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SHAPE-PRESERVING PROJECTIONS IN LOW-DIMENSIONAL SETTINGS AND THE q-MONOTONE CASE

ФОРМОЗБЕРІГАЮЧІ ПРОЕКЦІЇ У МАЛОВИМІРНІЙ ПОСТАНОВЦІ ТА q-МОНОТОННИЙ ВИПАДОК

Let $P\colon X \to V$ be a projection from a real Banach space X onto a subspace V and let $S \subset X$. In this setting, one can ask if S is left invariant under P, i.e., if $PS \subset S$. If V is finite-dimensional and S is a cone with particular structure, then the occurrence of the imbedding $PS \subset S$ can be characterized through a geometric description. This characterization relies heavily on the structure of S, or, more specifically, on the structure of the cone S^* dual to S. In this paper, ше remove the structural assumptions on S^* and characterize the cases where $PS \subset S$. We note that the (so-called) q-monotone shape forms a cone which (lacks structure and thus) serves as an application for our characterization.

Нехай $P\colon X \to V$ – проекція дійсного банахового простору X на підпростір V і, крім того, $S \subset X$. У цій постановці виникає питання: чи є S лівоінваріантним під дією P, тобто чи має місце вкладення $PS \subset S$? Якщо підпростір V є скінченновимірним, а S є конусом із певною структурою, то вкладення $PS \subset S$ може бути охарактеризовано шляхом геометричного опису. Ця характеризація істотно залежить від структури S, або, точніше, від структури конуса S^*, спряженого до S. У цій роботі усунено структурні припущення щодо S^* і охарактеризовано випадки, у яких $PS \subset S$. Відзначено, що (так звана) q-монотонна форма утворює конус, який (не має структури і тому) може бути використаний для застосування нашої характеризації.

1. Introduction. Denote the space of linear operators from real Banach space X into subspace $V \subset X$ by $\mathcal{L} = \mathcal{L}(X,V)$. For a given subset $S \subset X$, one can look to determine those $Q \in \mathcal{L}$ which leave S invariant; i.e., those Q such that $QS \subset S$. There are numerous settings in which $QS \subset S$ has important consequences and connections. For example, under the right conditions on S, X becomes a Banach lattice and Q such that $QS \subset S$ becomes a *positive operator* (see [7] for an overview). Existence of positive operators (or more precisely *positive extensions*) is employed, for example, in the Korovkin's classical theorem (described in [2]) and in its many generalizations (see, for example, [3]).

A natural assumption on S is that it is a *cone* – a convex set, closed under nonnegative scalar multiplication. And outside of the Banach lattice realm, $Q \in \mathcal{L}(X,V)$ such that $QS \subset S$ is often called a *cone-preserving* map (see [8] for an extensive description). Borrowing this terminology, for given cone S let us denote the set of all cone-preserving operators by $\mathcal{L}_S = \mathcal{L}_S(X,V)$. Not surprisingly, the determination of whether or not a given $Q \in \mathcal{L}$ belongs to $\mathcal{L}_S$ can be quite difficult. Indeed, one finds in the literature that existence of cone-preserving operators is frequently considered only in the case in which X is finite-dimensional. The fact that membership in $\mathcal{L}_S$ is very 'sensitive' to X, S and Q certainly contributes to the difficulty. For example, there is no finite-rank operator in $\mathcal{L}_S(X,V)$ which fixes V, where $X = (C[0,1], \|\cdot\|_\infty)$, S is the cone of nonnegative elements from X and $V = \Pi_2 = [1, x, x^2]$, the space of second-degree algebraic polynomials (spanned by $\{1, x, x^2\}$). However, if instead we require fixing Π_1 and $x^2 \mapsto (x + x^2)/2$, i.e., nearly fixing V, then such an operator does belong to $\mathcal{L}_S(X,V)$. Or instead, consider the fact that, while there exists no projection from X onto $V = \Pi_2$ preserving monotonicity, it is possible to project X_1 onto V and leave the cone of monotone functions (of X_1) invariant, where X_1 is the (Banach) space of C^1 functions on $[0,1]$ normed by $\|f\|_{X_1} := \max\{\|f\|_\infty, \|f'\|_\infty\}$.

When elements of X are to approximated from V such that the characteristic, or shape, described by (inclusion in) S should be maintained, then we say such a Q provides a *shape-preserving approximation* whenever $Q \in \mathcal{L}_S$ and Q is referred to as a *shape-preserving operator*. This paper considers the problem of existence of shape-preserving operators for a given S. From the viewpoint of shape-preserving approximation, we will be primarily interested in those $Q \in \mathcal{L}$ that projections, i.e.,

$$P \in \mathcal{L}(X,V) \quad \text{such that} \quad P_{|V} = id_V.$$

Let $\mathcal{P} = \mathcal{P}(X,V)$ denote the set of projections in $\mathcal{L}$ and let $\mathcal{P}_S$ be the set of *shape-preserving projections*. The paper [5] gives a characterization of $\mathcal{P}_S \neq \varnothing$ under so-called *high-dimensional* assumptions (which are explained below). As illustrated, for example, in [1, 4] and [6], there are many natural settings for which the high-dimensional assumptions are valid (and thus the characterization can be applied).

The main goal of this paper is to consider the existence question $\mathcal{P}_S \neq \varnothing$ without the assumptions of [5], that is, existence under *low-dimensional* assumptions, and to apply our results in a specific setting.

We divide this paper into four sections. Following this introductory section, we establish in Section 2 some basic notation involving convex cones and describe exactly our low-dimensional assumptions. In Section 3 we state, and subsequently prove, our main existence results. Within this section we describe a decomposition of subspace V which is used extensively in the consideration of shape-preserving operators. Finally in Section 4 we identify a very natural setting in which the low-dimensional assumptions hold and our existence results can be applied to yield some interesting results.

2. Preliminaries and low-dimensional assumptions. Throughout this paper, we will denote the ball and sphere of real Banach space X by $B(X)$ and $S(X)$, respectively. $V \subset X$ will always denote a finite-dimensional subspace of X. The dual space of X is denoted, as usual, by X^*. To emphasize bi-linearity, use $\langle x, \varphi\rangle$ to denote $\varphi(x)$ for $x \in X$ and $\varphi \in X^*$. In a (real) topological vector space, a *cone* K is a convex set, closed under nonnegative scalar multiplication. K is *pointed* if it contains no lines. For $\varphi \in K$, let $[\varphi]^+ := \{\alpha\varphi \mid \alpha \geq 0\}$. We say $[\varphi]^+$ is an *extreme ray* of K if $\varphi = \varphi_1 + \varphi_2$ implies $\varphi_1, \varphi_2 \in [\varphi]^+$ whenever $\varphi_1, \varphi_2 \in K$. We let $E(K)$ denote the union of all extreme rays of K. When K is a closed, pointed cone of finite dimension we always have $K = \mathrm{co}(E(K))$ (this need not be the case when K is infinite dimensional; indeed, we note in [6] that it is possible that $E(K) = \varnothing$ despite K being closed and pointed).

Definition 2.1. *Let $S \subset X$ denote a closed cone. We say that $x \in X$ has shape* (*in the sense of S*) *whenever $x \in S$. Denote the set of projections from X onto V by $\mathcal{P} = \mathcal{P}(X,V)$. If $P \in \mathcal{P}$ and $PS \subset S$ then we say P is a shape-preserving projection; denote the set of all such projections by $\mathcal{P}_S$. For a given cone S, define*

$$S^* = \big\{\varphi \in X^* \,\big|\, \langle x, \varphi\rangle \geq 0 \;\; \forall x \in S\big\}.$$

We will refer to S^* as the *dual cone* of S. A dual is always a weak*-closed cone in X^* but, in general, need not be pointed. The following lemma indicates that S^* is in fact "dual" to S.

Lemma 2.1. *Let $x \in X$. If $\langle x, \varphi \rangle \geq 0$ for all $\varphi \in S^*$ then $x \in S$.*

Proof. We prove the contrapositive; suppose $x \in X$ such that $x \notin S$. Then, since S is closed and convex, there exists a separating functional $\varphi \in X^*$ and $\alpha \in \mathbb{R}$ such that $\langle x, \varphi \rangle < \alpha$ and

$$\langle s, \varphi \rangle > \alpha \quad \forall s \in S. \tag{2.1}$$

Note that we must have $\alpha < 0$ because $0 \in S$. In fact, for every $s \in S$ we claim

$$\langle s, \varphi \rangle \geq 0 > \alpha. \tag{2.2}$$

To check this, suppose there exists $s_0 \in S$ such that $\langle s_0, \varphi \rangle = \beta < 0$; this would imply

$$\left\langle \frac{\alpha}{\beta} s_0, \varphi \right\rangle = \alpha$$

while $\frac{\alpha}{\beta} s_0 \in S$. And this is in contradiction to (2.1). The validity of (2.2) implies that $\varphi \in S^*$ and this completes the proof.

***Remark* 2.1.** Not surprisingly, characteristics of the cone S and the subspace V play a role in the existence of shape-preserving operators. In [5], it is assumed that both S and V have 'largest possible' dimension (the so-called high-dimensional assumptions). Specifically, in that paper it is assumed that a basis for V can be obtained from S $(\dim(V) = \dim(V \cap S))$ and that $S \subset X$ is 'so large' that the zero-functional is the *only* element of X^* that vanishes on S (and so, roughly speaking, $\dim(S) = \dim(X)$). This latter condition is clearly equivalent to the (geometric) condition that S^* is pointed.

In this paper we look to remove the assumptions described in the note above. Specifically, throughout the remainder of this paper we make the following *low-dimensional assumptions*: S^* is not pointed and $\dim(V \cap S) \leq \dim(V)$. By way of completeness, we note that the case S^* is pointed and $\dim(V \cap S) < \dim(V)$ is handled by Theorem 3.1 (below); in this case we always have $\mathcal{P}_S(X, V) = \varnothing$.

***Remark* 2.2.** We wish to distinguish between two types of (non-pointed) dual cones: those which can be made pointed and those which cannot. To this end, let $S^{\perp} \subset X^*$ denote the space of functionals that vanish against S and note $S^{\perp} \subset S^*$. We are interested in (potentially) 'sharpening' S^*, in the following sense.

Definition 2.2. *We say that S^* can be sharpened if*

$$\left(\overline{S^* \setminus S^{\perp}}\right) \cap S^{\perp} = \varnothing$$

where the closure is taken with respect to the weak topology. In this case, we define $S^{\sharp} := \overline{S^* \setminus S^{\perp}}$.*

This concept of sharpening a dual cone is motivated by a simple fact: $S^{\sharp}$ is a pointed cone, with a "pre-dual"cone nearly identical to cone S. And, as we illustrate in the next section, $S^{\sharp}$ can be employed to give a geometric characterization of when $\mathcal{P}_S = \varnothing$.

3. Main results. ***3.1. General existence results.*** In this section we give characterizations for $\mathcal{P}_S \neq \varnothing$; the proofs of these statements are given in Subsection 3.3. To understand when $\mathcal{P}_S \neq \varnothing$, we should consider the relationship between the shape to be preserved, S, and the range of our projection, V. Indeed, this relationship can be expressed by restricting S^* to V, denoted $S^*_{|V}$. This consideration can often completely characterize when $\mathcal{P}_S \neq \varnothing$.

Definition 3.1. *Let* $d := \dim(V)$. *Define* $V_0 := \{v \in V \mid \langle v, \varphi\rangle = 0 \ \forall \varphi \in S^*\}$ *and note* $V_0 \subset S$. *Now let* $k := \dim(V \cap S) - \dim(V_0)$. *Fix a basis* $\{v_1, \ldots, v_d\}$ *for* V *such that* $v_1, \ldots, v_r \notin S$, $V_0 = [v_{r+1}, \ldots, v_{d-(k+2)}]$, *and* $v_{d-(k+1)}, \ldots, v_d \in S$ (*where* $[a_1, \ldots, a_s]$ *denotes the linear span of* $\{a_1, \ldots, a_s\}$). *Using this basis, we define* $V_- := [v_1, \ldots, v_r]$ *and* $V_+ := [v_{d-(k+1)}, \ldots, v_d]$ *and decompose* V *as*

$$V = V_- \oplus V_0 \oplus V_+ = [v_1, \ldots, v_r, v_{r+1}, \ldots, v_{d-(k+2)}, v_{d-(k+1)}, \ldots, v_d].$$

***Remark* 3.1.** The following results rely on the decomposition of V given above. Note that once the cone $S \subset X$ is fixed, this decomposition is merely a convenient basis choice for V. Indeed, every $Q \in \mathcal{L}(X, V)$ can be expressed in terms of this basis as

$$Q = \sum_{i=1}^{d} u_i \otimes v_i, \qquad \text{where} \quad Qf = \sum_{i=1}^{d} \langle f, u_i\rangle v_i$$

with $u_i \in X^*$ for each i. Using the representation, we say that the action (up to similarity) of Q on V is the matrix $(\langle v_i, u_j\rangle)$. Evidently Q is a projection if and only if $(\langle v_i, u_j\rangle) = \delta_{ij}$.

Recall that $S^\perp \subset S^*$ denotes the space of functionals that vanish against S. We say subspace $M \subset X^*$ is *total* over subspace $Y \subset X$ if $\dim(M_{|_Y}) = \dim(Y)$. Without any assumptions on the dual cone S^* we have the following characterization.

Theorem 3.1. *Let* $S \subset X$ *be given and* $V = V_- \oplus V_0 \oplus V_+$. *Then* $\mathcal{P}_S(X, V) \neq \varnothing$ *if and only if* $S^\perp$ *is total over* V_- *and* $\mathcal{P}_S(X, V_+) \neq \varnothing$.

This characterization indicates that shape-preservation onto V is almost equivalent to shape-preservation onto V_+. And in Subsection 3.2, we establish existence results involving V_+. For the remainder of this section, we consider the case in which S^* can be sharpened, i.e., the case in which $S^\sharp$ is defined.

When a dual cone has a particular structure, existence of shape-preserving operators can be described in terms of that structure, which we now define. Note that, in the context of our current considerations, we say a finite (possibly) signed measure μ with support $E \subset X^*$ is a *generalized representing measure* for $\varphi \in X^*$ if $\langle x, \varphi\rangle = \int_E \langle s, x\rangle \, du(s)$ for all $x \in X$. A nonnegative measure μ satisfying this equality is simply a *representing measure*.

Definition 3.2. *Let* X *be a Hausdorff space over* $\mathbb{R}$. *We say that a pointed closed cone* $K \subset X^*$ *is simplicial if* K *can be recovered from its extreme rays* (*i.e.*, $K = \overline{\text{co}}\,(E(K))$) *and the set of extreme rays of* K *form an independent set* (*independent in the sense that any generalized representing measure for* $x \in K$ *supported on* $E(K)$ *must be a representing measure*).

Proposition 3.1. *A pointed closed cone* $K \subset X^*$ *of finite dimension* d *is simplicial if and only if* K *has exactly* d *extreme rays.*

Theorem 3.2 ([5], Theorem 1.1). *Let* $S^* \subset X^*$ *denote the dual cone of* $S \subset X$ *and suppose* S^* *is simplicial. Then* $\mathcal{P}_S(X, V) \neq \varnothing$ *if and only if the cone* $S^*_{|_V}$ *is simplicial.*

Theorem 3.3. *Let* $S \subset X$ *be given and suppose* $S^\sharp$ (*exists and*) *is simplicial. Then* $\mathcal{P}_S(X, V) \neq \varnothing$ *if and only if* $S^\perp$ *is total over* V *and* $S^\sharp_{|_{V_+}}$ *is simplicial.*

3.2. Preservation onto V_-, V_0, V_+. For any $Q \in \mathcal{L}(X, V)$ we can write (using Remark 3.1)

$$Q = \left(\sum_{i=1}^{r} u_i \otimes v_i\right) \oplus \left(\sum_{i=r+1}^{d-(k+2)} u_i \otimes v_i\right) \oplus \left(\sum_{i=d-(k+1)}^{d} u_i \otimes v_i\right) =: Q_- \oplus Q_0 \oplus Q_+.$$

In this section we consider these components of Q in the shape-preserving projection case. When Q is a projection, note that each component is also a projection (onto its specific range).

Lemma 3.1. *For a given $S \subset X$, let $V = V_- \oplus V_0 \oplus V_+$. Let $P \in \mathcal{P}(X, V)$ be any projection. Then $P_0 \in \mathcal{P}_S(X, V_0)$.*

Proof. For every $f \in S$ and every $\varphi \in S^*$ we have

$$\langle P_0 f, \varphi \rangle = \left\langle \sum_{i=r+1}^{d-(k+2)} \langle f, u_i \rangle v_i, \varphi \right\rangle = \sum_{i=r+1}^{d-(k+2)} \langle f, u_i \rangle \langle v_i, \varphi \rangle = 0$$

by definition of V_0. This implies, by Lemma 2.1, that $P_0 f \in S$ and, since P is a projection, we have $P_0 \in \mathcal{P}_S(X, V_0)$.

Lemma 3.1 is proved.

Lemma 3.2. *For a given $S \subset X$, let $V = V_- \oplus V_0 \oplus V_+$ and assume* $\dim (V_-) = r \neq 0$. *If $P = \sum_{i=1}^{d} u_i \otimes v_i \in \mathcal{P}_S(X, V)$ then $u_1, \ldots, u_r \in S^\perp$ and $S^\perp$ is total over V_-.*

Proof. Let $P \in \mathcal{P}_S(X, V)$ and write $P_- = \sum_{i=1}^{r} u_i \otimes v_i$. For every $f \in S$ we know

$$P_- f + P_0 f + P_+ f \in S.$$

But the decomposition of V (Definition 3.1) implies

$$P_- f = \sum_{i=1}^{r} u_i(f) v_i = 0, \tag{3.1}$$

for every $f \in S$, since otherwise we would have $\dim (V_+) > k$. Now (3.1) implies that for each i, $u_i(f) = 0$ for all $f \in S$ and thus $u_i \in S^\perp$. This, together with the fact that P is a projection, i.e., $u_i(v_j) = \delta_{ij}$, implies that $S^\perp$ is total over V_-.

Lemma 3.2 is proved.

***Remark* 3.2.** When $k = \dim (V_+) \neq 0$, note that $S^*_{|V_+}$ is a k-dimensional pointed cone. It is convenient to interpret this cone as a subset of $\mathbb{R}^k$ by associating each $\varphi_{|V_+} \in S^*_{|V_+}$ with the k-vector $[\varphi(v_{d-(k+1)}), \ldots, \varphi(v_d)]^T$. We will use this association throughout the remainder of the paper. And so by construction, we may regard $S^*_{|V_+}$ as a cone in the positive orthant of $\mathbb{R}^k$.

Lemma 3.3. *Let $S \subset X$ be given and let S^* denote its dual cone. Let $V = V_- \oplus V_0 \oplus V_+$ and assume* $\dim (V_+) = k \neq 0$. *If the* (k-*dimensional*) *cone $S^*_{|V_+}$ is simplicial then $\mathcal{P}_S(X, V_+) \neq \varnothing$.*

Proof. Recall that our fixed basis of V_+ is given by $\{v_{d-(k+1)}, \ldots, v_d\}$. For convenience within this proof, relabel these elements as $\{v_1, \ldots, v_k\}$. Now, by assumption, $S^*_{|V_+}$ has exactly k extreme rays. Label each ray as

$$[u_{1|_{V_+}}]^+, \ldots, [u_{k|_{V_+}}]^+,$$

where $u_{1|_{V_+}}, \ldots, u_{k|_{V_+}}$ are non-zero points chosen from distinct rays. Thus we have

$$S^*_{|V_+} = \mathrm{co}\left([\mathrm{u}_{1|_{V_+}}]^+, \ldots, [\mathrm{u}_{k|_{V_+}}]^+ \right). \tag{3.2}$$

Define the (row) vector $\boldsymbol{u} := (u_1, \ldots, u_k) \in (S^*)^k$, where each u_i restricts to extreme ray $[u_{i|_{V_+}}]^+$, and the (column) vector $\boldsymbol{v} = (v_1, \ldots, v_k)^T$. Using this notation, note that for any $\varphi \in S^*$ we may write

$$(\langle v_1, \varphi \rangle, \ldots, \langle v_k, \varphi \rangle)^T = \langle \boldsymbol{v}, \varphi \rangle = (\langle v_i, u_j \rangle)\, \boldsymbol{c}_\varphi = M \boldsymbol{c}_\varphi,$$

where $M := (\langle v_i, u_j \rangle)$ is a $k \times k$ matrix and $\boldsymbol{c}_\varphi$ is the vector of nonnegative coefficients guaranteed by (3.2). Since $S^*_{|V_+}$ has k independent elements, matrix M is non-singular. Thus we may solve for $\boldsymbol{c}_\varphi$ and write $\boldsymbol{c}_\varphi = M^{-1}\langle \boldsymbol{v}, \varphi \rangle$. Let $P_+ := \boldsymbol{u}M^{-1} \otimes \boldsymbol{v}$; obviously P is a projection from X into V_+. Moreover, for every $f \in S$ and $\varphi \in S^*$ we have

$$\langle P_+ f, \varphi \rangle = \big\langle \langle f, \boldsymbol{u}M^{-1} \rangle \boldsymbol{v}, \varphi \big\rangle = \langle f, \boldsymbol{u} \rangle M^{-1} \langle \boldsymbol{v}, \varphi \rangle = \langle f, \boldsymbol{u} \rangle \boldsymbol{c}_\varphi \geq 0$$

since $\langle f, \boldsymbol{u} \rangle \boldsymbol{c}_\varphi$ is a dot-product of two vectors with nonnegative entries. By Lemma 2.1, $P_+ f \in S$.

Lemma 3.3 is proved.

Lemma 3.4. *Let $S \subset X$ be given and let S^* denote its dual cone. Let $V = V_- \oplus V_0 \oplus V_+$ and assume $\dim(V_+) = k \neq 0$. If the (k-dimensional) cone $S^*_{|V_+}$ is not closed then $\mathcal{P}_S(X, V_+) = \varnothing$.*

Proof. We consider the contrapositive. Let $P \in \mathcal{P}_S(X, V_+)$ and let $\overline{P^*S^*}$ denote the (weak*) closure of $P^*S^* \subset X^*$. Choose $P^*\varphi \in \overline{P^*S^*} \subset P^*X^*$ and a sequence $\{P^*\varphi_k\}_{k=1}^\infty \subset P^*S^*$ such that $P^*\varphi_k \to P^*\varphi$. Notice, by Lemma 2.1, $\{P^*\varphi_k\}_{k=1}^\infty \subset S^*$. S^* is weak*-closed and therefore $P^*\varphi \in S^*$; this implies $P^*\varphi \in P^*S^*$ since $(P^*)^2 = P^*$. Thus P^*S^* is closed. Note that P^*S^* is homeomorphic to $(P^*S^*)_{|V_+}$ and thus $(P^*S^*)_{|V_+}$ is closed. Finally, we claim $(P^*S^*)_{|V_+} = S^*_{|V_+}$. To verify this, choose $\varphi \in S^*$, $v \in V_+$ and consider

$$\langle v, P^*\varphi \rangle = \langle Pv, \varphi \rangle = \langle v, \varphi \rangle,$$

where the last equality follows from the fact that P is a projection. But this equation simply says that $P^*\varphi$ and φ agree on V_+, thus establishing the claim. From here we can conclude that $S^*_{|V_+}$ is closed.

Lemma 3.4 is proved.

3.3. Proofs of existence results. ***Proof of Theorem 3.1.*** ($\Rightarrow$) Let $P \in \mathcal{P}_S(X, V)$ and write $P = P_- \oplus P_0 \oplus P_+$. By Lemma 3.2, $S^\perp$ is total over V_-. Furthermore, for every $f \in S$ and every $\varphi \in S^*$ we have

$$0 \leq \langle Pf, \varphi \rangle = \langle P_- f, \varphi \rangle + \langle P_0 f, \varphi \rangle + \langle P_+ f, \varphi \rangle = \langle P_+, \varphi \rangle$$

by Lemmas 3.1 and 3.2 and therefore $\mathcal{P}_S(X, V_+) \neq \varnothing$.

($\Leftarrow$) Let $Q = Q_- \oplus Q_0 \oplus Q_+$ be any projection onto V and define $P_0 := Q_0$. Choose $P_1 \in \mathcal{P}_S(X, V_+)$; we claim

$$P_0 \oplus P_1 \in \mathcal{P}_S(X, V_0 \oplus V_+). \tag{3.3}$$

The fact that this operator is shape-preserving is clear since $V_0 \subset S$. We need only verify that that the action of the operator on $V_0 \oplus V_+$ is the identity action. Note that we need only check that P_1 vanishes on V_0. But this is clear since $V_0 \subset S$ is a linear space, $P_1 V_0 \subset S$ and $V_0 \cap V_+ = \{\boldsymbol{0}\}$. This establishes (3.3). We now focus on V_-. Since $S^\perp$ is total over V_- (and assuming $r := \dim(V_-) > 0$), there exist $u_1, \ldots, u_r \in S^\perp$ such that $P_- := \sum_{i=1}^r u_i \otimes v_i$ is a projection onto V_- (in the case $r = 0$ define P_- to be the zero-operator). Now with P_1 chosen as above, write $P_1 = \sum_{i=d-(k+1)}^d u_i \otimes v_i$. Again using $S^\perp$ total over V_-, there exist functionals $\varphi_1, \ldots, \varphi_r \in S^\perp$ such that for each $j \in \{d-(k+1), \ldots, d\}$, there exist constants $\{c_{1j}, \ldots, c_{rj}\} \in \mathbb{R}$ such that

$$\left\langle v_i, \sum_{m=1}^r c_{m,j}\varphi_m \right\rangle = -\langle v_i, u_j \rangle \quad \text{for} \quad i = 1, \ldots, r.$$

Define $\Phi_j := \sum_{m=1}^{r} c_{m,j}\varphi_j$ and note that

$$\langle v, \Phi_j\rangle = -\langle v, u_j\rangle \quad \text{for any} \quad v \in V_-. \tag{3.4}$$

Let $U_j := u_j + \Phi_j$ for each $j = d-(k+1), \ldots, d$ and $P_+ := \sum_{i=d-(k+1)}^{d} U_i \otimes v_i$. We claim

$$P := P_- \oplus P_0 \oplus P_+$$

belongs to $\mathcal{P}_S(X, V)$. Consider first P_+; note, by construction each $\Phi_j \subset S^\perp$ vanishes S. Thus $P_+ \in \mathcal{P}_S(X, V_+)$ and so by (3.3), we have

$$P_0 \oplus P_+ \in \mathcal{P}_S(X, V_0 \oplus V_+). \tag{3.5}$$

Regarding P_-, by construction this operator vanishes on S and this, combined with (3.5), implies $PS \subset S$. To see that P has the identity action on V, we need only check that P_- vanishes on $V_0 \oplus V_+$ and $P_0 \oplus P_+$ vanishes on V_-. The former condition holds since the basis we use for V_0 and V_1 belongs to S. To establish the latter, first note that P_0 vanishes on V_- by construction. And, by (3.4), for any $v \in V_-$ we have

$$P_+ v = \sum_{i=d-(k+1)}^{d} \langle v, U_i\rangle v_i = \sum_{i=d-(k+1)}^{d} \langle v, u_i + \Phi_i\rangle v_i =$$

$$= \sum_{i=d-(k+1)}^{d} \langle v, u_i - u_i\rangle v_i = 0$$

by the definition of each Φ_i. So P_+ vanishes on V_-. This establishes that P is a projection.

Theorem 3.1 is proved.

Proof of Theorem 3.3. By Theorem 3.1, the proof will be complete if we can show $\mathcal{P}_S(X, V_+) \neq$ $\neq \varnothing$ is equivalent to $S^\sharp_{|V_+}$ simplicial, which we now establish. Recall that $S^\sharp \subset S^*$ is a pointed, weak* closed cone and, as such, is exactly the dual cone of

$$S_1 := \{x \in X \mid \langle x, \psi\rangle \geq 0 \ \ \forall \psi \in S^\sharp\}.$$

Note that S_1 contains the cone S. By Theorem 3.2,

$$S^\sharp_{|V_+} \text{ is simplicial } \iff \mathcal{P}_{S_1}(X, V_+) \neq \varnothing$$

and thus we need only show

$$\mathcal{P}_S(X, V_+) \neq \varnothing \iff \mathcal{P}_{S_1}(X, V_+) \neq \varnothing. \tag{3.6}$$

Let $P \in \mathcal{P}_S(X, V_+)$; we claim $P(S_1) \subset S_1$. From Lemma 2.1, it follows that $P(S_1) \subset S_1$ if and only if $P^*(S^\sharp) \subset S^\sharp$, where P^* denotes the adjoint of P (defined by $\langle f, P^* u\rangle = \langle Pf, u\rangle$ for $f \in X$ and $u \in X^*$). We know that $P^*(S^\sharp) \subset S^*$ since (via Lemma 2.1) $P^* S^* \subset S^*$ and $S^\sharp \subset S^*$. Thus we need only show that, for each $\psi \in S^\sharp$, non-zero $P^*\psi$ does not vanish against S. But $P^*\psi = \sum_{j=1}^{k} \langle v_j, \psi\rangle u_j$, where (via relabeling) $\{v_1, \ldots, v_k\} \subset S$ is our fixed basis for V_+. And so $P^*\psi \neq 0$ implies $\langle v_i, \psi\rangle \neq 0$ for some i. Therefore $P^*\psi \in S^\sharp$, which establishes $P(S_1) \subset S_1$. Thus $P \in \mathcal{P}_{S_1}(X, V_+)$. To complete the proof, let $P \in \mathcal{P}_{S_1}(X, V_+)$. Arguing as above, it follows that $P^* S^* \subset S^*$ and thus $P \in \mathcal{P}_S(X, V_+)$, which establishes (3.6).

Theorem 3.3 is proved.

4. Application: the q-monotone case. In this section we consider the preservation of q-monotonicity (defined below) by a projection from $X = (C^q[-1.1], \|\cdot\|)$ onto $V = \Pi_n$ (the subspace of algebraic polynomials of degree less than or equal to n), where

$$\|f\| := \max_{j=0,\ldots,q} \{\|f^{(j)}\|_\infty\}.$$

For $s \in \mathbb{N}$, let $\mathbb{Y}_s$ denote the collection of s distinct points $Y = \{y_i\}_{i=1}^s$ where $y_0 = -1 < < y_1 < \ldots < y_s < 1 = y_{s+1}$. For $q \in \mathbb{N}$ and $Y \in \mathbb{Y}_s$, define

$$S_Y^q = \{f \in X \mid (-1)^j f^{(q)}(t) \geq 0 \text{ whenever } t \in [y_j, y_{j+1}], \ j = 0, \ldots, s\}.$$

We say $f \in X$ is q-monotone (with respect to $Y \in \mathbb{Y}_s$) exactly when $f \in S_Y^q$. We denote by $\mathcal{P}_{S_Y^q}$ the set of q-monotone preserving projections from X onto Π_n.

The main point of this section is the following characterization. The proof of this theorem considers the (topological) consequence of restricting a dual cone to subspace $V = \Pi_n$. For purposes of illustration, we include (in Subsection 4.1) two arguments that establish an existence result; Version 1 uses a "classical" approach to shape-preservation and Version 2 utilizes the restriction of a dual cone.

Theorem 4.1. *Let $s \in \mathbb{N}$. Then, for $Y \in \mathbb{Y}_s$,*

$$\mathcal{P}_{S_Y^q} \neq \varnothing \iff n - s - q \leq 1.$$

Proof. We prove this result through induction on q. The $q = 1$ case is verified (for all s and n) in the following section (see Lemma 4.1). We now proceed with the inductive step; for fixed q_0, we assume

$$\mathcal{P}_{S_Y^{q_0}} \neq \varnothing \iff n - s - q_0 \leq 1 \tag{4.1}$$

and show

$$\mathcal{P}_{S_Y^{q_0+1}} \neq \varnothing \iff n - s - (q_0 + 1) \leq 1. \tag{4.2}$$

Suppose $n - s - (q_0 + 1) \leq 1$; then we have $(n - 1) - s - q_0 \leq 1$ and so by (4.1) there exists $P \in \mathcal{P}_{S_Y^{q_0}}(X, \Pi_{n-1})$. Using the notation from Subsection 3.2, we may write $P = \sum_{k=1}^{n-1} u_k \otimes v_k$ where $Pf = \sum_{k=1}^{n-1} \langle f, u_k \rangle v_k \in \Pi_{n-1}$. Define $\widehat{P} := \sum_{k=0}^{n} \hat{u}_k \otimes \hat{v}_k$ where $\hat{u}_0 \otimes \hat{v}_0 := \delta_{-1} \otimes 1$ and, for $k > 0$, $\hat{u}_k := u_k \circ D_t$ (D_t is the differential operator), $\hat{v}_k := I_t \circ v_k$ (I_t is the integral operator). Thus

$$(\widehat{P}f)(t) = \sum_{k=0}^{n} \langle f, \hat{u}_k \rangle \hat{v}_k(t) = f(-1) + \sum_{k=1}^{n} \langle f', u_k \rangle I_t(v_k) =$$

$$= f(-1) + \int_{-1}^{t} \sum_{k=1}^{n} \langle f', u_k \rangle v_k(x)\, dx = f(-1) + \int_{-1}^{t} (Pf')(x)\, dx.$$

Note that $\widehat{P} : C^{q_0+1}[-1, 1] \to \Pi_n$. Moreover, since P is a projection (onto Π_{n-1}), so is $\widehat{P}$ (onto Π_n). And finally, if $f \in S_Y^{q_0+1}$ then $f' \in S_Y^{q_0}$ which implies $Pf' \in S_Y^{q_0}$. Therefore, since $(\widehat{P}f)^{(q_0+1)} = = (Pf')^{(q_0)}$, we have $\widehat{P}f \in \mathcal{P}_{S_Y^{q_0+1}}$. Thus $\mathcal{P}_{S_Y^{q_0+1}} \neq \varnothing$. To establish the other direction of (4.2), consider $n - s - (q_0 + 1) > 1$; we show that this implies $\mathcal{P}_{S_Y^{q_0+1}} = \varnothing$. Suppose there exists $P \in \mathcal{P}_{S_Y^{q_0+1}}$.

Arguing as above, express P as $P = \sum_{k=0}^{n} u_k \otimes v_k$, where $v_k := x^k$. Define $\widehat{P} := \sum_{k=0}^{n-1} \hat{u}_k \otimes \hat{v}_k$ where $\hat{u}_k = u_k \circ I_t$ and $\hat{v}_k = D_t \circ v_k$. Then

$$\big(\widehat{P}f\big)(t) = \sum_{k=0}^{n} \langle f, \hat{u}_k \rangle \hat{v}_k(t) = D_t \left(\sum_{k=1}^{n} \langle I_t f, u_k \rangle v_k \right) = D_t \left(P(I_t f) \right).$$

Evidently $\widehat{P}$ is a projection from C^{q_0} onto Π_{n-1}. If $f \in S_Y^{q_0}$ then $\widehat{P}f \in S_Y^{q_0}$ since $P(I_t f) \in S_Y^{q_0+1}$ and this implies $\widehat{P} \in \mathcal{P}_{S_Y^{q_0}}(X, \Pi_{n-1})$. But from our supposition, we have $(n-1) - s - q_0 > 1$, which, from (4.1), implies $\mathcal{P}_{S_Y^{q_0}} = \varnothing$. This contradiction has resulted from assuming $P \in \mathcal{P}_{S_Y^{q_0+1}}$ and therefore we must have $\mathcal{P}_{S_Y^{q_0+1}} = \varnothing$. This establishes (4.2).

Theorem 4.1 is proved.

4.1. The $q = 1$ case. In this subsection we verify the $q_0 = 1$ case via the following lemma.

Lemma 4.1. $\mathcal{P}_{S_Y^1}(X, \Pi_n) \neq \varnothing \iff n - s \leq 2.$

To begin, denote S_Y^1 by S_Y and let $S^* \subset X^*$ denote the dual cone of S_Y. Recall the decomposition of V used above; relative to S_Y, we write $V = V_- \oplus V_0 \oplus V_+$. Note that V_0 is 1-dimensional and $V_0 = [1]$. As we will see below, $\dim(V_+) = n - s$; recall from above that we may assume $S^*_{|V_+} \subset \mathbb{R}^{n-s}$. For fixed Y, put

$$\Delta = \Delta(x) := \prod_{i=1}^{s} (y_i - x).$$

Proposition 4.1. $\dim(V_+) = \max\{0, n-s\}$. *If* $n - s > 0$ *then, for* $i = 1, \ldots, n-s$,

$$v_i(x) := \int_{-1}^{x} (1 - t^i) \Delta(t)\, dt \in S_Y$$

and $\{v_1, \ldots, v_{n-s}\}$ *forms a basis for* V_+.

Let $v \in V \cap S_Y$; then for $i = 1, \ldots, s$ we have $v'(y_i) = 0$. Thus if $n - s \leq 0$ then $\dim(V_+) = 0$. Assume $n - s > 0$; then by definition of S_Y we can write $v'(x) = p(x)\Delta(x)$ for some polynomial p. But $\deg(\Delta) = s$ and so $p \in \Pi_{n-(s+1)}$. Therefore $\dim(V_+) \leq n - s$. Finally, note that for $i = 1, \ldots, n - s$,

$$v_i = \int_{-1}^{x} (1 - t^i) \Delta(t)\, dt \in S_Y$$

and are independent. Thus $V_+ = [v_1, \ldots, v_{n-s}]$.

Note that in this application we have have labeled the basis elements for V_+ as $v_1, \ldots, v_{n-s}$. This departure from the labeling in the previous section is meant to simplify the notation in the current setting.

Lemma 4.2. *Suppose* $n - s > 2$. *Then* $S^*_{|V_+} \subset \mathbb{R}^{n-s}$ *is not closed and thus* $\mathcal{P}_{S_Y}(X, \Pi_n) = \varnothing$.

Proof. Fix y_j for some $j \in \{1, \ldots, s\}$. Since $n - s \geq 3$, it is clear from Proposition 4.1 that a basis for V_+ can be chosen as prescribed to include elements $v_1 := \int_{-1}^{x} \Delta(t)$ and $v_2 := \int_{-1}^{x} (1 - t^2)\Delta(t)$. Without loss, assume $\Delta(t) \geq 0$ for $t \in (y_{j-1}, y_j)$. And so, since $S^*_{|V_+}$ is a cone, it must contain, for each such t, the point (or vector) $\dfrac{(\delta'_t)_{|V_+}}{\Delta(t)}$. Thus by Proposition 4.1 there exists a vector

$$\boldsymbol{z} = [1, 1, z_3, \ldots, z_{n-s}] := \lim_{t \to y_j^-} \frac{(\delta_t')_{|V_+}}{\Delta(t)}$$

belonging to the closure of $S^*_{|V_+}$. Now, by way of contradiction, let us suppose there exists $\varphi \in S^*$ such that $\varphi_{|V_+} = \boldsymbol{z}$. Note that

$$1 = \varphi(v_1) = \varphi\left(\int_{-1}^{x} \Delta(t)\right) = \varphi(v_2) = \varphi\left(\int_{-1}^{x} (1 - t^2)\Delta(t)\right) \tag{4.3}$$

which implies

$$\varphi\left(\int_{-1}^{x} t^2 \Delta(t)\right) = 0.$$

Moreover, for every even integer $\nu \geq 2$ we have

$$\int_{-1}^{x} t^\nu \Delta(t) \in S \quad \text{and} \quad \int_{-1}^{x} (t^2 - t^\nu)\Delta(t) \in S$$

since $t^2 - t^\nu \geq 0$ on $[-1, 1]$. And thus for every ν

$$\varphi\left(\int_{-1}^{x} t^\nu \Delta(t)\right) = 0. \tag{4.4}$$

For convenience, assume $y_j = 0$. Define $\widehat{\Delta}(x)$ by $\Delta(x) = x\widehat{\Delta}(x)$. Let $T_O(x)$ be an odd Tchebyshev polynomial of (arbitrary odd) degree d. Consider the polynomial $p(x) := \int_{-1}^{x} T_O \widehat{\Delta} \in X$; the norm $\|p\|$ is clearly bounded independent of d. But by (4.3) and (4.4) we find

$$|\varphi(p)| = \left|\varphi\left(\int_{-1}^{x}\left(\sum_{\substack{i=1 \\ i \text{ odd}}}^{d} c_i t^i\right)\widehat{\Delta}(t)\right)\right| = \left|\varphi\left(\sum_{\substack{i=1 \\ i \text{ odd}}}^{d} \int_{-1}^{x} c_i t^{i-1} \Delta\right)\right| = d$$

since $|c_1| = d$. This implies that φ is unbounded and thus cannot be an element of S^*. Therefore $S^*_{|V_+}$ is not closed. Consequently, by Lemma 4.2 and Corollary 3.4, we have $\mathcal{P}_{S_Y}(X, V_+) = \varnothing$ and thus $\mathcal{P}_{S_Y}(X, V) = \varnothing$ by Theorem 3.1.

Lemma 4.2 is proved.

Lemma 4.3. *Suppose* $n - s \leq 2$. *Then* $\mathcal{P}_{S_Y}(X, V) \neq \varnothing$.

***Proof* (*Version* 1).** Set $y_{s+2} := y_0 = -1$. Fix $n \in \mathbf{N}$, $n - s \leq 2$. For each $g \in C[-1, 1]$ denote by $L_{n-1}(x, g) := L(x, g; y_1, \ldots, y_n)$ — the Lagrange polynomial of degree $< n$, that interpolates g at y_j's, $j = 1, \ldots, n$. First we remark, that the operator $P \in \mathcal{L}(C^1[-1, 1], \Pi_n)$, defined by

$$(Pg)(x) := g(0) + \int_{0}^{x} L_{n-1}(t, g')dt,$$

is a projection, that is $P \in \mathcal{P}(C^1[-1, 1], \Pi_n)$. This readily follows from the fact, that for each $p_{n-1} \in \Pi_{n-1}$ we have

$$L_{n-1}(x, p_{n-1}) \equiv p_{n-1}(x).$$

So, to end the proof we have to check, that if $f \in S_Y$, then $(Pf) \in S_Y$ as well, or, which is the same,

$$L_{n-1}(x, f')\Delta(x) \geq 0, \quad x \in [-1, 1], \tag{4.5}$$

where $\Delta(x) := \prod_{j=1}^{s}(y_j - x)$. Indeed, if $n \leq s$, then $L_{n-1}(x, f') \equiv 0$, that yields (4.5). If $n = s + 1$, then $L_{n-1}(x, f') = A\Delta(x)$, where $A \geq 0$, that yields (4.5). Finally, if $n = s + 2$, then $L_{n-1}(x, f') = (ax + b)\Delta(x)$. Let us show, that

$$ax + b \geq 0, \quad x \in [-1, 1]. \tag{4.6}$$

If $x = -1$, then

$$-a + b = \frac{L_{n-1}(-1, f')}{\Delta(-1)} = \frac{f'(-1)}{\Delta(-1)} \geq 0.$$

Similarly $a + b \geq 0$. Thus (4.6) holds, that yields (4.5).

***Proof* (*Version* 2).** We claim that (regardless of the value $n - s$) $S^{\perp}$ is total over V_-. Indeed note that in our setting we have $r := \dim(V_-) = \min\{s, n\}$ and $V_- = [x, x^2, \ldots, x^r]$. And since $\{\delta'_{y_i}\}_{i=1}^{s} \subset S^{\perp}$ we have that $S^{\perp}$ is total over V_-. Now in the case $n - s \leq 0$ we have $\dim(V_+) = 0$ and so trivially $\mathcal{P}_S(X, V_+) \neq \varnothing$ since the zero-operator belongs to this set. Suppose $n - s > 0$; by Proposition 4.1, $n - s$ is exactly the dimension of $S^*_{|V_+}$. We claim, in the cases $n - s = 1, 2$, the cone $S^*_{|V_+}$ is simplicial. This is clear in the $n - s = 1$ case, since every 1-dimensional pointed cone is (trivially) simplicial. For $n - s = 2$, note that a 2-dimensional pointed cone is simplicial if and only if it is closed. We now show $S^*_{|V_+} \subset \mathbb{R}^2$ is closed. Recall that $S^*_{|V_+}$ belongs to the positive quadrant of $\mathbb{R}^2$. And it will suffice to show that for some basis for V_+, there exist functionals $\varphi_1, \varphi_2 \in S^*$ such that $(\varphi_i)_{|V_+}$ belongs to the ray determined by $\boldsymbol{e}_i$ (the standard basis element) for $i = 1, 2$. To this end, note that

$$v_1 := \int_{-1}^{x} -(t-1)\Delta(t) \qquad \text{and} \qquad v_2 := \int_{-1}^{x} (t+1)\Delta(t)$$

are elements of S and form a basis for V_+. Moreover $(\delta'_{-1})_{|V_+} = [a, 0]$ and $(\delta'_1)_{|V_+} = [0, b]$ for some $a, b > 0$. Therefore $S^*_{|V_+}$ is exactly the positive quadrant of $\mathbb{R}^2$. Thus, in the cases $n - s = 1, 2$ we have $S^*_{|V_+}$ simplicial, which implies $\mathcal{P}_S(X, V_+) \neq \varnothing$ by Theorem 3.3. By Theorem 3.1 we conclude $\mathcal{P}_S(X, V) \neq \varnothing$.

Lemma 4.3 is proved.

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