UDC 512.4

O. L. Horbachuk (Lviv. Nat. Univ.),
Yu. P. Maturin (Inst. Appl. Problems Mech. and Math., Lviv)

I-RADICALS, THEIR LATTICES AND SOME CLASSES OF RINGS

I-РАДИКАЛИ, ЇХ ГРАТКИ І ДЕЯКІ КЛАСИ КІЛЕЦЬ

We describe some I-radicals in the categories of modules over semilocal rings. We give the characterization of rings over which the set of I-radicals coincides with the set of hereditary idempotent radicals. We prove that the lattices of I-radicals in the categories of "modules over Morita-equivalent rings are isomorphic".

Описуються деякі I-радикали в категоріях модулів над напівлокальними кільцями. Наведено характеризацію кілець, над якими множина I-радикалів збігається з множиною скрутів. Доводиться, що гратки I-радикалів у категоріях модулів над Моріта-еквівалентними кільцями є ізоморфними.

The notion of an I-radical was introduced by O. Horbachuk in 1972. It was applied in torsion theory and in structural theory of rings (see, for example, [1]).

Our paper is also devoted to describing properties of rings with the help of I-radicals. The main results of this paper are Theorem 4 and Theorem 5.

We shall generally follow the notation and the terminology of [1 – 5]. Let R be a ring and let $\Omega_1 \subseteq \Omega_L$. The idempotent radical cogenerated by $\overline{\Omega_1}$ is designated as $r(\Omega_1)$.

Theorem 1. *Let R be a semilocal ring and let $\Omega_1 \subseteq \Omega_L$. Then $r(\Omega_1)$ is the $J_L(\Omega_1)$-radical in R-*Mod.

Proof. Set $T := T(r(\Omega_1))$. It suffices to show that $T = \{M \mid M \in R\text{-Mod}, J_L(\Omega_1)M = M\}$. Let $M \in T$. Suppose that $M \neq J_L(\Omega_1)M$. We set $K := M/(J_L(\Omega_1)M)$. Therefore $K \neq 0$. Since $J_L(\Omega_1)K = 0$, $K = r(K, J_L(\Omega_1))$. By Theorem 1 [5], $K \in T(S\Omega_1)$. Thus, there exists an isomorphism $\beta: K \cong \oplus_A P_\alpha$, where $\{P_\alpha \mid \alpha \in A\} \subseteq \overline{\Omega_1}$. But $K \neq 0$. This implies that $A \neq \varnothing$.

Let $u \in A$ and let $\Pi_u: \oplus_A P_\alpha \to P_u$ be a natural projection. It is clear that $\Pi_u \beta \alpha \neq 0$, where $\alpha: M \to M/(J_L(\Omega_1)M) = K$ is a natural epimorphism. It follows from this that $\mathrm{Hom}_R(M, P_u) \neq 0$. But $P_u \in \overline{\Omega_1}$. Therefore $M \notin T$. This contradicts the assumption that $M \in T$. Thus, $M \in T \Rightarrow J_L(\Omega_1)M$. Let D be a left R-module such that $J_L(\Omega_1)D = D$ and let $P \in \overline{\Omega_1}$. If $d \in D$, then $d = i_1 d_1 + \ldots + i_k d_k$, where $\{i_1, \ldots, i_k\} \subseteq J_L(\Omega_1)$, $\{d_1, \ldots, d_k\} \subseteq D$. Since $J_L(\Omega_1)P = 0$, $f(d) = f(i_1 d_1 + \ldots + i_k d_k) = i_1 f(d_1) + \ldots + i_k f(d_k) = 0$, where $f \in \mathrm{Hom}_R(D, P)$. Hence $\mathrm{Hom}_R(D, P) = 0$.

Lemma 1. *Let R be a ring and let I be an ideal of R. If $I^2 = I$, then r_I preserves epimorphisms and $r_I(M) = IM$ for every left R-module M.*

Proof. Define S by the rule $S: M \to IM$ for each left R-module M. It is easy to see that S is an idempotent radical in R-Mod and $T(S) = T(r_I)$. Hence $S = r_I$. It is clear that S preserves epimorphisms.

Lemma 2. *Suppose R is a ring and* $\mathrm{Ir}(1, R)$ *is Boolean. Then*:

I. *Every idempotent ideal of R has the form ReR, where e is an idempotent of R which is central modulo $J(R)$.*

II. *Card* $\{I \mid I$ *is an idempotent ideal of* $R\} = \mathrm{Card}(\mathrm{Ir}(1, R))$.

III. *For every left ideal S of R there exists an idempotent ideal I of R such that $r_S = r_I$.*

IV. *Every I-radical in R-*Mod *preserves epimorphisms.*

Proof. I $\Leftrightarrow$ Corollary 1 [2].

Let $A, B \subseteq \{1, \ldots, n\}$ and let $A \neq B$. This implies $RE_A R \neq RE_B R$ (see proof of Corollary 1 [2]). Therefore $\mathrm{Card}\{RE_T R \mid T \subseteq \{1, \ldots, n\}\} = 2^n$. By Lemma 4 [2], $\mathrm{Card}\{r_I \mid I = RE_T R, T \subseteq \{1, \ldots, n\}\} = 2^n$. And now by Theorems 2, 3 [2] $\mathrm{Card}(\mathrm{Ir}(1, R)) = 2^n$. Hence $\{r_1 \mid I = RE_T R, T \subseteq \{1, \ldots, n\}\} = \mathrm{Ir}(1, R)$. Now apply Lemma 1.

Corollary 1. *Let R be a left perfect ring. Then there is a bijective correspondence*

$$r \to T(r),$$

$$F \to r^F$$

*between Jans hereditary torsions in R-*Mod *and torsion-free classes of torsion theories cogenerated by classes of simple left R-modules.*

Proof. It is clear that $\mathrm{Card}(\{r(\Omega_1) \mid \Omega_1 \subseteq \Omega_L\}) = 2^n$, where $n = \mathrm{Card}(\Omega_L)$. By Theorem 1 and by Theorem 3 [2], $\{r(\Omega_1) \mid \Omega_1 \subseteq \Omega_L\} = \mathrm{Ir}(1, R)$. By Lemmas 1, 2, we have that $\{F(r(\Omega_1)) \mid \Omega_1 \subseteq \Omega_L\} = \{\{M \mid M \in R\text{-Mod}, IM = 0\} \mid I$ is an idempotent ideal of $R\}$. Therefore $\{F(r(\Omega_1)) \mid \Omega_1 \subseteq \Omega_L\} = \{T(r) \mid r$ is a Jans hereditary torsion in R-Mod$\}$ (see [4, p. 58, 59], Theorem 3^0).

Now apply Proposition 2.3 [4].

Corollary 2. *Let R be a perfect ring. Then for every hereditary torsion r in R-*Mod

$$T(r) = F(r(\Omega_1)),$$

where $\Omega_1 = \{w \mid w \in \Omega_L, \exists M \in T(r) : M \in w\}$.

Proof. Since R is right perfect, every hereditary torsion in R-Mod is Jans semisimple.

Now apply Corollary 1.

Theorem 2. *Let R be a semilocal ring. Then every I-radical in R-*Mod *is a semisimple torsion if and only if* $R \cong M_{n_1}(T_1) \times \ldots \times M_{n_k}(T_K)$ *for some local perfect rings* $T_1, \ldots, T_K$.

Proof. ($\Rightarrow$) Suppose that every I-radical in R-Mod is a semisimple torsion. By Corollary 1 [1], $J(R)$ is right T-nilpotent. Thus, R is left perfect. By Theorem 3 [2] and by Lemma 2, every semisimple torsion in R-Mod is exact. Therefore R is a left semiartinian ring (see Theorem 3.1 [6], Corollary 3.2 [6]). Then every hereditary torsion in R-Mod is semisimple. Thus, every hereditary torsion in R-Mod is exact. By Theorems 3.1, 3.3 [6], $R \cong M_{n_1}(T_1) \times \ldots \times M_{n_k}(T_K)$ for some local right perfect rings $T_1, \ldots, T_K$. But R is left perfect. Whence $T_1, \ldots, T_K$ are perfect rings.

($\Leftarrow$). Now apply Corollary 2 [1], Proposition 8.16 [4] and Proposition 1.9.1 [7].

Theorem 3. *Let R be a ring. If every hereditary torsion in R-*Mod *is an I-radical, the* $\dot{R}$ *is left semiartinian.*

Proof. Suppose that every hereditary torsion in R-Mod is an I-radical. Then $\overline{S\Omega_L}$ is an I-radical. Whence $\overline{S\Omega_L} = r_S$ for some ideal S of R. It is clear that for every maximal left ideal m of R $R/m \in T(\overline{S\Omega_L}) = T(r_S)$. It follows from this that $R/m = S(R/m) = (S + m)/m$. Therefore $S + m = R$ for every maximal left ideal m of R. Since every proper ideal is contained in a maximal left ideal of R, $S = R$. Thus $\overline{S\Omega_L}(M) = M$ for every left R-module M. By Proposition 8.15 [4], R is left semiartinian.

Proposition 1. *Let T be a left semiartinian ring. Then* $\mathrm{Ir}(1, T)$ *is Boolean if and only if T is left-right perfect.*

Proof. ($\Rightarrow$) Suppose that $\mathrm{Ir}(1, T)$ is Boolean. By Theorem 2 [2], $J(T)$ is right T-nilpotent. Since T is left semiartinian, $J(T)$ is left T-nilpotent. In view of Theorem 2 [2], $\mathrm{soc}({}_R R)$ is a ring direct summand of $R = T/J(T)$. By Proposition 2.8 [3, p. 184], R is left semiartinian. But $\mathrm{soc}({}_R R)$ is a ring direct summand of R. Therefore $R = \mathrm{soc}({}_R R)$.

($\Leftarrow$) This is clear.

Set

$$\mathrm{tor}(1, R) := \{r \mid r \text{ is a hereditary torsion in } R\text{-Mod}\},$$

$$\mathrm{tor}(r, R) := \{r \mid r \text{ is a hereditary torsion in } \text{Mod-}R\},$$

$$\mathrm{Ir}(r, R) := \{r \mid r \text{ is an } I\text{-radical Mod-}R\}.$$

Theorem 4. *Let R be a ring. Then the following are equivalent*:

1) $\mathrm{tor}(1, R) = \mathrm{Ir}(1, R)$;

2) $\mathrm{tor}(r, R) = \mathrm{Ir}(r, R)$;

3 $R \cong M_{n_1}(T_1) \times \ldots \times M_{n_k}(T_K)$

for some local left-right perfect rings $T_1, \ldots, T_K$.

Proof. 1) $\Rightarrow$ 3). Assume 1). By Theorem 3, R is left semiartinian. Therefore $\mathrm{tor}(1, R)$ is Boolean (see Proposition 8.15 [4]). Since $\mathrm{tor}(1, R) = \mathrm{Ir}(1, R)$, $\mathrm{Ir}(1, R)$ is Boolean. By Proposition 1, R is perfect. Now apply Theorem 2.

3) $\Rightarrow$ 1). Apply Corollary 2 [1], Proposition 8.16 [4], Proposition 1.9.1 [7].

***Corollary* 3.** *Let R be a ring. Then the following are equivalent*:

1) *R is semilocal and every I-radical in R-*Mod (Mod-R) *is a semisimple torsion;*

2) $\mathrm{tor}(1, R) = \mathrm{Ir}(1, R)$ ($\mathrm{tor}(r, R) = \mathrm{Ir}(r, R)$);

3) $R \cong M_{n_1}(T_1) \times \ldots \times M_{n_k}(T_K)$

for some local left-right perfect rings $T_1, \ldots, T_K$.

Let R and S be a pair of equivalent rings. Specifically, assume that there are covariant functors

$$F: R\text{-Mod} \to S\text{-Mod}, \tag{1}$$

$$G: S\text{-Mod} \to R\text{-Mod} \tag{2}$$

such that there exist natural isomorphisms

$$f: FG \to 1_{S\text{-Mod}}, \tag{3}$$

$$g: GF \to 1_{R\text{-Mod}}. \tag{4}$$

Let $PR(1, R)$ denote the class of preradicals of R-Mod and let $p \in PR(1, R)$, $t \in PR(1, S)$. Set

$$_S p(M) = \mathrm{Im}\, f_M F(i_{M,p}) \quad \text{for every left-module } M, \tag{5}$$

$$_R t(N) = \mathrm{Im}\, g_N G(i_{N,t}) \quad \text{for every left-module } N, \tag{6}$$

where

$$i_{M,p}: p(G(M)) \subseteq G(M),$$

$$\mu_{N,t}: t(F(N)) \subseteq F(N).$$

It is well known that $_S p \in PR(1, S)$, $_R t \in PR(1, R)$, $_R(_S p) = p$, $_S(_R t) = t$ (see [7]).

Lemma 3. *Let R, S be a pair of equivalent rings (see (1) – (4)) and let* $p \in \mathrm{Ir}(1, R)$, $t \in \mathrm{Ir}(1, S)$. *Then* $_S p \in \mathrm{Ir}(1, S)$, $_R t \in \mathrm{Ir}(1, R)$ *(see (5), (6)).*

Proof. It suffices to show that $_S p \in \mathrm{Ir}(1, S)$. Let I be an ideal of R such that $p = r_I$. Set $d(M) = IM$ for every left R-module M. It is easy to see that d is a radical of R-Mod. It is clear that d preserves epimorphisms. We shall prove that ${}_S d$ also preserves epimorphisms.

Let

$$\varphi : M \to N \to 0 \tag{7}$$

be an exact sequence in S-Mod. It is well known, that G is an exact functor (see Proposition 21.4 [8]).

Then the sequence

$$G(\varphi) : G(M) \to G(N) \to 0 \tag{8}$$

is exact, because (7) is exact. Since d preserves epimorphisms, the sequence

$$d(G(\varphi)) : d(G(M)) \to d(G(N)) \to 0 \tag{9}$$

is exact.

Since f is a natural isomorphism,

$$\varphi f_M = f_N F G(\varphi), \tag{10}$$

d is a preradical. Hence

$$G(\varphi) i_{M,d} = i_{N,d} d(G(\varphi)). \tag{11}$$

It follows from (10), (11) that $\varphi f_M F(i_{M,d}) = (\varphi f_M) F(i_{M,d}) = (f_N F G(\varphi)) F(i_{M,d}) = f_N(FG(\varphi))F(i_{M,d})) = f_N F(G(\varphi)) i_{M,d}) = f_N F(i_{N,d} d(G(\varphi))) = f_N F(i_{N,d}) \times F(d(G(\varphi)))$. But F is exact (see Proposition 21.4 [8]) and (9) is an exact sequence. Thus $F(d(G(\varphi)))$ is an epimorphism in S-Mod. It follows from this that

$$\operatorname{Im} f_N F(i_{N,d}) = \operatorname{Im} f_N F(i_{N,d}) F(d(G(\varphi))).$$

Therefore $\operatorname{Im} \varphi f_M F(i_{M,d}) = \operatorname{Im} f_N F(i_{N,d})$. But $\operatorname{Im} f_N F(i_{N,d}) = {}_S d(N)$, $\operatorname{Im} \varphi f_M F(i_{M,d}) = \varphi(\operatorname{Im} f_M F(i_{M,d})) = \varphi({}_S d(M))$. Then we have $\varphi({}_S d(M)) = {}_S d(N)$. Thus ${}_S d(\varphi) : {}_S d(M) \to {}_S d(N)$ is an epimorphism in S-Mod. Then ${}_S d(M) = {}_S d(S) M$ for every left S-module M (see [7, p. 21]). Thus $T({}_S d) = \{ M \mid M \in S\text{-Mod}, {}_S d(S) M = M \} = T(r_V)$, where r_V is an V-radical in S-Mod, $V = {}_S d(S)$. But $T({}_S d) = T(\widehat{({}_S d)})$ (see [3, p. 137]). Since ${}_S d$ is a radical, $\widehat{({}_S d)}$ is also a radical (see [3, p. 138]). Thus $\widehat{({}_S d)}$ is an idempotent radical (see [3, p. 137]). But $T(\widehat{({}_S d)}) = T(r_V)$ and r_V is an idempotent radical. Therefore $\widehat{({}_S d)} = r_V$. But $\widehat{({}_S d)} = {}_S \widehat{(d)}$ (see [7, p. 50]). It is clear that ${}_S p = {}_S \widehat{(d)}$. Thus ${}_S p = {}_S \widehat{(d)} = \widehat{({}_S d)} = r_V$. It follows from this that ${}_S p$ is an ${}_S d(S)$-radical in S-Mod.

Theorem 5. *Let R and S be a pair of equivalent rings. Then the lattices* $\operatorname{Ir}(1, R)$, $\operatorname{Ir}(1, S)$ *are isomorphic.*

Proof. See Lemma 3, Proposition 1.9.2 [7].

5. *Горбачук Е. Л., Комарницкий Н. Я.* I-радикалы и их свойства // Укр. мат. журн. – 1978. – **30**, № 2. – С. 212 – 217.
2. *Maturin Yu.* When is the lattice of I-radicals Boolean? // Third Int. Algebraic Conf. in Ukraine. – Sumy (Ukraine), 2001. – P. 71 – 72.
3. *Stenström Bo.* Rings of quontiens. – Berlin: Springer, 1975. – 309 p.
4. *Кашу А. И.* Радикалы и кручения в модулях. – Кишинев: Штиинца, 1983. – 156 с.
5. *Maturin Yu.* On semisimple torsion // Вісн. Київ. ун-ту. Сер. фіз.-мат. – 2000. – **1**. – С. 42 – 46.
6. *Teply M. L.* Homological dimension and splitting torsion theories // Pacif. J. Math. – 1970. – **34**. – P. 193 – 205.
7. *Bican L., Kepka T., Nemec P.* Rings, modules and preradicals // Lect. Notes. Appl. Math. – New York: Marcell Dekker, 1982. – **75**. – 241 p.
8. *Anderson F. W., Fuller K. R.* Rings and categories of modules. – Berlin: Springer, 1973. – 340 p.

Received 06.02.2002