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PROPERLY POSED AND REGULAR NONLOCAL BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

КОРЕКТНІ ТА РЕГУЛЯРНІ НЕЛОКАЛЬНІ ГРАНИЧНІ ЗАДАЧІ ДЛЯ РІВНЯНЬ З ЧАСТИННИМИ ПОХІДНИМИ

The present paper deals with the proper posedness and regularity of a class of 1D time dependent boundary-value problems with global boundary conditions through all time interval. The conditions of the proper posedness of boundary-value problems for partial differential equations is established in the class of bounded differentiable functions. The criterion of the regularity of the problem under consideration is also established.

Розглядаються питання коректності та регулярності для одного класу нестаціонарних граничних задач із однією просторовою змінною та глобальними граничними умовами за часом. Встановлено умови коректності задачі в класі обмежених диференційовних функцій, а також умови її регулярності.

Introduction. By investigating the real processes and phenomena, governed by differential equations springs up, not only the necessity in the construction of the solutions, but also the necessity in the study of different properties of these solutions. In the case where the desired solutions are constructed, it is easy to study its properties. Unfortunately such cases are very scare. Therefore it is necessary to establish the properties of the solutions of the differential problems by the indirect methods, with respect to properties of the differential equations. One of these outlines is the proper posedness of the problems.

The problems of the proper posedness of differential problems in the mathematical simulation of real objects, phenomena, and processes are the essential and the very important part of the problems of description of real objects in the mathematics' means.

The present papers deal with the regularity and proper posedness of a class of 1D time dependent of boundary-value problems with global boundary conditions through the entire time interval.

1. Formulation of problems. Consider in the stripe $\Pi = \mathbf{R} \times [0, Y]$ the following nonlocal boundary-value problem for partial differential equations:

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} - P\left(\frac{\partial}{\partial x}\right)\right]u(x, y) = 0, \qquad (x, y) \in \Pi, \tag{1}$$

$$\begin{aligned} &\alpha_1 u(x, 0) + \alpha_2 u_y(x, 0) + \alpha_3 u(x, Y) + \alpha_4 u_y(x, Y) = u_0(x), \\ &\beta_1 u(x, 0) + \beta_2 u_y(x, 0) + \beta_3 u(x, Y) + \beta_4 u_y(x, Y) = u_1(x). \end{aligned} \tag{2}$$

Here $P(s)$ is an arbitrary polynomial with constant (complex-valued) coefficients, α_i, $\beta_i \in \mathbf{C}$, $i = 1, 2, 3, 4$, $u_j(x)$: $\mathbf{R} \to \mathbf{C}$, $j = 0, 1$, — two given functions, and $u(x, y)$: $y \to \mathbf{C}$ — the unknown.

The uniqueness of solutions of problem (1), (2) is investigated in [1]. The present work investigates the proper posedness and the regularity of problem (1), (2) under the condition that

$$\operatorname{rang}\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} = 2.$$

The conditions of the proper posedness of problem (1), (2) are establish in Section 2. Section 3 investigates the regularity of the problem under consideration.

First of all, let us introduce the following notations and definitions

$$H_m = \left\{ \varphi \in C^m(\mathbf{R}) : \|\varphi\|_m = \max_{0 \le j \le m} \sup_{\mathbf{R}} \left| \varphi^{(j)}(x) \right| < +\infty \right\}.$$

$\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the real part and the imaginary part of the complex number z; $P_1(\sigma) = \mathrm{Re}(YP(\sigma))$; $P_2(\sigma) = \mathrm{Im}(YP(i\sigma))$; $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $A_{ij} = \alpha_i \beta_j - \beta_j \alpha_i$, $1 \le i < j \le 4$; $\rho(A, B)$ — the distance between the sets A and B;

$$N_f = \{x \in \mathbf{R} : f(x) = 0\}, \qquad N[f] = \{z \in \mathbf{C} : f(z) = 0\},$$

$$f(z) \equiv -A_{13} + \frac{A_{12} - A_{23}}{Y} z + \left[A_{13} + \frac{A_{14} + A_{34}}{Y} z \right] e^z, \qquad z \in \mathbf{C},$$

$$\Delta(\sigma) \equiv f(YP(i\sigma)),$$

$$R_0(\sigma, y) \equiv \frac{\beta_1 + \beta_2 P(i\sigma) + [\beta_3 + \beta_4 P(i\sigma)] e^{YP(i\sigma)} - [\beta_1 + \beta_3] e^{YP(i\sigma)}}{\Delta(\sigma)},$$

$$R_1(\sigma, y) \equiv \frac{\alpha_1 + \alpha_2 P(i\sigma) + [\alpha_3 + \alpha_4 P(i\sigma)] e^{YP(i\sigma)} - [\alpha_1 + \alpha_3] e^{YP(i\sigma)}}{\Delta(\sigma)},$$

$$\delta(z) \equiv Az - C + (C + Bz)e^z, \qquad A, B, C \in \mathbf{C}.$$

According to G. I. Petrovskii [2], we introduce the following definition.

Definition 1. *Problem* (1), (2) *is said to be properly posed in* Π *if for every* $m \in \mathbf{N}_0$ *there exists* p *and* q *in* $\mathbf{N}_0$ *so that to every boundary functions* $u_0(x) \in H_p$ *and* $u_1(x) \in H_q$ *there corresponds a unique solution* $u_1(x, y) \in H_m$ $(\forall y \in [0, Y])$ *of problem* (1), (2) *satisfying the condition that*

$$\max_{[0, Y]} \|u(x, y)\|_m \le C_0 \|u_0(x)\|_p + C_1 \|u_1(x)\|_q, \qquad C_0 > 0, \qquad C_1 > 0.$$

Definition 2. *We say that problem* (1), (2) *is regular if the uniqueness of its solutions in the class of bounded functions in* Π *implies its proper posedness in the said class.*

2. The proper posedness of problem (1), (2).

Theorem 1. *For the proper posedness of problem in* Π, *it is sufficient that* $\Delta(\sigma) \ne 0$ *(for every real* σ*).*

Lemma 1. *If* $A \times B \ne 0$ *the large zeros,* z_k *of* $\delta(z)$ *take the form*

$$z_k = \mathrm{Ln}\left(-\frac{A}{B}\right) + \sum_{j=1}^{\infty} \zeta_j k^{-j}, \qquad \zeta_j \in \mathbf{C}, \qquad j \in \mathbf{N}.$$

Proof. Because $\tilde{\delta}(z) = z(A + Be^z)$ is the congruence function [3] for $\delta(z)$, we have $\lim_{k \to +\infty} |z_k - \tilde{z}_k| = 0$, where $\{z_k\}_{k=1}^{+\infty}$ and $\{\tilde{z}_k\}_{k=1}^{+\infty}$ are two sequences of zeros of $\delta(z)$ and $\delta(\tilde{z})$ respectively. Because $\tilde{z}_k = \ln|A/B| + i(\gamma_0 + 2 \times k \times \pi) + o(1)$, $|k| \to +\infty$, $\gamma_0 = \arg(-A/B)$, we have $z_k = \ln|A/B| + i(\gamma_0 + 2 \times k \times \pi) + \mu_k$, $\mu_k \to 0$, $|k| \to +\infty$. If we insert z_k in $\delta(z)$ we obtain

$$0 = \delta(z_k) = \delta(\chi, \mu) \Big|_{\substack{\chi = k \\ \mu = \mu_k}},$$

where

$$\delta(\chi, \mu) = -C + A\left[\left\{ \ln\left|\frac{A}{B}\right| + i(\gamma_0 + 2 \times \chi \times \pi) + \mu \right\}(1 - e^{\mu}) - \frac{C}{B} e^{\mu} \right].$$

Let $\delta_1(\chi, \mu)$ be a function defined as follows

$$\delta_1 \equiv \chi \delta(1/\chi, \mu).$$

Then $\delta_1(0,0) = 0$ and $\partial\delta_1(0,0)/\partial\mu = -2 \times \pi \times i \times A \neq 0$. This means that equation $\delta_1(\chi, \mu) = 0$ defines μ as an implicit function of χ:

$$\mu = \mu(\chi) = \sum_{j=1}^{+\infty} \zeta_j \chi^j, \quad \zeta_j \in \mathbf{C}, \quad j \in \mathbf{N}.$$

Therefore

$$\mu_k = \sum_{j=1}^{+\infty} \zeta_j k^{-j}, \quad \zeta_j \in \mathbf{C}, \quad j \in \mathbf{N}.$$

Consequently

$$z_k = \ln\left|\frac{A}{B}\right| + i(\gamma_0 + 2 \times k \times \pi) + \mu_k = \mathrm{Ln}\left(-\frac{A}{B}\right) + \sum_{j=1}^{\infty} \zeta_j k^{-j},$$

and this completes the proof of Lemma 1.

Lemma 2. *If the condition of Theorem 1 is satisfied then for some* $M > 0$ *and* $\mu \in \mathbf{R}$, *the following inequality*

$$|\Delta(\sigma)| \geq M(1 + |\sigma|)^{\mu} \tag{3}$$

holds for every σ *in* $\mathbf{R}$.

Proof. The establishment of the formula (3) is obvious if $P(i\sigma) \equiv \mathrm{const}$.

If $A_{12} - A_{23} = A_{14} + A_{34} = 0$ then $A_{13} \neq 0$ and (3) follows from [4].

The estimate (3) will follow from [5] if $A_{13} = 0$ and $(A_{12} - A_{23})(A_{14} + A_{34}) \neq 0$.

Let us examine the case where $P(i\sigma) \neq \mathrm{const}$ and $A_{13}(|A_{12} - A_{23}| + |A_{14} + A_{34}|) \neq 0$. Here we distinguish the following three cases:

1) $(A_{12} - A_{23})(A_{14} + A_{34}) \neq 0$;

2) $A_{12} - A_{23} = 0$, $A_{14} + A_{34} \neq 0$;

3) $A_{12} - A_{23} \neq 0$, $A_{14} + A_{34} = 0$.

Let $\wp = \{YP(i\sigma) : \sigma \in \mathbf{R}\}$. It follows from the Weierstrass' theorem that to every $\delta_0 > 0$ there corresponds $M_0 > 0$ so that $|\Delta(\sigma)| \geq M_0$ for every $\sigma \in \mathbf{R}$ satisfying the condition that $|\sigma| \leq \delta_0$. From here we conclude that in order to establish the required estimate (3), it is sufficient to establish it for large scale values of σ so that $|\sigma| \geq \delta_0$.

***Case* 1.** $(A_{12} - A_{23})(A_{14} + A_{34}) \neq 0$.

If $P_1(\sigma) \neq \ln\left|\frac{A_{12} - A_{23}}{A_{14} + A_{34}}\right|$ then the inequality

$$\left|P_1(\sigma) - \ln\left|\frac{A_{12} - A_{23}}{A_{14} + A_{34}}\right|\right| \geq C_0, \quad |\sigma| \geq \delta_1 > 0,$$

occurs for some $C_0 > 0$, $\delta_1 > 0$. From the condition $N_\Delta = \varnothing$ we have $\wp \cap N[f] = \varnothing$, from where we obtain the inequality $\rho(\wp, N[f]) \geq C_1 > 0$. Hence [3]

$$|\Delta(\sigma)| = |f(YP(i\sigma))| \geq M > 0.$$

Let us consider the case where $P_1(\sigma) \equiv \ln\left|\dfrac{A_{12}-A_{23}}{A_{14}+A_{34}}\right|$ and let $\sigma \in \mathbf{R}$ is foxed. Then

$$\rho(yP(i\sigma), N[f]) = |yP(i\sigma) - z_k|, \tag{4}$$

where $k = k(\sigma)$, $z_{k(\sigma)} \in N[f]$. If $|YP(i\sigma) - z_{k(\sigma)}| \geq C_1$ then $|\Delta(\sigma)| \geq M > 0$. Let us examine the case where $\lim_{|\sigma| \to +\infty} |YP(i\sigma) - z_{k(\sigma)}| = 0$. It follows from Lemma 1 that $2 \times k(\sigma) \times \pi + \arg(-A/B) - P_2(\sigma) = o(1)$, $|\sigma| \to +\infty$, from where we have

$$k(\sigma) = C_2 \sigma^p (1 + O(1)), \quad |\sigma| \to +\infty, \tag{5}$$

where $p = \deg P_2(\sigma)$ (> 0). By virtue of Lemma 1 we obtain

$$|YP(i\sigma) - z_{k(\sigma)}| \geq |P_1(\sigma) - \mathrm{Re}(z_{k(\sigma)})| = \left|\sum_{j=1}^{+\infty} \mathrm{Re}(\zeta_j) k^{-j}\right|. \tag{6}$$

If $\mathrm{Re}(\zeta_j) = 0$ $(\forall j \in \mathbf{N})$ then

$$z_{k(\sigma)} = \ln\left|\frac{A_{12}-A_{23}}{A_{14}+A_{34}}\right| + i[2 \times k \times \pi + \arg(-A/B) + \varepsilon(\sigma)]$$

where $\varepsilon(\sigma) = o(1)$, $|\sigma| \to +\infty$. Consequently $YP(i\sigma') = z_{k(\sigma')}$ for some $\sigma' \in \mathbf{R}$, and this contradicts the condition that $\wp \cap N[f] = \varnothing$.

Let $\mathrm{Re}(\zeta_j) = 0$ for $1 \leq j \leq j_0 - 1$ and $\mathrm{Re}(\zeta_{j_0}) \neq 0$. Then for large-scale values of $|k|$ we have

$$\left|\sum_{j=1}^{+\infty} \zeta_j k^{-j}\right| \geq \frac{|\mathrm{Re}(\zeta_{j_0})|}{2} k^{-j_0}.$$

From the formulae (4), (5), and (6) we obtain the following inequality

$$\rho(YP(i\sigma), N[f]) \geq C_3 \times (1 + |\sigma|)^{-pj_0}, \quad C_3 > 0. \tag{7}$$

Now we can evaluate $|\Delta(\sigma)|$ in the examine case, that is, in the case where $\lim_{|\sigma| \to +\infty} |YP(i\sigma) - z_{k(\sigma)}| = 0$:

$$|\Delta(\sigma)| = |f(z)|_{z=YP(i\sigma)} = \left|f(z) - f(z_{k(\sigma)})\right|_{z=YP(i\sigma)} =$$

$$= |z - z_k| \left|\frac{A_{12}-A_{23}}{Y} + A_{13}\frac{(e^{z-z_k}-1)e^{z_k}}{z-z_k} + \frac{A_{14}+A_{34}}{Y}\left(1 + \frac{e^{z-z_k}-1}{z-z_k} z\right) e^{z_k}\right|_{\substack{z=YP(i\sigma) \\ k=k(\sigma)}},$$

and we have

$$\lim_{|\sigma| \to +\infty} e^{z_{k(\sigma)}} = -\frac{A_{12}-A_{23}}{A_{14}+A_{34}} \quad \text{and} \quad \lim_{|\sigma| \to +\infty} \frac{e^{z-z_{k(\sigma)}}-1}{z-z_{k(\sigma)}} = 1.$$

Hence

$$|\Delta(\sigma)| \geq \frac{1}{2}\left|\frac{A_{13}(A_{12}-A_{23})}{A_{14}+A_{34}}\right| |z - z_k|_{\substack{z=YP(i\sigma) \\ k=k(\sigma)}}.$$

By using (7) we obtain the required result.

***Case* 2.** Let $A_{12} - A_{23} = 0$, $A_{14} + A_{34} \neq 0$.

$f(z)$ in the present case reads

$$f(z) = -A_{13} + \left[\frac{A_{12} + A_{34}}{Y} z + A_{13}\right] e^{z},$$

and [3],

$$N[f] =$$

$$= \left\{ -z_k = x_k + iy_k : x_k = \ln\left|\frac{YA_{13}}{A_{14} + A_{34}}\right| - \ln\left|2 \times k \times \pi + \arg\left(\frac{YA_{13}}{A_{14} + A_{34}}\right) \mp \frac{\pi}{2}\right| + o(1), \right.$$

$$\left. y_k = 2 \times k \times \pi + \arg\left(\frac{YA_{13}}{A_{14} + A_{34}}\right) \mp \frac{\pi}{2} + o(1),\ |k| \to +\infty \right\},$$

where the signs "+" and "–" are used if $k \to -\infty$ and $k \to +\infty$ respectively. Thus the large zeros z_k of $f(z)$ on the plane $\mathbf{C} = \{z = x + iy\}$ take the form

$$z_k = x_k + iyk = -\ln|k|(1 + o(1)) + 2i \times k \times \pi(1 + o(1)), \qquad |k| \to +\infty.$$

That is the large scale zeros z_k of $f(z)$ are asymptotically close to the curve $x = -\ln|y/2\pi|$ (as $|k| \to +\infty$), while the values of the polynomial $YP(i\sigma) = P_1(\sigma) + P_2(\sigma)$ (under the condition that $P(i\sigma) \neq \text{const}$) are very close to the curve $y = A_0 x^{\alpha_0}$ $(A_0, \alpha_0 \in \mathbf{R})$ (for large scale values of $|\sigma|$). Hence $\rho(\wp, N[f]) \geq C_0' > 0$, from where we have $|\Delta(\sigma)| \geq M > 0$.

***Case* 3.** Let us consider that $A_{12} - A_{23} \neq 0$, $A_{14} + A_{34} = 0$. Then

$$f(z) = \frac{A_{12} - A_{23}}{Y} z + A_{13}(e^{z} - 1).$$

Let us denote by

$$\tilde{f}(z) = \frac{A_{12} - A_{23}}{Y} z + A_{13} e^{z}$$

the congruence function for $f(z)$. Then the large scale zeros z_k of $f(z)$ [3] are very close to the zeros $\tilde{z}_k$ of $\tilde{f}(z)$ (as $|k| \to +\infty$). That is $z_k - \tilde{z}_k = o(1)$, $|k| \to +\infty$.

On the other hand, the zeros $\tilde{z}_k = \xi_k + i\eta_k$ of $\tilde{f}(z)$ are solutions of the following system

$$\begin{aligned} e^{\xi}\cos\eta &= a\xi - b\eta, \\ e^{\xi}\sin\eta &= a\eta + b\xi \end{aligned} \tag{8}$$

where

$$a = \operatorname{Re}\left(\frac{A_{23} - A_{12}}{YA_{13}}\right), \qquad b = \operatorname{Re}\left(\frac{A_{23} - A_{12}}{YA_{13}}\right).$$

Consequently

$$\sin(\eta - \varphi_0) = \xi e^{-\xi}, \tag{9}$$

where

$$\sin\varphi_0 = \frac{-a}{|a + ib|^2}, \qquad \cos\varphi_0 = \frac{b}{|a + ib|^2}.$$

It is seen from (9) that for the large scale values of $k > 0$, system (8) will not have any solution (ξ, η) satisfying the condition that $\xi < -k$. It is obvious that the said system (8) is unsolvable in the stripe $|\xi| \le k$ for which the quantity $|\xi + i\eta|$ is very large, and we are interesting ourselves only in the solutions of (8), that coincide with the zeros of $\tilde{f}(z)$. Because we are interesting ourselves only in the large scale zeros of $f(z)$, we may take into account only the solutions (ξ, η) of system (8) for which $\xi \to \to +\infty$. Under this consideration (9) gives

$$\eta = \eta_k = \varphi_0 + k \times \pi + o(1), \quad k = k(\xi) \in \mathbf{Z}, \quad \xi \to +\infty. \tag{10}$$

If $a \ne 0$ the second equation of (8) gives

$$\xi = \xi_k = \ln|k|(1 + o(1)), \quad |k| \to +\infty. \tag{11}$$

A comparison of (10) and (11) shows that for large scale values of $|\sigma|$, $\rho(\wp, N[f]) \ge C_0 > 0$, from where we obtain $|\Delta(\sigma)| \ge M > 0$.

If $a = 0$ the second equation (8) gives

$$\eta = \eta_k = k \times \pi + o(1), \quad |k| \to +\infty, \tag{10'}$$

and from the first equation of the said system we obtain

$$\xi = \xi_k = \ln|k|(1 + o(1)), \quad |k| \to +\infty. \tag{11'}$$

Formulae (10′) and (11′) give the required result, and Lemma 2 is completely proven.

Lemma 3. *If condition* (3) *is satisfied then to every* $j \in \mathbf{N}_0$ *there correspond* $m_i^j \in \mathbf{R}$ *and* $C_i^j > 0$, $i = 0, 1$, *so that* $\forall y \in [0, Y]$, $\forall \sigma \in \mathbf{R}$,

$$\left| \frac{\partial^j R_i(\sigma, y)}{\partial \sigma^j} \right| \le C_i^j (1 + |\sigma|)^{m_i^j}. \tag{12}$$

Proof. In order to prove (12), it is sufficient to establish the following result

$$\left| \frac{\partial^j R(\sigma, y)}{\partial \sigma^j} \right| \le C_j (1 + |\sigma|)^{m_j} \qquad (\forall \sigma \in \mathbf{R}, \quad \forall y \in [0, Y]) \tag{12'}$$

where $R(\sigma, y) = Q(i\sigma) e^{yP(i\sigma)} / \Delta(\sigma)$, and $Q(i\sigma)$ — an arbitrary polynomial with constant (complex-valued) coefficients.

It is easy to verify by induction that

$$\frac{\partial^j R(\sigma, y)}{\partial \sigma^j} = (\Delta(\sigma))^{-1-j} \sum_{k=0}^{j} H_{kj}(\sigma, y) e^{(y+kY)P(i\sigma)} \tag{13}$$

where $H_{kj}(\sigma, y)$ are polynomials with respect to σ and y.

If $\operatorname{Re}(P(i\sigma)) \le 0$ then estimate (12′) follows from (13) and Lemma 2.

If $\operatorname{Re}(P(i\sigma)) \ge 0$ then by rewriting (13) in the form

$$\frac{\partial^j R(\sigma, y)}{\partial \sigma^j} = \left(\Delta^1(\sigma)\right)^{-1-j} \sum_{k=0}^{j} H_{kj}(\sigma, y) e^{[y+(k-j-1)Y]P(i\sigma)}$$

(with $\Delta^1(\sigma) = \Delta(\sigma) e^{-YP(i\sigma)}$), and by applying Lemma 2 on $\Delta^1(\sigma)$, we obtain the required result, and this completes the proof of Lemma 3.

Lemma 4. *If* $N_\Delta \ne \varnothing$ *then the uniqueness of solutions of problem* (1), (2) *will be violated in* H_m $(\forall m \in \mathbf{N}_0)$.

Proof. It is clear that problem (1), (2) will have one and only one solution in H_m if and only if the only solution in the said space of the corresponding homogeneous problem (1), (2) $(u_0(x) \equiv u_1(x) \equiv 0)$ is the trivial function $u(x, y) \equiv 0$.

Let $\sigma_0 \in N_\Delta$ and let (A, B) be a nontrivial solution of the following homogeneous system

$$(\alpha_1+\alpha_3)A + \left[\alpha_1+\alpha_2 P(i\sigma_0)+(\alpha_3+\alpha_4 P(i\sigma_0))e^{YP(i\sigma_0)}\right]B = 0,$$

$$(\beta_1+\beta_3)A + \left[\beta_1+\beta_2 P(i\sigma_0)+(\beta_3+\beta_4 P(i\sigma_0))e^{YP(i\sigma_0)}\right]B = 0$$

whose determinant coincides with $\Delta(\sigma_0)=0$. It is easily seen that

$$u(x,y) = \left(A+Be^{yP(\sigma_0)}\right)e^{ix\sigma_0}$$

is a nontrivial solution of the homogeneous problem (1), (2) $(u_0(x)\equiv u_1(x)\equiv 0)$. Moreover $\forall j\in \mathbf{N}_0$, $\exists C_j>0$ such that $\left|\partial^j u(x,y)/\partial x^j\right| \le C_p$ $(\forall (x,y)\in \Pi)$ and this completes the proof of Lemma 4.

Proof of Theorem **1.** If the solution $u(x,y)$ of problem (1), (2) and all its derivatives appearing in equation (1), and also the boundary functions $u_0(x)$ and $u_1(x)$ are absolutely integrable, and $\nu(\sigma,y)$, $\nu_0(\sigma)$, and $\nu_1(\sigma)$ are their Fourier transform respectively, then it is easily seen that $\nu(\sigma,y)$ will be solution of the following nonlocal boundary-value problem for the ordinary differential equations with the parameter $\sigma\in\mathbf{R}$:

$$\frac{d^2\nu(\sigma,y)}{dy^2} - P(i\sigma)\frac{d\nu(\sigma,y)}{dy} = 0, \tag{1'}$$

$$\begin{aligned}\alpha_1\nu(\sigma,0)+\alpha_2\nu'_y(\sigma,0)+\alpha_3\nu(\sigma,Y)+\alpha_4\nu'_y(\sigma,Y) &= \nu_0(\sigma),\\ \beta_1\nu(\sigma,0)+\beta_2\nu'_y(\sigma,0)+\beta_3\nu(\sigma,Y)+\beta_4\nu'_y(\sigma,Y) &= \nu_1(\sigma).\end{aligned} \tag{2'}$$

Therefore

$$\nu(\sigma,y) = R_0(\sigma,y)\nu_0(\sigma) + R_1(\sigma,y)\nu_1(\sigma). \tag{14}$$

On the basis of the formula (14) and the estimate (12) we obtain [2, 6 – 10] the proper posedness of problem (1), (2) in the class of bounded smooth functions. Theorem 1 is completely proven.

3. Criterion of the regularity of problem (1), (2) and conclusions.

Theorem 2. *The nonlocal boundary-value problem* (1), (2) *under the condition*

$$\operatorname{rang}\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\end{pmatrix} = 2 \quad \textit{is always regular.}$$

Proof. If for every polynomial $P(s)$ problem (1), (2) possesses one and only one solution in the class of bounded functions, then by applying Lemma 4, we have $N_\Delta = \varnothing$. By applying Theorem 1 we obtain that the problem under consideration is properly posed (in Π). According to Definition 2, problem (1), (2) is regular, and this completes the proof of Theorem 2.

It follows from Theorems 1 and 2 that the algebraic properties of the polynomial $P(s)$ do not affect the proper posedness of nonlocal problem (1), (2) in the class of bounded smooth functions, and therefore, do not affect the regularity of this problem. Contrary to other nonlocal problems for partial differential equations [5, 6, 10], for every polynomial $P(s)$, there exists at least a properly posed and regular problem (1), (2). In other words, for every polynomial $P(s)$, one can indicate the boundary conditions (2) for which

$$\operatorname{rang}\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} = 2$$

and $N_\Delta = \varnothing$.

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