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A NEW APPLICATION OF GENERALIZED QUASI-POWER INCREASING SEQUENCES

НОВЕ ЗАСТОСУВАННЯ УЗАГАЛЬНЕНИХ ПОСЛІДОВНОСТЕЙ КВАЗІСТЕПЕНЕВОГО ЗРОСТАННЯ

We prove a theorem dealing with $\mid \bar{N}, p_n, \theta_n \mid_k$-summability using a new general class of power increasing sequences instead of a quasi-η-power increasing sequence. This theorem also includes some new and known results.

Доведено теорему про $\mid \bar{N}, p_n, \theta_n \mid_k$-сумовність із використанням нового загального класу послідовностей степеневого зростання замість послідовності квазі-η-степеневого зростання. Окремими випадками цієї теореми є деякі нові та відомі результати.

1. Introduction. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). We write $\mathcal{BV}_{\mathcal{O}} = \mathcal{BV} \cap \mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}} = \left\{ x = (x_k) \in \Omega \colon \lim_k |x_k| = 0 \right\}$, $\mathcal{BV} = \left\{ x = (x_k) \in \Omega \colon \sum_k |x_k - x_{k+1}| < \infty \right\}$ and Ω being the space of all real-valued sequences. A positive sequence (δ_n) is said to be a quasi-η-power increasing sequence if there exists a constant $K = K(\eta, \delta) \geq 1$ such that $Kn^\eta \delta_n \geq m^\eta \delta_m$ holds for all $n \geq m \geq 1$ (see [9]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n the nth (C,1) mean of the sequence (na_n), that is, $t_n = \dfrac{1}{n} \sum_{v=1}^{n} va_v$. A series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \qquad (P_{-i} = p_{-i} = 0, \ \ i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}|\Delta\sigma_{n-1}|^k < \infty, \tag{4}$$

where

$$\Delta\sigma_{n-1} = \sigma_n - \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v, \quad n \geq 1. \tag{5}$$

In the special case $p_n = 1$ for all values of n, $|\bar{N}, p_n|_k$ summability is the same as $|C,1|_k$ summability. Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n, \theta_n|_k$, $k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty}\theta_n^{k-1}|\Delta\sigma_{n-1}|^k < \infty. \tag{6}$$

If we take $\theta_n = \dfrac{P_n}{p_n}$, then $|\bar{N}, p_n, \theta_n|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability. Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $|C,1|_k$ summability.

Furthermore, if we take $\theta_n = n$, then $|\bar{N}, p_n, \theta_n|_k$ summability reduces to $|R, p_n|_k$ (see [4]) summability.

2. Known result. In [6], we have proved the following main theorem dealing with $|\bar{N}, p_n, \theta_n|_k$ summability factors of infinite series.

Theorem A. *Let* $\left(\dfrac{\theta_n p_n}{P_n}\right)$ *be a non-increasing sequence,* $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$ *and* (X_n) *be a quasi-η-power increasing sequence for some* η $(0 < \eta < 1)$. *Suppose also that there exist sequences* (β_n) *and* (λ_n) *such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{7}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{8}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{9}$$

$$|\lambda_n|X_n = O(1). \tag{10}$$

If

$$\sum_{v=1}^{n}\theta_v^{k-1}v^{-k}|s_v|^k = O(X_n) \quad \text{as} \quad n \to \infty, \tag{11}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{12}$$

$$P_n\Delta p_n = O(p_n p_{n+1}), \tag{13}$$

then the series $\sum_{n=1}^{\infty} a_n \dfrac{P_n \lambda_n}{n p_n}$ is summable $|\bar{N}, p_n, \theta_n|_k$, $k \geq 1$. If we take (X_n) as an almost increasing sequence and $\theta_n = \dfrac{P_n}{p_n}$ in Theorem A, then we get a result which was published in [3], in this case the condition "$\left(\dfrac{\theta_n p_n}{P_n}\right)$ is a non-increasing sequence" is automatically satisfied and the condition $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$ is not needed.

Remark. It should be noted that, we can take $(\lambda_n) \in \mathcal{BV}$ instead of $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$ and it is sufficient to prove Theorem A.

3. Main result. In the present paper, we have generalized Theorem A by using a quasi-f-power increasing sequence instead of a quasi η-power increasing sequence. For this purpose, we need the concept of a quasi-f-power increasing sequence. A positive sequence $\alpha = (\alpha_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(\alpha, f) \geq 1$ such that $K f_n \alpha_n \geq f_m \alpha_m$, holds for $n \geq m \geq 1$, where $f = (f_n) = \left[n^{\eta} (\log n)^{\sigma},\ \sigma \geq 0,\ 0 < \eta < 1\right]$ (see [12]). It should be noted that, if we take σ=0, then we get a quasi-η-power increasing sequence.

Now, we shall prove the following general theorem.

Theorem. *Let* $\left(\dfrac{\theta_n p_n}{P_n}\right)$ *be a non-increasing sequence,* $(\lambda_n) \in \mathcal{BV}$ *and* (X_n) *be a quasi-f-power increasing sequence. If the conditions* (7)–(13) *of Theorem A are satisfied, then the series* $\sum_{n=1}^{\infty} a_n \dfrac{P_n \lambda_n}{n p_n}$ *is summable* $|\bar{N}, p_n, \theta_n|_k$, $k \geq 1$.

If we take $\sigma = 0$, then we have Theorem A.

We require the following lemmas for the proof of the theorem.

Lemma 1. *Except for the condition* $(\lambda_n) \in \mathcal{BV}$, *under the conditions on* (X_n), (β_n) *and* (λ_n) *as expressed in the statement of the theorem, we have the following* :

$$n X_n \beta_n = O(1), \tag{14}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{15}$$

Proof. Since $\beta_n \to 0$, then we have $\Delta\beta_n \to 0$, and hence

$$\sum_{n=1}^{\infty} \beta_n X_n \leq \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty} |\Delta\beta_v| = \sum_{v=1}^{\infty} |\Delta\beta_v| \sum_{n=1}^{v} X_n =$$

$$= \sum_{v=1}^{\infty} |\Delta\beta_v| \sum_{n=1}^{v} n^{\eta} (\log n)^{\sigma} X_n n^{-\eta} (\log n)^{-\sigma} =$$

$$= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\eta} (\log v)^{\sigma} X_v \sum_{n=1}^{v} n^{-\eta} (\log n)^{-\sigma} =$$

$$= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\eta} (\log v)^{\sigma} X_v \sum_{n=1}^{v} n^{\epsilon} (\log n)^{-\sigma} n^{-\eta-\epsilon} =$$

$$= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\eta} X_v (\log v)^{\sigma} v^{\epsilon} (\log v)^{-\sigma} \sum_{n=1}^{v} n^{-\eta-\epsilon} =$$

$$= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\eta+\epsilon} X_v \int\limits_0^v x^{-\eta-\epsilon} dx =$$

$$= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\eta+\epsilon} X_v v^{1-\eta-\epsilon} =$$

$$= O(1) \sum_{v=1}^{\infty} v |\Delta\beta_v| X_v = O(1), \quad 0 < \epsilon < \eta + \epsilon < 1.$$

Again, we have that

$$n\beta_n X_n = nX_n \sum_{v=n}^{\infty} \Delta\beta_v \leq nX_n \sum_{v=n}^{\infty} |\Delta\beta_v| =$$

$$= n^{1-\eta} (\log n)^{-\sigma} n^{\eta} (\log n)^{\sigma} X_n \sum_{v=n}^{\infty} |\Delta\beta_v| \leq$$

$$\leq n^{1-\eta} (\log n)^{-\sigma} \sum_{v=n}^{\infty} v^{\eta} (\log v)^{\sigma} X_v |\Delta\beta_v| \leq$$

$$\leq \sum_{n=v}^{\infty} v^{1-\eta} (\log v)^{-\sigma} X_v v^{\eta} (\log v)^{\sigma} |\Delta\beta_v| =$$

$$= \sum_{v=1}^{\infty} v X_v |\Delta\beta_v| = O(1).$$

Lemma 1 is proved.

Lemma 2 [10]. *If the conditions* (12) *and* (13) *are satisfied*, *then we have that*

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{16}$$

4. Proof of the theorem. Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \dfrac{a_n P_n \lambda_n}{np_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{rp_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{vp_v}. \tag{17}$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{vp_v}, \quad n \geq 1. \tag{18}$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} s_v \Delta \left(\frac{P_{v-1} P_v \lambda_v}{v p_v} \right) + \frac{\lambda_n s_n}{n} =$$

$$= \frac{s_n \lambda_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \frac{P_{v+1} P_v \Delta \lambda_v}{(v+1) p_{v+1}} +$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta \left(\frac{P_v}{v p_v} \right) - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \lambda_v \frac{1}{v} =$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show for $k \geq 1$

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty \quad \text{for} \quad r = 1, 2, 3, 4. \tag{19}$$

Firstly, by using Abel's transformation, we have that

$$\sum_{n=1}^{m} \theta_n^{k-1} |T_{n,1}|^k = \sum_{n=1}^{m} \theta_n^{k-1} n^{-k} |\lambda_n|^{k-1} |\lambda_n| |s_n|^k =$$

$$= O(1) \sum_{n=1}^{m} |\lambda_n| \theta_n^{k-1} n^{-k} |s_n|^k =$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \theta_v^{k-1} v^{-k} |s_v|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \theta_n^{k-1} n^{-k} |s_n|^k =$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m =$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty$$

by virtue of (7), (10), (11) and (15).

Now, using the fact that $P_{v+1} = O\left((v+1) p_{v+1}\right)$ by (12), and applying Hölder's inequality we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}^k} \left| \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_v \right|^k =$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} |s_v| p_v |\Delta \lambda_v| \right\}^k =$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k|s_v|^kp_v\ (\beta_v)^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} =$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k|s_v|^kp_v\ (\beta_v)^k\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}} =$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k|s_v|^kp_v\ (\beta_v)^k\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} =$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k|s_v|^k\ (\beta_v)^k\left(\frac{p_v}{P_v}\right)\theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1} =$$

$$= O(1)\sum_{v=1}^{m}\ (v\beta_v)^{k-1}v\beta_v\frac{1}{v^k}\theta_v^{k-1}|s_v|^k =$$

$$= O(1)\sum_{v=1}^{m}v\beta_v\theta_v^{k-1}v^{-k}|s_v|^k =$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\theta_r^{k-1}r^{-k}|s_r|^k + O(1)m\beta_m\sum_{v=1}^{m}\theta_v^{k-1}v^{-k}|s_v|^k =$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_mX_m =$$

$$= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_vX_v + O(1)m\beta_mX_m = O(1)$$

as $m\to\infty$, in view of (7), (9), (11), (14) and (15).

Again, we have that

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|T_{n,3}|^k = O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v|s_v||\lambda_v|\frac{1}{v}\right\}^k =$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^kv^{-k}p_v|s_v|^k|\lambda_v|^k\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} =$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^kv^{-k}|s_v|^kp_v|\lambda_v|^k\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}} =$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} v^{-k}\theta_v^{k-1}\left(\frac{p_v}{P_v}\right)^{k-1}|\lambda_v|^{k-1}|\lambda_v||s_v|^k =$$

$$= O(1)\sum_{v=1}^{m}|\lambda_v|\theta_v^{k-1}v^{-k}|s_v|^k =$$

$$= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)|\lambda_m|X_m = O(1) \quad \text{as} \quad m\to\infty,$$

in view of (7), (10), (11), (15) and (16).

Finally, using Hölder's inequality, as in $T_{n,3}$ we have that

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|T_{n,4}|^k = \sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\left|\sum_{v=1}^{n-1}s_v\frac{P_v}{v}\lambda_v\right|^k =$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}^k}\left|\sum_{v=1}^{n-1}s_v\frac{P_v}{vp_v}p_v\lambda\right|^k =$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}|s_v|^k\left(\frac{P_v}{p_v}\right)^k v^{-k}p_v|\lambda_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} =$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k v^{-k}|s_v|^k p_v|\lambda_v|^k\frac{1}{P_v}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} =$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} v^{-k}\left(\frac{p_v}{P_v}\right)^{k-1}\theta_v^{k-1}|\lambda_v|^{k-1}|\lambda_v||s_v|^k =$$

$$= O(1)\sum_{v=1}^{m}|\lambda_v|\theta_v^{k-1}v^{-k}|s_v|^k =$$

$$= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)|\lambda_m|X_m = O(1) \quad \text{as} \quad m\to\infty.$$

Therefore, we get that

$$\sum_{n=1}^{m}\theta_n^{k-1}|T_{n,r}|^k = O(1) \quad \text{as} \quad m\to\infty, \quad \text{for} \quad r=1,2,3,4.$$

This completes the proof of the theorem. If we take $p_n = 1$ for all values of n, then we have a new result for $|C,1,\theta_n|_k$ summability. Furthermore, if we take $\theta_n = n$, then we have another new result for $|R,p_n|_k$ summability. Finally, if we take $p_n = 1$ for all values of n and $\theta_n = n$, then we get a new result dealing with $|C,1|_k$ summability factors.

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