction via interatomic potentials, which is a natural 2d analogy of the 1d Fermi – Pasta – Ulam lattice.

In this paper, we focus on forced 2d Frenkel – Kontorova models and their generalizations. Motivated by [4], we consider an isotropic two-dimensional planar model where rigid molecules rotate in the plane of a square lattice. At site (n, m) the angle of rotation is $u_{n,m}$, each molecule interacts linearly with its first nearest neighbors and with a nonlinear periodic substrate potential. If χ is the linear coupling coefficient, ω^2 is the strength of the potential barrier or square of the frequency of small oscillations in the bottom of the potential wells and $\gamma\cos\mu t$ is the forcing then the equation of motion of the rotator at site (n, m) is (see Figure)

$$\ddot{u}_{n,m} = \chi[u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}] - \omega^2 \sin u_{n,m} + \gamma\cos\mu t. \tag{1.1}$$

J. M. Tamga et al. [4] studied how a weak initial uniform perturbation can evolve spontaneously into nonlinear localized modes with large amplitudes and investigated the solitary-wave and particle-like properties of these robust nonlinear entities.

More general countable systems of nonlinear ordinary differential equations like (1.1) are investigated in the book [5] focusing on the existence and stability of invariant tories. We also refer the reader for more motivations to study equations on lattices to [6].

Our paper has the following structure: Section 2 discusses the mathematical formulation of the periodic moving wave solutions in two-dimensional lattices and its connection to a small divisor problem. The existence of weak periodic moving waves in equations like (1.1) is given in Section 3. More regular and classical periodic moving waves are shown in Section 4. Final Section 5 is devoted to damped and periodically

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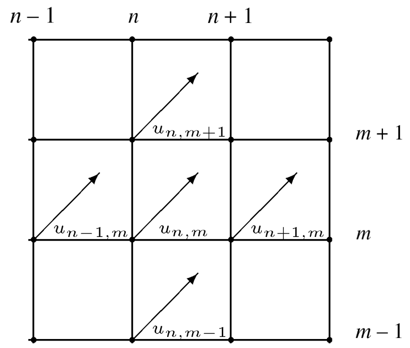
forced differential equations on two-dimensional lattices.



The two-dimensional lattice model of rigid rotation molecules with orientation $u_{n,m}$ at site (n, m).

2. Periodic moving waves. In this section, we consider the infinite system of ODEs

$$\ddot{u}_{n,m} = \chi(\Delta u)_{n,m} - f(u_{n,m}) + h(\mu t), \quad (n, m) \in \mathbb{Z}^2, \tag{2.1}$$

on the two-dimensional integer lattice $\mathbb{Z}^2$ for $f \in C^1(\mathbb{R}, \mathbb{R})$, $h \in C(\mathbb{R}, \mathbb{R})$, $\chi > 0$, $\mu > 0$ under the following conditions:

(H_1) f is odd and 2π-periodic, i.e., $f(-x) = -f(x)$ and $f(x + 2\pi) = f(x)$ $\forall x \in \in \mathbb{R}$;

(H_2) $h \neq 0$ is π-antiperiodic, i.e., $h(x + \pi) = -h(x)$ for all $x \in \mathbb{R}$.

Δ denotes the discrete Laplacian defined as

$$(\Delta u)_{n,m} = u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}.$$

For $f(u) = \omega^2 \sin u$ and $\chi = 1$, we get the 2d discrete sine-Gordon lattice equation (1.1). Clearly f is globally Lipschitz continuous on $\mathbb{R}$, i. e., $|f(x) - f(y)| \leq L|x - y|$ $\forall x, y \in \mathbb{R}$ with $L := \max_{\mathbb{R}} |f'(x)|$.

We are interested in the existence of *periodic moving wave* solutions of the form

$$u_{n,m}(t) = U(n\cos\theta + m\sin\theta - \nu t, \mu t) \tag{2.2}$$

for some $\nu, \theta \in \mathbb{R}$ and $U \in C^2(\mathbb{R}^2, \mathbb{R})$ which is 2π-periodic in the both variables. We may consider solutions of (2.2) to be moving waves on the lattice $\mathbb{Z}^2$, in the direction $e^{i\theta}$. Substitution of (2.2) into (2.1) leads to the equation

$$\nu^2 U_{zz}(z, \upsilon) - 2\mu\nu U_{z\upsilon}(z, \upsilon) + \mu^2 U_{\upsilon\upsilon}(z, \upsilon) = \chi\big(U(z + \cos\theta, \upsilon) + U(z - \cos\theta, \upsilon) + \\ + U(z + \sin\theta, \upsilon) + U(z - \sin\theta, \upsilon) - 4U(z, \upsilon)\big) - f\big(U(z, \upsilon)\big) + h(\upsilon) \tag{2.3}$$

with $z = n\cos\theta + m\sin\theta - \nu t$ and $\upsilon = \mu t$.

The linear part of (2.3) has the form

$$\mathcal{L}U := -\nu^2 U_{zz}(z, \upsilon) + 2\mu\nu U_{z\upsilon}(z, \upsilon) - \mu^2 U_{\upsilon\upsilon}(z, \upsilon) + \\ + \chi\big(U(z + \cos\theta, \upsilon) + U(z - \cos\theta, \upsilon) + \\ + U(z + \sin\theta, \upsilon) + U(z - \sin\theta, \upsilon) - 4U(z, \upsilon)\big). \tag{2.4}$$

Taking $e_{n,m} := \dfrac{1}{\pi} e^{i(nz + m\upsilon)}$ we derive

$$\mathcal{L} e_{n,m} = \lambda_{n,m} e_{n,m}$$

with

$$\lambda_{n,m} := (n\nu - m\mu)^2 - 4\chi\left(\sin^2\frac{n\cos\theta}{2} + \sin^2\frac{n\cos\theta}{2}\right).$$

We see that in general $\mathcal{L}$ is not invertible, since we are led to a problem of small divisors [7 – 9]. To avoid this difficulty, we use the symmetry of f and h in the next sections.

Finally, the unforced case of (2.1) with the form

$$\ddot{u}_{n,m} = (\Delta u)_{n,m} - f(u_{n,m}), \qquad (n, m) \in \mathbb{Z}^2, \tag{2.5}$$

is investigated in [10] by looking for *traveling waves* of (2.5) of the form

$$u_{n,m}(t) = U(n\cos\theta + m\sin\theta - \nu t), \tag{2.6}$$

when f satisfies assumption (H_1) in (2.5). Conditions are found in [10] to show the existence of *uniform sliding states* and periodic traveling waves of (2.5). Comparing formulas (2.2) and (2.6), this paper is a natural continuation of [10] to the periodically forced case (2.1) of (2.5). Next, we are also motivated to study periodic moving waves by the paper [11] where 1d undamped and periodically forced Frenkel – Kontorova model is investigated.

3. Weak periodic moving waves. A function $U : \mathbb{R}^2 \to \mathbb{R}$ is π-*antiperiodic* if

$$U(z+\pi, v) = U(z, v+\pi) = -U(z, v) \quad \forall (z, v) \in \mathbb{R}^2. \tag{3.1}$$

Note that any such U satisfying (3.1) is also 2π-periodic in the both variables.

Let

$$H^r := \left\{ U \in W^{r,2}_{\text{loc}}(\mathbb{R}^2) \,\middle|\, U \text{ is } \pi\text{-antiperiodic} \right\}$$

be Hilbert spaces for $r \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ with scalar products

$$(u, w)_r := (\partial_z^r u, \partial_z^r w)_0 + (\partial_v^r u, \partial_v^r w)_0$$

for $r \in \mathbb{N}$ and $(u, w)_0 := \int_\Omega u(z, v) w(z, v)\, dz\, dv$ with $\Omega := (0, \pi) \times (0, \pi)$ (see cf. [7, 12]). The corresponding norms are denoted by $\|\cdot\|_r$.

In the first part of this section, we are interested in the existence of *weak* π-*antiperiodic solutions* U of (2.3), i.e., $U \in H^0$ satisfying

$$\int_\Omega \Big\{ -\nu^2 U(z, v) w_{zz}(z, v) + 2\mu\nu U(z, v) w_{zv}(z, v) - \mu^2 U(z, v) w_{vv}(z, v) +$$
$$+ \big(\chi\big(U(z+\cos\theta, v) + U(z-\cos\theta, v) + U(z+\sin\theta, v) +$$
$$+ U(z-\sin\theta, v) - 4U(z, v)\big) - f\big(U(z, v)\big) + h(v)\big) w(z, v) \Big\}\, dz\, dv = 0 \tag{3.2}$$

for all $w \in H^0 \cap C^2(\mathbb{R}^2, \mathbb{R})$. Since the integration by parts formula holds for π-antiperiodic functions, if U is π-antiperiodic and C^2-smooth solving (2.3) then U is also a weak π-antiperiodic solution of (2.3). Clearly, (3.2) has the form

$$\mathcal{L}U + N(U) + h = 0, \tag{3.3}$$

where $\mathcal{L} : D(L) \subset H^0 \to H^0$ is defined by (2.4) and $N : H^0 \to H^0$ is a Nemytskij operator $N(U) := -f(U)$. Note that assumptions (H_1) and (H_2) imply that really N maps H^0 to itself and $h \in H^0$. Now we are ready to prove the following result.

Theorem 3.1. *Suppose* (H_1) *and* (H_2) *hold. If* $\mu > \sqrt{L + 8\chi}$ *and one of the following conditions holds*:

i) $\nu = \mu \dfrac{2p+1}{2k}$ *for some* $p \in \mathbb{Z}$ *and* $k \in \mathbb{N}$ *such that* $2k < \dfrac{\mu}{\sqrt{L+8\chi}}$,

ii) $\nu = \mu \dfrac{2k}{2p+1}$ *for some* $k \in \mathbb{Z}$ *and* $p \in \mathbb{Z}_+$ *such that* $2p+1 < \dfrac{\mu}{\sqrt{L+8\chi}}$,

then for any $\theta \in \mathbb{R}$, (2.3) *has a unique weak* π*-antiperiodic solution.*

Proof. We expand $u \in H^0$ in the Fourier series

$$u(z, \upsilon) = \sum_{n,m\in\mathbb{Z}} c_{n,m} e_{2n-1,2m-1}, \qquad \bar{c}_{n,m} = c_{-n+1,-m+1}.$$

Then $\|u\|_0^2 = \sum_{n,m\in\mathbb{Z}} \left| c_{n,m} \right|^2$ and $\mathcal{L}\, u = \sum_{n,m\in\mathbb{Z}} c_{n,m} \lambda_{2n-1,2m-1} e_{2n-1,2m-1}$. If i) holds, then we have

$$\lambda_{2n-1,2m-1} \ge ((2n-1)\nu - (2m-1)\mu)^2 - 8\chi =$$

$$= \left((2n-1)\mu \frac{2p+1}{2k} - (2m-1)\mu \right)^2 - 8\chi =$$

$$= \frac{\mu^2}{4k^2} ((2n-1)(2p+1) - 2k(2m-1))^2 - 8\chi \ge \frac{\mu^2}{4k^2} - 8\chi > 0.$$

If ii) holds, then we have

$$\lambda_{2n-1,2m-1} \ge ((2n-1)\nu - (2m-1)\mu)^2 - 8\chi =$$

$$= \left((2n-1)\mu \frac{2k}{2p+1} - (2m-1)\mu \right)^2 - 8\chi =$$

$$= \frac{\mu^2}{(2p+1)^2} ((2n-1)2k - (2p+1)(2m-1))^2 - 8\chi \ge \frac{\mu^2}{(2p+1)^2} - 8\chi > 0.$$

Consequently, $\mathcal{L}^{-1} : H^0 \to H^0$ satisfies

$$\| \mathcal{L}^{-1} \| \le \frac{4k^2}{\mu^2 - 32k^2\chi} \quad \text{under condition i)},$$

$$\| \mathcal{L}^{-1} \| \le \frac{(2p+1)^2}{\mu^2 - 8(2p+1)^2\chi} \quad \text{under condition ii)}.$$

Note that $N : H^0 \to H^0$ is Lipschitz continuous with a constant L. Next, we rewrite (3.3) as a fixed point problem

$$U = F(U) := -\mathcal{L}^{-1} N(U) - \mathcal{L}^{-1} h. \tag{3.4}$$

Clearly, $F : H^0 \to H^0$ is Lipschitz continuous with a constant $\| \mathcal{L}^{-1} \| L$. The assumptions of Theorem 3.1 ensure that $\| \mathcal{L}^{-1} \| L < 1$. So the Banach fixed point theorem gives a unique solution U of (3.4) in H^0. This is a unique weak π-antiperiodic solution of (2.3).

The theorem is proved.

Remark **3.1.** We prove in [10] that if $\nu > \sqrt{L + 8}$ then (2.5) has a unique uniform sliding state, i.e., a solution of the form (2.6) satisfying $U(z+2\pi) = U(z) + 2\pi$ $\forall z \in \mathbb{R}$.

Now we look for solutions of (2.3) satisfying

$$U(z+\pi, v) = U(z, v+\pi) = -U(z, v) + 2\pi \quad \forall (z, v) \in \mathbb{R}^2. \tag{3.5}$$

Note that any U satisfying (3.5) is 2π-periodic in the both variables. So we change $U(z, v) = u(z, v) + \pi$, where u is π-antiperiodic. Substituting this into (2.3), we get

$$\nu^2 u_{zz}(z, v) - 2\mu\nu u_{zv}(z, v) + \mu^2 u_{vv}(z, v) = \chi\big(u(z+\cos\theta, v) + u(z-\cos\theta, v) + \\ + u(z+\sin\theta, v) + u(z-\sin\theta, v) - 4u(z, v)\big) - f\big(u(z, v)+\pi\big) + h(v). \tag{3.6}$$

Assumption (H_1) implies that

$$f(\pi - x) = -f(\pi + x) \quad \forall x \in \mathbb{R}. \tag{3.7}$$

Using (3.7) we easily check that if u is π-antiperiodic, then also $f(u(z, v)+\pi)$ is π-antiperiodic. So the Nemytskij operator $\tilde{N}(u)(z, v) := f(u(z, v)+\pi)$ maps H^0 to H^0. Consequently, by repeating the proof of Theorem 3.1, we obtain the following result.

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *for any* $\theta \in \mathbb{R}$, (2.3) *has a unique weak solution satisfying* (3.5), *i.e.,* (3.6) *possesses a weak solution* $u \in H^0$.

Hence under the assumptions of Theorem 3.1, for fixed involved parameters, we have at least two 2π-periodic weak solutions of (2.3): one satisfying (3.1) and other satisfying (3.5).

We can further utilize the symmetries of f and h as follows. We look for solutions of (2.3) satisfying

$$U(z+\pi, v) = U(z, v), \qquad U(z, v+\pi) = -U(z, v) \qquad \forall (z, v) \in \mathbb{R}^2. \tag{3.8}$$

Again, any U satisfying (3.8) is 2π-periodic in the both variables.

Instead of Hilbert spaces H^r, we consider similar ones defined by

$$X^r := \left\{U \in W^{r,2}_{\mathrm{loc}}(\mathbb{R}^2) \,\middle|\, U \text{ satisfies (3.8)}\right\}$$

keeping the scalar products $(\cdot, \cdot)_r$.

Theorem 3.3. *Suppose* (H_1) *and* (H_2) *hold. If*

$$\mu > \sqrt{L+8\chi} \quad \text{and} \quad \nu = \mu\frac{k}{2p+1}$$

for some $p \in \mathbb{Z}_+$ *and* $k \in \mathbb{Z}$ *such that* $2p+1 < \dfrac{\mu}{\sqrt{L+8\chi}}$, *then for any* $\theta \in \mathbb{R}$, (2.3) *has a unique weak solution satisfying* (3.8), *i.e., a solution* $U \in X^0$ *satisfying* (3.2) *for all* $w \in X^0 \cap C^2(\mathbb{R}^2, \mathbb{R})$.

Proof. We expand $u \in X^0$ in the Fourier series

$$u(z, v) = \sum_{n,m\in\mathbb{Z}} c_{n,m} e_{2n,2m-1}, \qquad \bar{c}_{n,m} = c_{-n,-m+1}.$$

Then $\|u\|_0^2 = \sum_{n,m\in\mathbb{Z}} |c_{n,m}|^2$ and $\mathcal{L}u = \sum_{n,m\in\mathbb{Z}} c_{n,m}\lambda_{2n,2m-1}e_{2n,2m-1}$. From our assumptions, we derive

$$\lambda_{2n,2m-1} \ge (2n\nu - (2m-1)\mu)^2 - 8\chi = \left(2n\mu\frac{k}{2p+1} - (2m-1)\mu\right)^2 - 8\chi =$$

$$= \frac{\mu^2}{(2p+1)^2}\left(2nk-(2p+1)(2m-1)\right)^2 - 8\chi \geq \frac{\mu^2}{(2p+1)^2} - 8\chi > 0.$$

The rest of the proof is similar to one of Theorem 3.1, so we omit it.

Similarly to (3.5), we look for solutions of (2.3) satisfying

$$U(z+\pi, v) = U(z, v), \quad U(z, v+\pi) = -U(z, v) + 2\pi \quad \forall (z, v) \in \mathbb{R}^2. \tag{3.9}$$

Clearly, any U satisfying (3.9) is 2π-periodic in the both variables. Repeating the proof of Theorem 3.3, we obtain the following result.

Theorem 3.4. *Under the assumptions of Theorem* 3.3, *for any* $\theta \in \mathbb{R}$, (2.3) *has a unique weak solution satisfying* (3.9), *i.e.,* (3.6) *possesses a weak solution* $u \in X^0$.

***Corollary* 3.1.** *Suppose* (H_1) *and* (H_2) *hold. If*

$$\mu > \sqrt{L + 8\chi} \quad \text{and} \quad \nu = \mu \frac{2k}{2p+1}$$

for some $p \in \mathbb{Z}_+$ *and* $k \in \mathbb{Z}$ *such that* $2p+1 < \frac{\mu}{\sqrt{L+8\chi}}$, *then for any* $\theta \in \mathbb{R}$, (2.3) *has at least four weak* 2π*-periodic solutions*: *ones satisfying conditions* (3.1), (3.5), (3.8), *and* (3.9), *respectively.*

Clearly, if (H_2) holds, then h is 2π-periodic. Now we only suppose

(H_3) $h \neq 0$ is 2π-periodic, i.e., $h(x+2\pi) = h(x)$ $\forall x \in \mathbb{R}$.

Then we look for solutions of (2.3) satisfying either

$$U(z+\pi, v) = -U(z, v), \quad U(z, v+2\pi) = U(z, v) \quad \forall (z, v) \in \mathbb{R}^2, \tag{3.10}$$

or

$$U(z+\pi, v) = -U(z, v) + 2\pi, \quad U(z, v+2\pi) = U(z, v) \quad \forall (z, v) \in \mathbb{R}^2. \tag{3.11}$$

Clearly, any U satisfying either (3.10) or (3.11) is 2π-periodic in the both variables. Now we consider Hilbert spaces defined by

$$Y^r := \left\{ U \in W^{r,2}_{\mathrm{loc}}(\mathbb{R}^2) \,\middle|\, U \text{ satisfies (3.10)} \right\}$$

with scalar products

$$\langle u, w \rangle_r := \langle \partial_z^r u, \partial_z^r w \rangle_0 + \langle \partial_v^r u, \partial_v^r w \rangle_0$$

for $r \in \mathbb{N}$ and $\langle u, w \rangle_0 := \int_{\tilde{\Omega}} u(z, v) w(z, v)\, dz\, dv$ with $\tilde{\Omega} := (0, \pi) \times (0, 2\pi)$.

Theorem 3.5. *Suppose* (H_1) *and* (H_3) *hold. If* $\mu > \sqrt{L + 8\chi}$ *and condition* i) *of Theorem* 3.1 *holds, then for any* $\theta \in \mathbb{R}$, (2.3) *has a unique weak solution satisfying* (3.10) *and other unique weak solution satisfying* (3.11), *i.e., there are two functions* $U_{1,2} \in Y^0$ *satisfying* (3.2) *and* (3.6) *for all* $w \in Y^0 \cap C^2(\mathbb{R}^2, \mathbb{R})$, *respectively.*

Proof. We expand $u \in Y^0$ in the Fourier series

$$u(z, v) = \sum_{n,m \in \mathbb{Z}} c_{n,m} \frac{1}{\sqrt{2}} e_{2n-1,m}, \quad \bar{c}_{n,m} = c_{-n+1,-m}.$$

Then the rest of the proof is the same as for Theorem 3.3, so we omit it.

***Remark* 3.2.** The function h in (2.1) could be arbitrary satisfying either (H_2) or (H_3). Next, the above results are applied to (1.1) for any $\gamma > 0$ and $\mu > 0$, $\chi > 0$, $\omega >$ > 0 satisfying $\mu > \sqrt{\omega^2 + 8\chi}$.

4. More regular periodic moving waves. Now we study (2.3) in H^r for $r \in \mathbb{N}$. First, we note the Sobolev embedding result [7, 12]

$$H^2 \subset C_\pi(\mathbb{R}^2) := H^0 \cap C(\mathbb{R}^2, \mathbb{R}).$$

Moreover, we have

$$|U|_0 := \max_{z,v\in\mathbb{R}} |U(z, v)| \leq \left| \sum_{n,m\in\mathbb{Z}} c_{n,m} e_{2n-1,2m-1} \right| \leq \frac{1}{\pi} \sum_{n,m\in\mathbb{Z}} |c_{n,m}| \leq$$

$$\leq \frac{1}{\pi} \sqrt{\sum_{n,m\in\mathbb{Z}} |c_{n,m}|^2 \left((2n-1)^4 + (2m-1)^4\right)} \sqrt{\sum_{n,m\in\mathbb{Z}} \frac{1}{(2n-1)^4 + (2m-1)^4}} \leq$$

$$\leq \frac{1}{\pi} c_1 \|U\|_2$$

with

$$c_1^2 = \sum_{n,m\in\mathbb{Z}} \frac{1}{(2n-1)^4 + (2m-1)^4} \leq 4 \sum_{n,m\in\mathbb{N}} \frac{1}{n^4 + m^4} \leq 8 \sum_{n,m\in\mathbb{N}} \frac{1}{(n^2 + m^2)^2} \leq$$

$$\leq 8 \sum_{n\in\mathbb{N}} \int_0^\infty \frac{dy}{(n^2 + y^2)^2} = 2\pi \sum_{n\in\mathbb{N}} \frac{1}{n^3} \leq 2\pi \left(1 + \int_1^\infty \frac{dx}{x^3}\right) = 3\pi.$$

Hence we get the Sobolev inequality [7, 12]

$$|U|_0 \leq \frac{\sqrt{3}}{\sqrt{\pi}} \|U\|_2 \quad \forall U \in H^2. \tag{4.1}$$

Next, supposing $f \in C^2(\mathbb{R}^2, \mathbb{R})$, we compute for $U \in C_\pi^\infty := H^0 \cap C^\infty(\mathbb{R}^2, \mathbb{R})$

$$\|f(U)\|_2 \leq \|f(U)_{zz}\|_0 + \|f(U)_{vv}\|_0. \tag{4.2}$$

Furthermore,

$$\|f(U)_{zz}\|_0 \leq \|f'(U)U_{zz}\|_0 + \|f''(U)U_z^2\|_0 \leq L\|U_{zz}\|_0 + L_2\|U_z^2\|_0,$$

where $L_2 := \max_{x\in\mathbb{R}} |f''(x)|$. Similarly we derive

$$\|f(U)_{vv}\|_0 \leq L\|U_{vv}\|_0 + L_2\|U_v^2\|_0.$$

Hence by (4.2)

$$\|f(U)\|_2 \leq L\left(\|U_{zz}\|_0 + \|U_{vv}\|_0\right) + L_2\left(\|U_z^2\|_0 + \|U_v^2\|_0\right).$$

Next, using integration by parts, we derive

$$\int_0^\pi U_z^4(z, v)\,dz = U_z^3(z, v)U(z, v)\Big|_{z=0}^\pi - 3\int_0^\pi U_z^2(z, v)U_{zz}(z, v)U(z, v)\,dz \leq$$

$$\leq 3|U|_0 \sqrt{\int_0^\pi U_z^4(z, v)\,dz} \sqrt{\int_0^\pi U_{zz}^2(z, v)\,dz}$$

which implies

$$\int_0^\pi U_z^4(z, v)\,dz \leq 9|U|_0^2 \int_0^\pi U_{zz}^2(z, v)\,dz.$$

Consequently, we obtain

$$\left\| U_z^2 \right\|_0 = \sqrt{\int_\Omega U_z^4(z, v)\,dz\,dv} \le 3|U|_0 \|U_{zz}\|_0 .$$

Similarly, we get

$$\left\| U_v^2 \right\|_0 \le 3|U|_0 \|U_{vv}\|_0 .$$

Summarizing, we arrive at

$$\|f(U)\|_2 \le \left(L + 3L_2|U|_0\right)\left(\|U_{zz}\|_0 + \|U_{vv}\|_0\right) \le \sqrt{2}\left(L + 3L_2|U|_0\right)\|U\|_2 . \quad (4.3)$$

Of course, (4.3) is the well-known Moser inequality [7, 12]. Assuming that $\mu > \sqrt{\sqrt{2}L + 8\chi}$ and one of the following conditions holds:

I) $\nu = \mu \dfrac{2p+1}{2k}$ for some $p \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $2k < \dfrac{\mu}{\sqrt{\sqrt{2}L + 8\chi}}$,

II) $\nu = \mu \dfrac{2k}{2p+1}$ for some $k \in \mathbb{Z}$ and $p \in \mathbb{Z}_+$ such that $2p + 1 < \dfrac{\mu}{\sqrt{\sqrt{2}L + 8\chi}}$,

and using (2.4), we get

$$(\mathcal{L}U, U)_r \ge \tilde{c}\|U\|_r^2 \quad \forall U \in C_\pi^\infty \quad (4.4)$$

for any $r \ge 0$ and with either $\tilde{c} = \dfrac{\mu^2}{4k^2} - 8\chi > 0$ or $\tilde{c} = \dfrac{\mu^2}{(2p+1)^2} - 8\chi > 0$. Supposing $h \in C^2(\mathbb{R}, \mathbb{R})$, from (4.1), (4.3), and (4.4) we derive

$$(\mathcal{L}U - f(U) + h, U)_2 \ge \left[\tilde{c} - \sqrt{2}\left(L + 3L_2 \frac{\sqrt{3}}{\sqrt{\pi}}\|U\|_2\right)\right]\|U\|_2^2 - \|h\|_2\|U\|_2 .$$

If

$$\tilde{c} > \sqrt{2}L, \quad (4.5)$$

then we get

$$(\mathcal{L}U - f(U) + h, U)_2 \ge \left((A - B\|U\|_2)\|U\|_2 - \|h\|_2\right)\|U\|_2$$

with

$$A := \tilde{c} - \sqrt{2}L > 0, \qquad B := 3L_2 \frac{\sqrt{6}}{\sqrt{\pi}} > 0. \quad (4.6)$$

The quadratic function $x \to (A - Bx)x$ has its maximum $\dfrac{A^2}{4B^2}$ at $x_0 = \dfrac{A}{2B}$. So if $\|h\|_2 < \dfrac{A^2}{4B^2}$, then there is $\kappa \in \left(0, \dfrac{A}{2B}\right)$ such that for any $U \in C_\pi^\infty$ with $\|U\|_2 = \kappa$, the following inequality holds:

$$(\mathcal{L}U - f(U) + h, U)_2 > 0. \quad (4.7)$$

Now we take the finite-dimensional Banach spaces $H_k \subset C_\pi^\infty$, $k \in \mathbb{N}$, given by

$$H_k := \left\{ \sum_{n,m=-k}^{k} c_{n,m} e_{n,m} \mid n, m \text{ are odd integers and } \bar{c}_{n,m} = c_{-n,-m} \right\}.$$

Next, like in [13], we take a convex set

$$V_k := \{U \in H_k \mid \|U\|_2 \le \kappa\}.$$

Then, on the boundary ∂V_k of V_k, (4.7) holds. Hence we consider the homotopy

$$H(\lambda, U) := \lambda P_k(\mathcal{L}U - f(U) + h) + (1-\lambda)U, \quad U \in V_k, \tag{4.8}$$

where $P_k : H^2 \to H_k \subset H^2$ is the orthogonal projection. According to (4.7), we see that

$$(H(\lambda, U), U)_2 = \lambda(\mathcal{L}U - f(U) + h, U)_2 + (1-\lambda)\|U\|_2^2 > 0 \tag{4.9}$$

for any $\lambda \in [0, 1]$ and $U \in \partial V_k$. Then $0 \notin H(\lambda, \partial V_k)$ for any $\lambda \in [0, 1]$. Consequently, using the Brouwer topological degree theory [9], we derive

$$\deg(P_k(\mathcal{L}U - F(U) + h), V_k, 0) = \deg(\mathbb{I}_k, V_k, 0) = 1,$$

where $\mathbb{I}_k : H_k \to H_k$ is the identity mapping. This gives a solution $U_k \in V_k$ of

$$P_k(\mathcal{L}U_k - F(U_k) + h) = 0. \tag{4.10}$$

Since H^2 is compactly embedded into H^0 [12], we can suppose that $U_k \to U_0 \in H^2$ in H^0. We note that

$$P_k u = \sum_{|2n-1|,|2m-1| \le k} c_{n,m} e_{2n-1,2m-1}$$

when

$$u(z, v) = \sum_{n,m \in \mathbb{Z}} c_{n,m} e_{2n-1,2m-1}$$

with $\bar{c}_{n,m} = c_{-n+1,-m+1}$. It is easy to check that $P_k : H^r \to H_k \subset H^r$ is an orthogonal projection for any $r \ge 0$. Then (4.10) gives $(\mathcal{L}U_k - f(U_k) + h, w)_0 = 0$ $\forall w \in H_{k_1}$ and $k \ge k_1$. But this means that (3.2) holds with $U = U_k$ for any $w \in H_{k_1}$ and $k \ge$ $\ge k_1$. Since $N(U) = -F(U)$ is continuous from H^0 to itself, fixing k_1 and passing to the infinity with $k \to \infty$, we see that (3.2) holds with $U = U_0$ for any $w \in H_{k_1}$ and $k_1 \ge 1$. So $U_0 \in H^2$ is a weak π-antiperiodic solution of (2.3) which is continuous. Summarizing, we get the following result.

Theorem 4.1. *Suppose* $f, h \in C^2(\mathbb{R}, \mathbb{R})$ *satisfy* (H_1) *and* (H_2). *If*

$$\mu > \sqrt{\sqrt{2}L + 8\chi}$$

and one of the conditions I) *and* II) *holds, and*

$$\|h\|_2 < \frac{\pi(\tilde{c} - \sqrt{2}L)^2}{216 L_2^2}, \quad \textit{i.e.,} \quad \|h\|_2 \ \textit{is sufficiently small,}$$

where $\tilde{c} = \dfrac{\mu^2}{4k^2} - 8\chi > 0$ *for* I) *and* $\tilde{c} = \dfrac{\mu^2}{(2p+1)^2} - 8\chi > 0$ *for* II), *then for any* $\theta \in \mathbb{R}$, (2.3) *has a unique* π*-antiperiodic solution* U *belonging to* H^2, *i.e.,* (2.3) *is satisfied when generalized derivatives of* U *are considered.*

Proof. Note that $\mu > \sqrt{\sqrt{2}L + 8\chi}$ and I), II) imply assumptions of Theorem 3.1. So (2.3) has a unique weak π-antiperiodic solution U. Since also (4.5) is satisfied, from the above consideration we also known that $U \in H^2$.

The theorem is proved.

The same arguments can be applied to show that if $f, h \in C^4(\mathbb{R}, \mathbb{R})$ and $\|h\|_2$ is sufficiently small in Theorem 4.1, then $U \in H^4 \subset C^2(\mathbb{R}, \mathbb{R})$. So we get a unique classical solution of (2.3). Indeed, we have for $u \in C_\pi^\infty$

$$\|f(U)\|_4 \le \|f(U)_{zzzz}\|_0 + \|f(U)_{vvvv}\|_0 \le$$

$$\le L_4\left(\|U_z^4\|_0 + \|U_v^4\|_0\right) + 6L_3\left(\|U_z^2 U_{zz}\|_0 + \|U_v^2 U_{vv}\|_0\right) +$$

$$+ 4L_2\left(\|U_z U_{zzz}\|_0 + \|U_v U_{vvv}\|_0\right) + 3L_2\left(\|U_{zz}^2\|_0 + \|U_{vv}^2\|_0\right) +$$

$$+ L\left(\|U_{zzzz}\|_0 + \|U_{vvvv}\|_0\right), \tag{4.11}$$

where $L_i := \max_{x \in \mathbb{R}} |f^{(i)}(x)|$, $i = 2, 3, 4$. Using the Sobolev inequality (4.1) and the Nirenberg ones like in [7, p. 273, 274; 12], we get from (4.11)

$$\|f(U)\|_4 \le \left[c\left(L_4\|U\|_2^3 + L_3\|U\|_2^2 + L_2\|U\|_2\right) + \sqrt{2}L\right]\|U\|_4 \tag{4.12}$$

for a constant $c > 0$. Supposing either I) or II), from (4.4) and (4.12), we derive

$$(\mathcal{L}U - f(U) + h, U)_4 \ge$$

$$\ge \left[\tilde{c} - \sqrt{2}L - c\left(L_4\|U\|_2^3 + L_3\|U\|_2^2 + L_2\|U\|_2\right)\right]\|U\|_4^2 - \|h\|_4\|U\|_4. \tag{4.13}$$

Since $\tilde{c} > \sqrt{2}L$, the equation

$$\tilde{c} - \sqrt{2}L - c(L_4x^3 + L_3x^2 + L_2x) = 0$$

has a unique positive root $\tilde{x}_0$. Finally, we define a function $G : [0, \infty) \to [0, \infty)$ by

$$G(x) = \begin{cases} \dfrac{A^2}{4B^2} & \text{for } x \ge \dfrac{A}{2B}, \\ (A - Bx)x & \text{for } 0 \le x \le \dfrac{A}{2B}, \end{cases}$$

where constants A and B are given in (4.6) and $\tilde{c} = \dfrac{\mu^2}{4k^2} - 8\chi > 0$ for I) and $\tilde{c} = \dfrac{\mu^2}{(2p+1)^2} - 8\chi > 0$ for II). Now we are ready to prove the following result.

Theorem 4.2. *Suppose* (H_1) *and* (H_2) *for* $f, h \in C^4(\mathbb{R}, \mathbb{R})$. *If*

$$\mu > \sqrt{\sqrt{2}L + 8\chi}$$

and one of the conditions I) *and* II) *holds, and*

$$\|h\|_2 < G(\tilde{x}_0), \quad \textit{i.e.,} \quad \|h\|_2 \ \textit{is sufficiently small,}$$

then for any $\theta \in \mathbb{R}$, (2.3) *has a unique classical* π*-antiperiodic solution* U.

Proof. From $\|h\|_2 < G(\tilde{x}_0)$ we infer the existence of $0 < \kappa < \min\left\{\dfrac{A}{2B}, \tilde{x}_0\right\}$ and $R \gg 1$ such that (4.7) holds for any $U \in C_\pi^\infty$ with $\|U\|_2 = \kappa$, and for any $U \in C_\pi^\infty$ with $\|U\|_2 \le \kappa$ and $\|U\|_4 = R$, the following inequality holds:

$$(\mathcal{L}U - f(U) + h, U)_4 > 0. \tag{4.14}$$

Next, like in [13], we take a convex set

$$W_k := \left\{ U \in H_k \mid \|U\|_2 \le \kappa,\ \|U\|_4 \le R \right\}.$$

We consider the homotopy $H(\lambda, U)$ defined in (4.8). We recall that $P_k : H^r \to H_k \subset \subset H^r$ is an orthogonal projection for any $r \ge 0$. On the boundary ∂W_k of W_k, either $\|U\|_2 = \kappa$ which, by (4.7), implies (4.9) for any $\lambda \in [0, 1]$, or $\|U\|_4 = R$ which, by (4.14), implies $\big(H(\lambda, U), U\big)_4 > 0$ for any $\lambda \in [0, 1]$. Summarizing, we see that $0 \notin \notin H(\lambda, \partial W_k)$ for any $\lambda \in [0, 1]$. Consequently, using the Brouwer topological degree theory [9], we derive

$$\deg\big(P_k(\mathcal{L}U - F(U) + h), W_k, 0\big) = \deg(\mathbb{I}_k, W_k, 0) = 1.$$

This gives a solution $U_k \in W_k$ of $P_k\big(\mathcal{L}U_k - F(U_k) + h\big) = 0$ for any $k \in \mathbb{N}$. So we can suppose that weakly $U_k \rightharpoonup U \in H^4$, and so strongly $U_k \to U$ in H^2. The rest of the proof is the same as for Theorem 4.1, consequently, $U \in H^4$ is a unique weak π-antiperiodic solution of (2.3). Since $U \in H^4 \subset C^2(\mathbb{R}^2, \mathbb{R})$, we get a unique classical solution of (2.3).

The theorem is proved.

***Remark* 4.1.** i) We note that in Theorem 4.2 we need only to control the norm $\|h\|_2$ in spite of the fact that $h \in C^4(\mathbb{R}^2, \mathbb{R})$. For instance, for

$$h_{p,q,\varepsilon}(z, v) := \varepsilon \frac{\sin(2p+1)z}{(2p+1)^2} + \varepsilon \frac{\sin(2q+1)v}{(2q+1)^2}, \quad p, q \in \mathbb{Z}, \quad \varepsilon \ne 0,$$

we have $\left\|h_{p,q,\varepsilon}\right\|_2 = |\varepsilon|\pi$, while $\left\|h_{p,q,\varepsilon}\right\|_4 \to \infty$ as $|p| + |q| \to \infty$. So for $\varepsilon \ne 0$ sufficiently small, Theorem 4.2 can be applied with $h = h_{p,q,\varepsilon}$ for any $p, q \in \mathbb{Z}$.

ii) Next, it seems to be awkward to find the constant c in (4.12) and subsequently the root $\tilde{x}_0$, for this reason we present Theorem 4.1 with concrete and explicit values of involved constants.

5. Damped and periodically forced systems. In this section, we consider the infinite system of ODEs

$$\ddot{u}_{n,m} = -\delta \dot{u}_{n,m} + \chi(\Delta u)_{n,m} - f(u_{n,m}) + h(\mu t), \quad (n, m) \in \mathbb{Z}^2, \tag{5.1}$$

on the two-dimensional integer lattice $\mathbb{Z}^2$ for $f \in C^1(\mathbb{R}, \mathbb{R})$, $h \in C(\mathbb{R}, \mathbb{R})$, $\delta > 0$, $\chi > 0$, $\mu > 0$ under conditions (H_1) and (H_2). Inserting (2.2) into (5.1), we get

$$\begin{aligned} \nu^2 U_{zz}(z, v) - 2\mu\nu U_{zv}(z, v) + \mu^2 U_{vv}(z, v) + \delta\big(\mu U_v(z, v) - \nu U_z(z, v)\big) = \\ = \big(\chi\big(U(z + \cos\theta, v) + U(z - \cos\theta, v) + \\ + U(z + \sin\theta, v) + U(z - \sin\theta, v) - 4U(z, v)\big) - f\big(U(z, v)\big) + h(v). \end{aligned} \tag{5.2}$$

Now we write (5.2) as follows:

$$\tilde{\mathcal{L}}U + \tilde{N}(U) + h = 0 \tag{5.3}$$

with

$$\tilde{\mathcal{L}}U := -\nu^2 U_{zz}(z, v) + 2\mu\nu U_{zv}(z, v) - \mu^2 U_{vv}(z, v) + \delta\big(\nu U_z(z, v) - \mu U_v(z, v)\big) \tag{5.4}$$

and

$$\tilde{N}(U) := \chi\big(U(z + \cos\theta, v) + U(z - \cos\theta, v) +$$

$$+ U(z+\sin\theta, v) + U(z-\sin\theta, v) - 4U(z, v)\big) - f\big(U(z, v)\big).$$

We have $\tilde{\mathcal{L}}e_{n,m} = \tilde{\lambda}_{n,m}e_{n,m}$ with $\tilde{\lambda}_{n,m} := (n\nu - m\mu)^2 + i\delta(n\nu - m\mu)$. Clearly, $N : H^0 \to H^0$ is Lipschitz continuous with constant $L + 8\chi$. A weak π-antiperiodic solution $U \in H^0$ of (5.2) is formulated like in (3.2), so we omit that formula.

Theorem 5.1. *Suppose* (H_1) *and* (H_2) *hold. If one of the following conditions holds*:

a) $\nu = \mu\dfrac{2p+1}{2k}$ *for some* $p \in \mathbb{Z}$ *and* $k \in \mathbb{N}$ *such that* $\mu^4 + 4\delta^2\mu^2k^2 > 16k^4(L+8\chi)^2$,

b) $\nu = \mu\dfrac{2k}{2p+1}$ *for some* $k \in \mathbb{Z}$ *and* $p \in \mathbb{Z}_+$ *such that* $\mu^4 + (2p+1)^2\delta^2\mu^2 > (2p+1)^4(L+8\chi)^2$,

then for any $\theta \in \mathbb{R}$, (5.2) *has a unique weak* π*-antiperiodic solution.*

Proof. We expand $u \in H^0$ in the Fourier series

$$u(z, v) = \sum_{n,m\in\mathbb{Z}} c_{n,m}e_{2n-1,2m-1}, \qquad \bar{c}_{n,m} = c_{-n+1,-m+1}.$$

Then $\|u\|_0^2 = \sum_{n,m\in\mathbb{Z}}|c_{n,m}|^2$ and $\tilde{\mathcal{L}}u = \sum_{n,m\in\mathbb{Z}} c_{n,m}\tilde{\lambda}_{2n-1,2m-1}e_{2n-1,2m-1}$. If a) holds, then we have

$$\left|\tilde{\lambda}_{2n-1,2m-1}\right| \geq \sqrt{\frac{\mu^4}{16k^4} + \delta^2\frac{\mu^2}{4k^2}}.$$

If b) holds, then we have

$$\left|\tilde{\lambda}_{2n-1,2m-1}\right| \geq \sqrt{\frac{\mu^4}{(2p+1)^4} + \delta^2\frac{\mu^2}{(2p+1)^2}}.$$

Consequently, $\tilde{\mathcal{L}}^{-1} : H^0 \to H^0$ satisfies

$$\|\tilde{\mathcal{L}}^{-1}\| \leq \frac{4k^2}{\sqrt{\mu^4 + 4\delta^2k^2\mu^2}} \quad \text{under condition a)},$$

$$\|\tilde{\mathcal{L}}^{-1}\| \leq \frac{(2p+1)^2}{\sqrt{\mu^4 + (2p+1)^2\delta^2\mu^2}} \quad \text{under condition b)}.$$

In both cases we get $\|\tilde{\mathcal{L}}\|(L+8\chi) < 1$, so rewriting (5.3) as a fixed point problem

$$U = -\tilde{\mathcal{L}}^{-1}\tilde{N}(U) - \tilde{\mathcal{L}}^{-1}h, \tag{5.5}$$

and applying the Banach fixed point theorem to (5.5), we get the desired unique weak π-antiperiodic solution of (5.2).

The theorem is proved.

Of course, other results of Sections 3 and 4 can be extended for (5.2), but since it is straightforward, we omit details.

We note that for sufficiently large $\delta > 0$, equation (5.2) has a weak π-antiperiodic solution. Indeed, if

$$\delta\mu > 2(L+8\chi), \tag{5.6}$$

then condition a) is satisfied with $k = 1$ and any $p \in \mathbb{Z}$, and condition b) holds as well with $p = 0$ and any $k \in \mathbb{Z}$. So we get an expected result that if the damping $\delta > 0$ is sufficiently large, for instance, if (5.6) holds, then the system (5.1) has a (weak) periodic moving wave solution. Moreover, they are infinitely many in any direction $(\cos\theta, \sin\theta)$. Indeed, for different values of ν in (5.2), the above derived weak solutions are different. To show this, suppose that (5.2) has a weak π-antiperiodic solution U for two parameters $\nu_1 \neq \nu_2$ with the same μ, χ, δ and θ satisfying assumptions of Theorem 5.1. Then according to (5.5) we get

$$U = -\tilde{\mathcal{L}}_1^{-1}(\tilde{N}(U) - h) = -\tilde{\mathcal{L}}_2^{-1}(\tilde{N}(U) - h), \tag{5.7}$$

where $\mathcal{L}_{1,2}^{-1}$ are linear maps of (5.4) for parameters μ, δ, $\nu_{1,2}$ with eigenvalues $\tilde{\lambda}_{2n-1,2m-1,1,2}$, respectively. Then we derive

$$\operatorname{Im}\tilde{\lambda}_{2n-1,2m-1,1} = \delta\big((2n-1)\nu_1 - (2m-1)\mu\big) \neq$$

$$\neq \delta\big((2n-1)\nu_2 - (2m-1)\mu\big) = \operatorname{Im}\tilde{\lambda}_{2n-1,2m-1,2}.$$

So for any $n, m \in \mathbb{Z}$ we see that $\tilde{\lambda}_{2n-1,2m-1,1} \neq \tilde{\lambda}_{2n-1,2m-1,2}$. But then $\tilde{\mathcal{L}}_1^{-1}\tilde{h} \neq \tilde{\mathcal{L}}_2^{-1}\tilde{h}$ for any $0 \neq \tilde{h} \in H^0$. Clearly $\tilde{N}(U) - h \neq 0$ in (5.7), since otherwise $U = 0$ and then $h = 0$, which is excluded in (H_2). But then $\tilde{\mathcal{L}}_1^{-1}(\tilde{N}(U) - h) \neq \tilde{\mathcal{L}}_2^{-1}(\tilde{N}(U) - h)$, which contradicts to (5.7). So solutions in Theorem 5.1 are different for different values of ν, i.e., we have infinitely many of them.

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