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I-RADICALS AND RIGHT PERFECT RINGS

I-РАДИКАЛИ ТА ДОСКОНАЛІ СПРАВА КІЛЬЦЯ

We determine the rings for which every hereditary torsion theory is an S-torsion theory in the sense of Komarnitskiy. We show that such rings admit a primary decomposition. Komarnitskiy obtained this result in the special case of left duo rings.

Визначено кільця, для яких кожна теорія скруту з успадкуванням є теорією S-скруту у сенсі Комарницького. Показано, що такі кільця допускають первинний розклад. Комарницький отримав цей результат у частинному випадку лівих дуо-кілець.

The concept of I-radical (defined below) was introduced by O. Horbachuk (see [1]) and further developed and applied in collaboration with Yu. Maturin [2–5]. Any ideal I of a ring R gives rise to an I-radical, and the lattice of I-radicals is always distributive [4]. It is natural to ask about the relationship between I-radicals and Gabriel topologies, that is, left exact radicals of R. In [5], it is proved that a ring R with the property

(P) Every left exact radical in R-**Mod** is an I-radical

is right perfect, while the converse does not hold in general [6]. Komarnitskiy [7] proved the converse in case of a left duo ring R (see also [2]). He showed that such rings R admit a primary decomposition [8].

In this note, we prove that a ring satisfies (P) if and only if it decomposes into finitely many quasilocal right perfect rings. This shows that the rings with property (P) coincide with the rings studied by M. Teply [9], i.e., those for which the global dimension with respect to each hereditary torsion theory is zero.

Let R be a ring (associative with 1). The category of left (right) R-modules will be denoted by R-**Mod** (resp. **Mod**-R), and $N \le M$ will indicate that N is a submodule of M. Recall that a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of R-**Mod** is said to be a *torsion theory* [8] if $\mathcal{T}$ and $\mathcal{F}$ are maximal with respect to $\operatorname{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$. The *torsion class* $\mathcal{T}$ of $(\mathcal{T}, \mathcal{F})$ is characterized by the property that it is closed with respect to extensions, direct sums, and factor modules. If, in addition, $\mathcal{T}$ is closed with respect to submodules, then $\mathcal{T}$ and $(\mathcal{T}, \mathcal{F})$ are said to be *hereditary*. If $\mathcal{T}$ is also a torsion-free class, i.e., closed with respect to products, there is another torsion theory $(\mathcal{C}, \mathcal{T})$. Then $\mathcal{T}$ is called a *TTF class* [10]. Every torsion class $\mathcal{T}$ gives rise to a *radical*, that is, an endofunctor T of R-**Mod** with $T(M/T(M)) = 0$ for all $M \in R$-**Mod**. Namely, $T(M)$ is the largest submodule $T \le$ $\le M$ with $T \in \mathcal{T}$. In this way, the (hereditary) torsion theories correspond to the idempotent (left exact) radicals [11].

For any full subcategory $\mathcal{C}$ of R-**Mod**, the torsion class $\mathcal{T}(\mathcal{C})$ *generated* by $\mathcal{C}$ is defined to be the smallest torsion class $\mathcal{T}$ with $\mathcal{C} \subset \mathcal{T}$. If $\mathcal{C}$ is closed with respect to factor modules, then $\mathcal{T}(\mathcal{C})$ consists of the modules M such that each non-zero factor module of M has a non-zero submodule in $\mathcal{C}$. If $\mathcal{C}$ is also closed with respect to submodules, then $\mathcal{T}(\mathcal{C})$ is hereditary ([11|, VI, Propositions 2.5 ana 3.3). Note that a hereditary torsion class $\mathcal{T}$ is determined by its *Gabriel filter*, i.e., the set $\mathcal{F}$ of left ideals I with $R/I \in \mathcal{T}$.

In what follows, let R-**ss** denote the full subcategory of semisimple modules in R-

Mod. An R-module is said to be *semi-artinian* if it belongs to the torsion class generated by R-**ss**. By the above remark, this torsion class is hereditary. The ring R is called *left semi-artinian* if ${}_RR$ is so. A hereditary torsion class $\mathcal{T}$ consisting of semi-artinian modules is generated by $\mathcal{T} \cap R$-**ss**, hence by just one semisimple module. We call such a torsion class *semisimple* (see also [2], with slight modification of the terminology in [11]). Recall that R is said to be *right perfect* [12] if every right R-module has a projective cover. The following proposition is essentially well-known (cf. [11], VIII, Corollary 6.3).

Proposition 1. *A ring R is right perfect if and only if every hereditary torsion class in R-**Mod** is a semisimple TTF class.*

For any left ideal I of R, consider the torsion class

$$\mathcal{T}_I := \{M \in R\text{-}\mathbf{Mod} \mid IM = M\}. \tag{1}$$

In accordance with O. Horbachuk (see [5]), who studied the radical of $\mathcal{T}_I$, we call $\mathcal{T}_I$ an *I-torsion* class. A torsion theory $(\mathcal{T}, \mathcal{F})$ with $\mathcal{T} = \mathcal{T}_I$ will be called an *I-torsion* theory. Since $IM = IRM$, we may assume, without loss of generality, that I is an ideal. Thus if $\mathcal{T}_I$ is hereditary, the corresponding Gabriel filter $\mathcal{T}_I$ is given by

$$\mathcal{T}_I = \{H \le {}_RR \mid I + H = R\}. \tag{2}$$

More generally, a hereditary torsion class with a Gabriel filter (2) for some left ideal I is called an *S-torsion* class [2, 7, 13]. Thus we obtain the following proposition.

Proposition 2. *An I-torsion class is hereditary if and only if it is an S-torsion class qiven by a* (*two-sided*) *ideal.*

Not every left ideal I of R defines an S-torsion class. However, any left ideal I defines a multiplicative submonoid $1 + I$ of R. Hence

$$\mathcal{T}^I := \{M \in R\text{-}\mathbf{Mod} \mid \forall\, x \in M \;\; \exists\, a \in I\colon (1+a)x = 0\} \tag{3}$$

is a hereditary torsion class in R-**Mod**. In fact, have the following proposition.

Proposition 3. *Every S-torsion class in R-**Mod** is of the form* (3). *Precisely, a left ideal I of R defines an S-torsion class if and only if $1 + I$ satisfies the left Ore condition.*

Proof. Let I be a left ideal of R. Then a left ideal H satisfies $R/H \in \mathcal{T}^I$ if and only if for each $a \in R$, there is an element $b \in I$ with $(1+b)a \in H$. The latter condition means that $1 + b \in (H : a)$. Therefore, H belongs to the Gabriel filter of $\mathcal{T}^I$ if and only if $I + (H : a) = R$ for all $a \in R$. Thus if I defines an S-torsion class $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}^I$. Moreover, we infer that $R/H \in \mathcal{T}^I$ implies $I + H = R$. Hence I defines an S-torsion class if and only if the reverse implication

$$I + H = R \Rightarrow R/H \in \mathcal{T}^I \tag{4}$$

holds for all left ideals H. Explicitly, condition (4) states that if $1 + a \in H$ holds for some $a \in I$, then each $x \in R$ satisfies $(1 + b)x \in H$ for some $b \in I$. In other words, for any $a \in I$ and $x \in R$, there exists an element $b \in I$ with $(1 + b)x \in$ $\in R(1 + a)$. But this is just the left Ore condition for $1 + I$.

The proposition is proved.

Let us define a *torsion sequence* to be a sequence $(\mathcal{T}_0, \ldots, \mathcal{T}_n)$ of full subcategories $\mathcal{T}_i \subset R$-**Mod** such that $(\mathcal{T}_{i-1}, \mathcal{T}_i)$ a torsion theory for $i \in \{1, \ldots, n\}$. For $n = 2$, this means that $\mathcal{T}_1$ is a TTF-class, and then $(\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2)$ is said to be a *TTF-theory*.

For any torsion class $\mathcal{T}$ with radical T, the ideal $I := T(R)$ of R satisfies

$$IM \subset T(M) \tag{5}$$

for all $M \in R$-**Mod**. The following proposition shows that TTF-theories correspond to a particular class of I-radicals.

Proposition 4. *Let $\mathcal{T}$ be a torsion class in R-**Mod** with radical T and $I := T(R)$. The following are equivalent*:

(a) *there is a TTF-theory $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ in R-**Mod***;

(b) *$\mathcal{T}$ is an I-torsion class*;

(c) $T(M) = IM$ for all $M \in R$-**Mod**.

Proof. (a) $\Rightarrow$ (b): For a given $M \in \mathcal{T}$, consider an epimorphism $p\colon F \twoheadrightarrow M$ with a free R-module F. Then $p(IF) = IM$. Hence M/IM is an epimorphic image of $F/IF \in \mathcal{F}$. Since $\mathcal{F}$ is a torsion class, $M/IM \in \mathcal{F} \cap \mathcal{T}$, and thus $M/IM = 0$. Conversely, $M = IM$ implies $M \in \mathcal{T}$ by (5).

(b) $\Rightarrow$ (c): By (5), we have $T(M) = IT(M) \subset IM \subset T(M)$.

(c) $\Rightarrow$ (a): There is a torsion theory $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{F}$ is closed with respect to factor modules. Hence $\mathcal{F}$ is a TTF class.

***Corollary* 1.** *There is a one-to-one correspondence between TTF-theories in R-**Mod** and I-torsion classes given by an idempotent ideal.*

Proof. An I-torsion class given by an ideal $I = I^2$ satisfies (c) of Proposition 4. This establishes the correspondence.

A torsion theory $(\mathcal{T}_1, \mathcal{T}_2)$ in R-**Mod** is said to be *centrally splitting* if $R = R_1 \times R_2$ such that $\mathcal{T}_i$ coincides with the full subcategory R_i-**Mod** of R-**Mod**.

Proposition 5. *For a torsion theory $(\mathcal{T}, \mathcal{F})$ in R-**Mod**, the following are equivalent*:

(a) *$(\mathcal{T}, \mathcal{F})$ is centrally splitting*;

(b) *$(\mathcal{F}, \mathcal{T})$ is a torsion theory*;

(c) *there is a torsion sequence $(\mathcal{B}, \mathcal{C}, \mathcal{T}, \mathcal{F}, \mathcal{D})$*.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. Thus let (c) be satisfied. Consider a morphism $f\colon F \to T$ with $F \in \mathcal{F}$ and $T \in \mathcal{T}$. Then $f(F) \in \mathcal{F} \cap \mathcal{T} = 0$. Hence $\mathrm{Hom}_R(\mathcal{F}, \mathcal{T}) = 0$, which gives $\mathcal{F} \subset \mathcal{C}$. Similarly, we get $\mathrm{Hom}_R(\mathcal{T}, \mathcal{C}) = 0$, which yields $\mathcal{C} \subset \mathcal{F}$. Hence $\mathcal{C} = \mathcal{F}$. Let T be the radical of $\mathcal{T}$ and F the radical of $\mathcal{F}$ with respect to the torsion theory $(\mathcal{F}, \mathcal{T})$. Then $I := T(R)$ and $J := F(R)$ are ideals of R. For any $M \in R$-**Mod**, Proposition 4 implies that $M/JM = I(M/JM) = (IM + JM)/JM$, whence $M = IM \oplus JM$. Thus (a) holds.

Now we are ready to prove our main result. Recall that a ring R with Jacobson radical $\mathrm{Rad}\, R$ is said to be *quasilocal* if $R/\mathrm{Rad}\, R$ is a simple artinian ring.

Theorem 1. *For a ring R, the following are equivalent*:

(a) *every hereditary torsion class in R-**Mod** is an I-torsion class*;

(b) *every hereditary torsion class in R-**Mod** is an S-torsion class*;

(c) $R = R_1 \times \ldots \times R_n$ *with quasilocal right perfect rings* R_i.

Proof. The implication (a) $\Rightarrow$ (b) follows by Proposition 2.

(b) $\Rightarrow$ (c): By [2], Corollary 3, the ring R is right perfect. Thus let P be an indecomposable projective left R-module, and let $\mathcal{T}$ be the torsion class generated by the simple R-module $S := P/\mathrm{Rad}\, P$. By Proposition 3, there exists a left ideal I of R with $\mathcal{T} = \mathcal{T}^I$. Moreover, Proposition 1 implies that there is a torsion theory $(\mathcal{C}, \mathcal{T})$ in R-**Mod**. So there is a smallest submodule M of P such that $P/M \in \mathcal{T}$. Choose

$e \in P \setminus \operatorname{Rad} P$. Then (3) implies that $(1 + a)e \in \operatorname{Rad} P$ for some $a \in I$. Hence $P = Re \subset Ie + \operatorname{Rad} P$, and thus $Ie = P$. Let $p: R \twoheadrightarrow P$ denote the epimorphism given by the right multiplication $p(x) := xe$. Then $p(I) = Ie = P$, whence $I + \operatorname{Ker} p = R$. By (2), we get $P \cong R/\operatorname{Ker} p \in \mathcal{T}$. On the other hand, let P' be an indecomposable projective R-module with $P'/\operatorname{Rad} P' \not\cong S$. Then $P'/\operatorname{Rad} P' \notin \mathcal{T}$, which yields $P' \in C$. Consequently, we get a decomposition ${}_R R = P^m \oplus Q$ with $P^m \in \mathcal{T}$ and $Q \in C$. Thus $\operatorname{Hom}_R(Q, P^m) = 0$. By symmetry, this implies that $\operatorname{Hom}_R(P, P') = 0$ for each indecomposable projective R-module $P' \not\cong P$. Hence $\operatorname{Hom}_R(P^m, Q) = 0$. Since $R \cong \operatorname{End}({}_R R)^{\mathrm{op}}$, we get a ring decomposition $R = R_1 \times R_1'$ with $R_1 \cong P^m$ as left R-modules. By induction, this gives the desired decomposition $R = R_1 \times \ldots \times R_n$.

(c) $\Rightarrow$ (a): Since R is right perfect, every hereditary torsion class $\mathcal{T}$ in R-**Mod** is semisimple by Proposition 1. Hence $\mathcal{T}$ defines a centrally splitting torsion theory. Now Proposition 4 completes the proof.

As a consequence, we get a relationship between left and right I-radicals (cf. [5], Theorem 5). Recall that a torsion theory $(\mathcal{T}, \mathcal{F})$ in R-**Mod** is said to be *splitting* if every $M \in R$-**Mod** admits a decomposition $M = T \oplus F$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

***Corollary* 2.** *For a right perfect ring* R, *the following are equivalent*:

(a) *every hereditary torsion class in* R-**Mod** *is an I-torsion class*;

(b) *every* *I-torsion theory in* **Mod**-R *splits.*

Proof. (a) $\Rightarrow$ (b): By Theorem 1, we have $R = R_1 \times \ldots \times R_n$ with quasilocal R_i. So we can assume, without loss of generality, that R is quasilocal. Then it suffices to show that there is no nontrivial *I*-torsion theory in **Mod**-R. In fact, every proper ideal I of R is superfluous in **Mod**-R by [14], Lemma 28.3. Hence I defines the zero torsion class.

(b) $\Rightarrow$ (a): Let $P \in$ **Mod**-R be indecomposable and projective, and $R_r = P^m \oplus Q$ with m maximal. Then $I := P^m + \operatorname{Rad} R$ is an ideal of R with $PI = P$. As the *I*-torsion theory in **Mod**-R splits, we infer that P^m is the *I*-torsion part of R, hence an ideal. By symmetry, this yields a decomposition $R = R_1 \times \ldots \times R_n$ into quasilocal R_i. Now Theorem 1 completes the proof.

As a second consequence, we get an extension of Teply's result (cf. [9], Theorem 3.3).

Theorem 2. *For a ring* R, *the following are equivalent*:

(a) *every hereditary torsion class in* R-**Mod** *is an* *I-torsion class*;

(b) *every left exact radical is exact*;

(c) *each hereditary torsion class* $\mathcal{T}$ *in* R-**Mod** *extends to a TTF-theory* $(\mathcal{T}, \mathcal{F}, \mathcal{D})$;

(d) *every hereditary torsion theory in* R-**Mod** *is centrally splitting.*

Proof. (a) $\Rightarrow$ (b): By Theorem 1 and Proposition 1, R decomposes into quasilocal rings, and every hereditary torsion class in R-**Mod** is semisimple. Whence (b) holds.

(b) $\Rightarrow$ (c): Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in R-**Mod**, and let T be the radical of $\mathcal{T}$. For an epimorphism $M \twoheadrightarrow N$ in R-**Mod** with $M \in \mathcal{F}$, the exactness of T implies that $T(N) = 0$. Hence $\mathcal{F}$ is a torsion class.

(c) $\Rightarrow$ (d) $\Rightarrow$ (a): This follows by Propositions 5 and 4, respectively.

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