

## THE SPECTRAL THEORY AND THE WIENER – ITÔ DECOMPOSITION FOR THE IMAGE OF A JACOBI FIELD\*

### СПЕКТРАЛЬНА ТЕОРІЯ ТА РОЗКЛАД ВІНЕРА – ІТО ДЛЯ ЗОБРАЖЕННЯ ПОЛЯ ЯКОБІ

Assume that  $K^+ : H_- \rightarrow T_-$  is a bounded operator, where  $H_-$  and  $T_-$  are Hilbert spaces and  $\rho$  is a measure on the space  $H_-$ . Denote by  $\rho_K$  the image of the measure  $\rho$  under  $K^+$ . This paper aims to study the measure  $\rho_K$  assuming  $\rho$  to be the spectral measure of a Jacobi field. We obtain a family of operators whose spectral measure equals  $\rho_K$ . We also obtain an analogue of the Wiener – Itô decomposition for  $\rho_K$ . Finally, we illustrate the results obtained by carrying out the explicit calculations for the case, where  $\rho_K$  is a Lévy noise measure.

Припустимо, що  $K^+ : H_- \rightarrow T_-$  є обмеженим оператором, де  $H_-$  та  $T_-$  — гільбертові простори, і  $\rho$  — міра на просторі  $H_-$ . Позначимо через  $\rho_K$  зображення міри  $\rho$  під дією  $K^+$ . Метою цієї роботи є вивчення міри  $\rho_K$  за припущення, що  $\rho$  є спектральною мірою поля Якобі. Отримано сім'ю операторів із спектральною мірою, рівною  $\rho_K$ , а також аналог розкладу Вінера – Іто для  $\rho_K$ . Одержані результати проілюстровано явними розрахунками для випадку, коли  $\rho_K$  є мірою шуму Леві.

**1. Introduction.** Consider a real separable Hilbert space  $H$  and a rigging

$$H_- \supset H \supset H_+$$

with the pairing  $\langle \cdot, \cdot \rangle_H$ . We assume the embedding  $H_+ \hookrightarrow H$  to be a Hilbert – Schmidt operator.

Consider another real separable Hilbert space  $T$  and a rigging

$$T_- \supset T \supset T_+$$

with the pairing  $\langle \cdot, \cdot \rangle_T$ . Given a bounded operator  $K : T_+ \rightarrow H_+$ , define the operator  $K^+ : H_- \rightarrow T_-$  via the formula

$$\langle K^+\xi, f \rangle_T = \langle \xi, Kf \rangle_H, \quad \xi \in H_-, \quad f \in T_+.$$

Let  $\rho$  be a Borel probability measure on the space  $H_-$ . We denote by  $\rho_K$  the image of the measure  $\rho$  under the mapping  $K^+$ . This paper aims to study the measure  $\rho_K$  assuming  $\rho$  to be the spectral measure of a Jacobi field  $J = (\tilde{J}(\phi))_{\phi \in H_+}$ . In particular, we want to explain the Wiener – Itô decomposition for  $\rho_K$  exploiting the operator theory point of view. Our approach is based on the well-known connection between Jacobi matrices and orthogonal polynomials and the infinite-dimensional version of this connection. The principal examples will be described below.

*De facto*, this paper develops the ideas of [22]. The assumption of the density of  $\text{Ran}(K)$  played a crucial role in [22]. This prevented the results from covering an important case of a Lévy noise measure. In the present paper, we show how to develop the theory without assuming the density of  $\text{Ran}(K)$ . The construction here is much more general, and it allows to study more complicated phenomena. In particular, it describes the Lévy noise measure as detailed below.

By definition, a Jacobi field  $J = (\tilde{J}(\phi))_{\phi \in H_+}$  is a family of commuting selfadjoint three-diagonal operators  $\tilde{J}(\phi)$  acting in the Fock space

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$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H), \quad \mathcal{F}_n(H) = H_{\mathbb{C}}^{\otimes n}$$

(we suppose  $H_{\mathbb{C}}^{\otimes 0} = \mathbb{C}$ ). The operators  $\tilde{J}(\phi)$  are assumed to depend on the indexing parameter  $\phi \in H_+$  linearly and continuously. In Section 2 of the present paper, we adduce the rigorous definition and the basic spectral theory of a Jacobi field. More details can be found in, e.g., [1–4]. Remark that the concept of a Jacobi field is relatively new, therefore the definitions given in different papers may differ in minor details.

Jacobi fields are actively used in non-Gaussian white noise analysis and theory of stochastic processes, see [2, 4–13]. In the case of a finite-dimensional  $H$ , the theory of Jacobi fields is closely related to some results in [14–17].

The most principal examples of spectral measures of Jacobi fields are the the Gaussian measure and the Poisson measure. The Jacobi field with the Gaussian spectral measure is the classical free field in quantum field theory, see, e.g., [2, 3, 5, 18]. The Jacobi field with the Poisson spectral measure is the so-called Poisson field (*de facto*, it has been independently discovered in [19, 20]). Section 4 of the present paper contains the rigorous definition of the Poisson field. More details can be found in, e.g., [2, 3, 5, 6].

For other examples of spectral measures of Jacobi fields, see [2, 4].

In Section 3 of the present paper, for a given operator  $K$  and a given Jacobi field  $J$ , we construct a Fock-type space

$$\mathcal{F}^{\text{ext}}(T_+, K) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n^{\text{ext}}(T_+, K)$$

and a family  $J_K = (\tilde{J}_K(f))_{f \in T_+}$  of operators in  $\mathcal{F}^{\text{ext}}(T_+, K)$  pursuing the three following goals:

1. To show that  $\rho_K$  is the spectral measure of the family  $J_K$ .
2. To show that the Fourier transform corresponding to the generalized joint eigenvector expansion of  $J_K$  coincides with the generalized Wiener – Itô – Segal transform associated with  $\rho_K$ .
3. To obtain an analogue of the Wiener – Itô orthogonal decomposition for  $\rho_K$  employing the generalized Wiener – Itô – Segal transform associated with  $\rho_K$ .

The family  $J_K$  can no longer appear as a Jacobi field. Basically, in order to introduce it and reach our destination, we need to extend the concept of a Jacobi field. The relation between  $J_K$  and the original  $J$  will become clear from the definitions given below.

Several versions of the third goal for concrete examples of the measure  $\rho_K$  were considered over the last decade. Different approaches were utilized. A collection of references will be given later. To a large extent, this paper is intended to give an operator theory approach to the problem within a rather general setting. We expect that our construction will be useful in explaining the complicated technical results obtained by other methods. It is also hopeful that it will be used for generalizations and extensions of the concrete classical theories. Of course, our construction allows to deal with a wider class of measures than those investigated in this context before.

We will now recall the classical concepts of the Wiener – Itô – Segal transform and the Wiener – Itô decomposition briefly. More details can be found in, e.g., [18] or [21].

Let  $\gamma$  stand for the Gaussian measure on  $H_-$ . Let  $\mathcal{P}_n$  stand for the set of all continuous polynomials on  $H_-$  with their degree less than or equal to  $n$ . Denote by  $\tilde{\mathcal{P}}_n$  the closure of  $\mathcal{P}_n$  in  $L^2(H_-, d\gamma)$ . The Wiener–Itô–Segal transform

$$\mathcal{F}_n(H_+) \ni \Phi_n \mapsto I\Phi_n = \frac{1}{\sqrt{n!}} \Pr_{\tilde{\mathcal{P}}_n \ominus \tilde{\mathcal{P}}_{n-1}} \langle \cdot, \Phi_n \rangle_H \in L^2(H_-, d\gamma), \quad n \in \mathbb{Z}_+ \tag{1.1}$$

(we suppose  $\tilde{\mathcal{P}}_{-1} = \{0\}$  and preserve the notation  $\langle \cdot, \cdot \rangle_H$  for the pairing between  $\mathcal{F}_n(H_+)$  and  $\mathcal{F}_n(H_-)$ ) can be extended to a unitary operator acting from  $\mathcal{F}(H)$  to  $L^2(H_-, d\gamma)$ . The unitarity of  $I$  implies

$$L^2(H_-, d\gamma) = \bigoplus_{n=0}^{\infty} (\tilde{\mathcal{P}}_n \ominus \tilde{\mathcal{P}}_{n-1}) = \bigoplus_{n=0}^{\infty} I(\mathcal{F}_n(H)).$$

This formula constitutes the Wiener–Itô orthogonal decomposition for the Gaussian measure. It is a powerful technical tool for carrying out the calculations in the space  $L^2(H_-, d\gamma)$ . Remark that analogous results are possible to obtain considering the Poisson measure instead of the Gaussian measure  $\gamma$ .

The Fourier transform corresponding to the generalized joint eigenvector expansion of the classical free field coincides with the Wiener–Itô–Segal transform  $I$ . An analogous result is possible to obtain for the Poisson field. These facts give the basis for investigating the concept of the Wiener–Itô decomposition from the viewpoint of spectral theory of Jacobi fields.

The operator  $I$  can be represented as a sum of operators of multiple stochastic integration. An analogous result is possible to obtain considering the Poisson measure instead of the Gaussian measure  $\gamma$ .

Our generalization of the classical picture is as follows. Let  $\mathcal{Q}_n$  stand for the set of all continuous polynomials on  $T_-$  with their degree less than or equal to  $n$ . Denote by  $\tilde{\mathcal{Q}}_n$  the closure of  $\mathcal{Q}_n$  in  $L^2(T_-, d\rho_K)$ . We suppose  $\tilde{\mathcal{Q}}_{-1} = \{0\}$  and preserve the notation  $\langle \cdot, \cdot \rangle_T$  for the pairing between  $\mathcal{F}_n(T_+)$  and  $\mathcal{F}_n(T_-)$ . One may introduce the generalized Wiener–Itô–Segal transform  $I_K : \mathcal{F}_n(T_+) \rightarrow L^2(T_-, d\rho_K)$  associated with the measure  $\rho_K$  by the formula analogous to (1.1). Of course, now  $\tilde{\mathcal{Q}}_n$  has to appear in the place of  $\tilde{\mathcal{P}}_n$  and the pairing used must be  $\langle \cdot, \cdot \rangle_T$ . But, in general,  $I_K$  cannot be extended to a unitary operator acting from  $\mathcal{F}(T)$  to  $L^2(T_-, d\rho_K)$ .

We construct the space  $\mathcal{F}^{\text{ext}}(T_+, K)$  so that  $I_K$  could be extended to a unitary operator acting from  $\mathcal{F}^{\text{ext}}(T_+, K)$  to  $L^2(T_-, d\rho_K)$ . The orthogonal component  $\mathcal{F}_n^{\text{ext}}(T_+, K)$  has to be constructed as the completion of  $\mathcal{F}_n(T_+)$  with respect to a new scalar product  $(\cdot, \cdot)_{\mathcal{F}_n^{\text{ext}}(T_+, K)}$ . Clearly, the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  should satisfy the equality

$$(F_n, G_n)_{\mathcal{F}_n^{\text{ext}}(T_+, K)} = (I_K F_n, I_K G_n)_{L^2(T_-, d\rho_K)}, \quad F_n, G_n \in \mathcal{F}_n(T_+).$$

Basically, the problem of constructing the space  $\mathcal{F}^{\text{ext}}(T_+, K)$  consists in identifying the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  explicitly.

If the range  $\text{Ran}(K)$  is dense in  $H_+$  and  $J$  is the classical free field or the Poisson field, then the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  satisfies the equality

$$(F_n, G_n)_{\mathcal{F}_n^{\text{ext}}(T_+, K)} = (K^{\otimes n} F_n, K^{\otimes n} G_n)_{\mathcal{F}_n(H)}, \quad F_n, G_n \in \mathcal{F}_n(T_+).$$

This case has undergone a detailed study in [22]. In the general case we are considering, the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  has a much more complicated form.

In order to construct the family  $J_K$ , we introduce an isometric operator  $A: \mathcal{F}^{\text{ext}}(T_+, K) \rightarrow \mathcal{F}(H)$ . The set  $\text{Ran}(A)$  is invariant with respect to the operators  $\tilde{J}(Kf)$ ,  $f \in T_+$ . We define  $\tilde{J}_K(f)$  via the formula

$$\tilde{J}_K(f) = A^{-1}\tilde{J}(Kf)A, \quad f \in T_+.$$

Importantly, this family is no longer a Jacobi field (at least, in general). If the range  $\text{Ran}(K)$  is dense in  $H_+$  and  $J$  is the classical free field or the Poisson field, then the operator  $A$  satisfies the equality

$$A = \bigoplus_{n=0}^{\infty} K^{\otimes n}$$

(we suppose  $K^{\otimes 0} = \text{Id}_{\mathbb{C}}$ .) In the general case we are considering,  $A$  has a much more complicated form.

The unitarity of  $I_K$  implies

$$L^2(T_-, d\rho_K) = \bigoplus_{n=0}^{\infty} (\tilde{\mathcal{Q}}_n \ominus \tilde{\mathcal{Q}}_{n-1}) = \bigoplus_{n=0}^{\infty} I_K(\mathcal{F}_n^{\text{ext}}(T_+, K)).$$

This formula constitutes an analogue of the Wiener – Itô orthogonal decomposition for the measure  $\rho_K$ . It discovers the Fock-type structure of the space  $L^2(T_-, d\rho_K)$  and enables one to carry out the calculations in  $L^2(T_-, d\rho_K)$ .

As mentioned above, the Wiener – Itô – Segal transform  $I$  can be represented as a sum of operators of multiple stochastic integration. Presumably, an analogous representation is possible to obtain for the generalized Wiener – Itô – Segal transform  $I_K$ . However, we do not concern ourselves with this problem in the present paper.

Noteworthy, if  $K$  is the operator of multiplication by a function of a new independent variable and  $J$  is the Poisson field, then  $\rho_K$  is a Lévy noise measure on  $T_-$ . This is an important illustrative example for us. Describing the operator theory approach to the Wiener – Itô decomposition for a Lévy noise measure is one of the objectives of the present work. Related questions were studied in many papers as detailed below. Our intention is to offer a construction that would help clarify the complicated phenomena occurring in this area. We will now discuss a few details of the definition. Namely, suppose  $T = L^2(\mathbb{R}^{d_2}, d\tau)$  and  $H = L^2(\mathbb{R}^{d_1+d_2}, d(\sigma \otimes \tau))$ . Fix a real-valued function  $\kappa$  on  $\mathbb{R}^{d_1}$  and denote by  $\sigma_{\kappa}$  the image of  $\sigma$  under  $\kappa$ . If  $K$  is defined via the formula

$$f(t) \mapsto (Kf)(s, t) = \kappa(s)f(t), \quad s \in \mathbb{R}^{d_1}, \quad t \in \mathbb{R}^{d_2}, \quad (1.2)$$

and  $J$  is the Poisson field, then  $\rho_K$  is the Lévy noise measure on  $T_-$  with the Lévy measure  $\sigma_{\kappa}$  and the intensity measure  $\tau$ . (A more complicated choice of  $K$  yields the fractional Lévy noise measure on  $T_-$ .) In this case, the space  $\mathcal{F}^{\text{ext}}(T_+, K)$  should be similar to the extended Fock space investigated in [11]. A special form of this space has been introduced in [23] in the framework of Gamma white noise analysis. Its further study has been carried out in [7, 8, 10, 24], see also [12].

A family of operators with a Lévy noise spectral measure has been constructed in [9], see also [11]. The case of the Gamma measure was studied in [7]. An analogue of the Wiener – Itô decomposition for a Lévy noise measure has been obtained in [11], see also [25 – 28]. The case of the Gamma measure was studied in [8, 10, 23]. Remark that the

works [11, 26–28] (respectively, [8, 10]) represent the generalized Wiener–Itô–Segal transform associated with a Lévy noise measure (respectively, the Gamma measure) as a sum of operators of stochastic integration. Once again we emphasize that the present paper does not attempt to obtain an analogous representation for the generalized Wiener–Itô–Segal transform associated with the measure  $\rho_K$  in the general case.

While constructing the space  $\mathcal{F}^{\text{ext}}(T_+, K)$  in Section 3, we identify the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  in terms of the operator  $K$  and the initial Jacobi field  $J$ . However, it remains quite a challenge to carry out the explicit calculations for a particular operator and a particular field. In Section 4 of the present paper, we illustrate the constructions of Section 3 by carrying out the explicit calculations for the operator (1.2) and the Poisson field. Theorem 3.1 and Theorem 4.1 explain the Fock-type structure of the space  $L^2(T_+, d\rho_K)$  in this case. Theorem 4.1 implies that  $\mathcal{F}^{\text{ext}}(T_+, K)$  is similar to the extended Fock space investigated in [11].

Remark that the riggings we consider in this paper are all quasinuclear. One may consider nuclear riggings instead.

**2. Commutative Jacobi fields.** This section contains the definition and the basic spectral theory of a Jacobi field.

Let  $H$  be a real separable Hilbert space. Denote by  $H_{\mathbb{C}}$  the complexification of  $H$ . Let  $\hat{\otimes}$  stand for the symmetric tensor product. Consider the symmetric Fock space

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H), \quad \mathcal{F}_n(H) = H_{\mathbb{C}}^{\hat{\otimes} n}$$

(we suppose  $H_{\mathbb{C}}^{\hat{\otimes} 0} = \mathbb{C}$ ). This space consists of the sequences  $\Phi = (\Phi_n)_{n=0}^{\infty}$ ,  $\Phi_n \in \mathcal{F}_n(H)$ . In what follows, we identify  $\Phi_n \in \mathcal{F}_n(H)$  with  $(0, \dots, 0, \Phi_n, 0, 0, \dots) \in \mathcal{F}(H)$  ( $\Phi_n$  standing at the  $n$ th position).

The finite vectors  $\Phi = (\Phi_1, \dots, \Phi_n, 0, 0, \dots) \in \mathcal{F}(H)$  form a linear topological space  $\mathcal{F}_{\text{fin}}(H) \subset \mathcal{F}(H)$ . The convergence in  $\mathcal{F}_{\text{fin}}(H)$  is equivalent to the uniform finiteness and coordinatewise convergence. The vector  $\Omega = (1, 0, 0, \dots) \in \mathcal{F}_{\text{fin}}(H)$  is called vacuum.

Let

$$H_- \supset H \supset H_+ \tag{2.1}$$

be a rigging of  $H$  with real separable Hilbert spaces  $H_+$  and  $H_- = (H_+)'$  (hereafter,  $X'$  denotes the dual of the space  $X$ ). We suppose the embedding  $H_+ \hookrightarrow H$  to be a Hilbert–Schmidt operator. The pairing in (2.1) can be extended naturally to a pairing between  $\mathcal{F}_n(H_+)$  and  $\mathcal{F}_n(H_-)$ . The latter can be extended to a pairing between  $\mathcal{F}_{\text{fin}}(H_+)$  and  $(\mathcal{F}_{\text{fin}}(H_+))'$ . In what follows, we use the notation  $\langle \cdot, \cdot \rangle_H$  for all of these pairings. Note that  $(\mathcal{F}_{\text{fin}}(H_+))'$  coincides with the direct product of the spaces  $\mathcal{F}_n(H_-)$ ,  $n \in \mathbb{Z}_+$ .

Throughout the paper,  $\text{Pr}_X F$  denotes the projection of a vector  $F$  onto a subspace  $X$ .

**2.1. Definition of a Jacobi field.** In the Fock space  $\mathcal{F}(H)$ , consider a family  $\mathcal{J} = (\mathcal{J}(\phi))_{\phi \in H_+}$  of operator-valued Jacobi matrices

$$\mathcal{J}(\phi) = \begin{pmatrix} b_0(\phi) & a_0^*(\phi) & 0 & 0 & 0 & \dots \\ a_0(\phi) & b_1(\phi) & a_1^*(\phi) & 0 & 0 & \dots \\ 0 & a_1(\phi) & b_2(\phi) & a_2^*(\phi) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the entries

$$\begin{aligned} a_n(\phi) &: \mathcal{F}_n(H) \rightarrow \mathcal{F}_{n+1}(H), \\ b_n(\phi) &= (b_n(\phi))^*: \mathcal{F}_n(H) \rightarrow \mathcal{F}_n(H), \\ a_n^*(\phi) &= (a_n(\phi))^*: \mathcal{F}_{n+1}(H) \rightarrow \mathcal{F}_n(H), \\ \phi &\in H_+, \quad n \in \mathbb{Z}_+ = 0, 1, \dots \end{aligned}$$

Each matrix  $\mathcal{J}(\phi)$  gives rise to a Hermitian operator  $J(\phi)$  in the space  $\mathcal{F}(H)$ : Given a vector  $\Phi = (\Phi_n)_{n=0}^\infty \in \text{Dom}(J(\phi)) = \mathcal{F}_{\text{fin}}(H_+)$ , we define

$$\begin{aligned} J(\phi)\Phi &= ((J(\phi)\Phi)_0, \dots, (J(\phi)\Phi)_n, \dots) \in \mathcal{F}(H), \\ (J(\phi)\Phi)_n &= a_{n-1}(\phi)\Phi_{n-1} + b_n(\phi)\Phi_n + a_n^*(\phi)\Phi_{n+1}, \quad n \in \mathbb{Z}_+ \end{aligned}$$

(we suppose  $a_{-1}(\phi) = 0$  and  $\Phi_{-1} = 0$ ).

Consider the following assumptions.

1. The operators  $a_n(\phi)$  and  $b_n(\phi)$ ,  $\phi \in H_+$ ,  $n \in \mathbb{Z}_+$ , are bounded and real, i.e., they take real vectors to real ones.
- 2 (smoothness). The inclusions

$$\begin{aligned} a_n(\phi)(\mathcal{F}_n(H_+)) &\subset \mathcal{F}_{n+1}(H_+), \quad b_n(\phi)(\mathcal{F}_n(H_+)) \subset \mathcal{F}_n(H_+), \\ a_n^*(\phi)(\mathcal{F}_{n+1}(H_+)) &\subset \mathcal{F}_n(H_+), \\ \phi &\in H_+, \quad n \in \mathbb{Z}_+, \end{aligned}$$

hold true. Basically, this axiom explains the relation between our family of operators and the fixed rigging. The tag “smoothness” appears because this axiom expresses the smoothness of coefficients when differential operators are considered for  $a_n(\phi)$  and  $b_n(\phi)$ .

3. The operators  $J(\phi)$ ,  $\phi \in H_+$ , are essentially selfadjoint and their closures  $\tilde{J}(\phi)$  are strongly commuting.

4. The functions

$$\begin{aligned} H_+ \ni \phi &\mapsto a_n(\phi)\Phi_n \in \mathcal{F}_{n+1}(H_+), \\ H_+ \ni \phi &\mapsto b_n(\phi)\Phi_n \in \mathcal{F}_n(H_+), \\ H_+ \ni \phi &\mapsto a_n^*(\phi)\Phi_{n+1} \in \mathcal{F}_n(H_+), \quad n \in \mathbb{Z}_+, \end{aligned}$$

are linear and continuous for any  $\Phi_n \in \mathcal{F}_n(H_+)$ ,  $\Phi_{n+1} \in \mathcal{F}_{n+1}(H_+)$ .

- 5 (regularity). This last axiom expresses a rather complicated technical requirement. At the same time, it deals with a set of operators that are extremely important for our further constructions.

The linear operators  $V_n: \mathcal{F}_n(H_+) \rightarrow \bigoplus_{j=0}^n \mathcal{F}_j(H_+)$  defined by the equalities

$$V_0 = \text{Id}_{\mathbb{C}}, \quad V_n(\phi_1 \hat{\otimes} \dots \hat{\otimes} \phi_n) = J(\phi_1) \dots J(\phi_n)\Omega,$$

$$\phi_1, \dots, \phi_n \in H_+, \quad n \in \mathbb{N},$$

must be continuous. The operators

$$\mathcal{F}_n(H_+) \ni F_n \mapsto V_{n,n}F_n = \text{Pr}_{\mathcal{F}_n(H_+)} V_n F_n \in \mathcal{F}_n(H_+), \quad n \in \mathbb{Z}_+,$$

must be invertible. Again, we point out that  $V_n$  and  $V_{n,n}$  play a crucial role in the further considerations.

The family  $J = (\tilde{J}(\phi))_{\phi \in H_+}$  is called a (commutative) *Jacobi field* if Assumptions 1–5 are satisfied. (Recall that  $\tilde{J}(\phi)$  stands for the closure of the operator  $J(\phi)$ .) Once again we should emphasize that the operators  $\tilde{J}(\phi)$  act in the Fock space  $\mathcal{F}(H)$ .

**2.2. Spectral theory of a Jacobi field.** One can apply the projection spectral theorem (see [18, 29]) to the field  $J = (\tilde{J}(\phi))_{\phi \in H_+}$ . We only adduce the result of such an application here. Proofs can be found in [2].

Given  $n \in \mathbb{Z}_+$ , let  $\mathcal{P}_n$  stand for the set of all continuous polynomials

$$H_- \ni \xi \mapsto \sum_{j=0}^n \langle \xi^{\otimes j}, a_j \rangle_H \in \mathbb{C}, \quad a_j \in \mathcal{F}_j(H_+)$$

(we suppose  $\xi^{\otimes 0} = 1$ .) The set  $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$  is a dense subset of  $L^2(H_-, d\rho)$ . The closure of  $\mathcal{P}_n$  in  $L^2(H_-, d\rho)$  will be denoted by  $\tilde{\mathcal{P}}_n$ .

**Theorem 2.1.** *There exist a vector-valued function  $H_- \ni \xi \mapsto P(\xi) \in (\mathcal{F}_{\text{fin}}(H_+))'$  and a Borel probability measure  $\rho$  on the space  $H_-$  (the spectral measure) such that the following statements hold:*

1. *For every  $\xi \in H_-$ , the vector  $P(\xi) \in (\mathcal{F}_{\text{fin}}(H_+))'$  is a generalized joint eigenvector of  $J$  with eigenvalue  $\xi$ , i.e.,*

$$\langle P(\xi), \tilde{J}(\phi)\Phi \rangle_H = \langle \xi, \phi \rangle_H \langle P(\xi), \Phi \rangle_H, \quad \phi \in H_+, \quad \Phi \in \mathcal{F}_{\text{fin}}(H_+).$$

2. *The Fourier transform*

$$\mathcal{F}_{\text{fin}}(H_+) \ni \Phi \mapsto I\Phi = \langle \Phi, P(\cdot) \rangle_H \in L^2(H_-, d\rho)$$

can be extended to a unitary operator acting from  $\mathcal{F}(H)$  to  $L^2(H_-, d\rho)$ . We preserve the notation  $I$  for this operator:

3. *The Fourier transform  $I$  satisfies the equality*

$$I\Phi_n = \text{Pr}_{\tilde{\mathcal{P}}_n \ominus \tilde{\mathcal{P}}_{n-1}} \langle V_{n,n}^{-1} \Phi_n, \cdot^{\otimes n} \rangle_H, \quad \Phi_n \in \mathcal{F}_n(H_+), \quad n \in \mathbb{Z}_+$$

(we suppose  $\mathcal{P}_{-1} = \{0\}$ ).

**Corollary 2.1.** *The equality*

$$L^2(H_-, d\rho) = \bigoplus_{n=0}^{\infty} (\tilde{\mathcal{P}}_n \ominus \tilde{\mathcal{P}}_{n-1}) = \bigoplus_{n=0}^{\infty} I(\mathcal{F}_n(H_+))$$

holds true.

**Remark 2.1.** If  $J$  is the classical free field, then

$$V_{n,n} = \sqrt{n!} \operatorname{Id}_{\mathcal{F}_n(H_+)}, \quad n \in \mathbb{Z}_+,$$

and  $\rho$  is the Gaussian measure. In this case, the Fourier transform  $I$  coincides with the Wiener–Itô–Segal transform and Corollary 2.1 constitutes the Wiener–Itô decomposition for the Gaussian measure. Analogous results hold for the Poisson field.

**Remark 2.2.** The equality

$$IV_n F_n = \langle F_n, \cdot^{\otimes n} \rangle_H, \quad F_n \in \mathcal{F}_n(H_+), \quad n \in \mathbb{Z}_+,$$

holds true. See [22] for the proof.

**3. Image of the spectral measure.** This section aims to study the image of the measure  $\rho$  under a bounded operator. Our main example for  $\rho$  will be the Poisson measure, as detailed in Section 4. We will now demonstrate how the image of  $\rho$  should be constructed for our purposes.

Consider a real separable Hilbert space  $T$ . Let

$$T_- \supset T \supset T_+ \tag{3.1}$$

be a rigging of  $T$  with real separable Hilbert spaces  $T_+$  and  $T_- = (T_+)'$ . As in the case of the rigging (2.1), the pairing in (3.1) can be extended to a pairing between  $\mathcal{F}_n(T_+)$  and  $\mathcal{F}_n(T_-)$ . The latter can be extended to a pairing between  $\mathcal{F}_{\text{fin}}(T_+)$  and  $(\mathcal{F}_{\text{fin}}(T_+))'$ . We use the notation  $\langle \cdot, \cdot \rangle_T$  for all of these pairings.

Consider a bounded operator  $K : T_+ \rightarrow H_+$  such that  $\operatorname{Ker}(K) = \{0\}$ . We preserve the notation  $K$  for the extension of this operator to the complexified space  $(T_+)_{\mathbb{C}}$ .

The adjoint of  $K$  with respect to (2.1) and (3.1) is a bounded operator  $K^+ : H_- \rightarrow T_-$  defined by the equality

$$\langle K^+ \xi, f \rangle_T = \langle \xi, Kf \rangle_H, \quad \xi \in H_-, \quad f \in T_+.$$

One can prove that  $\operatorname{Ran}(K^+)$  is dense in  $T_-$ .

We denote by  $\rho_K$  the image of the measure  $\rho$  under the mapping  $K^+$ . By definition,  $\rho_K$  is a probability measure on the  $\sigma$ -algebra

$$\mathcal{C} = \{ \Delta \subset T_- \mid (K^+)^{-1}(\Delta) \text{ is a Borel subset of } H_- \}$$

( $(K^+)^{-1}(\Delta)$  denoting the preimage of the set  $\Delta$ ).

**Remark 3.1.** Since the mapping  $K^+$  is Borel-measurable, the  $\sigma$ -algebra  $\mathcal{C}$  contains the Borel  $\sigma$ -algebra of the space  $T_-$ . If  $K^+$  takes Borel subsets of  $H_-$  to the Borel subsets of  $T_-$ , then  $\mathcal{C}$  coincides with the Borel  $\sigma$ -algebra of  $T_-$ .

**Remark 3.2.** The characteristic functional

$$\hat{\rho}_K(f) = \int_{T_-} e^{i\langle \omega, f \rangle_T} d\rho_K(\omega), \quad f \in T_+,$$

of the measure  $\rho_K$  satisfies the equality

$$\hat{\rho}_K(f) = \hat{\rho}(Kf), \quad f \in T_+,$$

with



$$\hat{\rho}(\phi) = \int_{H_-} e^{i\langle \xi, \phi \rangle_H} d\rho(\xi), \quad \phi \in H_+,$$

being the characteristic functional of the measure  $\rho$ .

**Remark 3.3.** The assumption  $\text{Ker}(K) = \{0\}$  is not essential. Indeed, the measure  $\rho_K$  proves to be lumped on the set of functionals which equal zero on  $\text{Ker}(K)$ . This set can be naturally identified with  $(\text{Ker}(K)^\perp)'$ . Thus we can always replace  $T_+$  with  $\text{Ker}(K)^\perp \subset T_+$ .

**3.1. The space  $\mathcal{F}^{\text{ext}}(T_+, K)$  and its rigging.** The goal is to produce a Fock-type space  $\mathcal{F}^{\text{ext}}(T_+, K)$  and a family of operators in it associated to the measure  $\rho_K$ . We must mention the paper [30] that offers a very broad generalization of the classical Fock space and discusses families of operators in it. That theory seems to be closely related to what we suggest in this section.

Before constructing  $\mathcal{F}^{\text{ext}}(T_+, K)$ , we have to introduce an auxiliary notation. Given  $n \in \mathbb{Z}_+$ , let  $\mathcal{V}_n$  stand for the subspace of  $\mathcal{F}(H)$  generated by the vectors

$$V_j K^{\otimes j} F_j, \quad F_j \in \mathcal{F}_j(T_+), \quad j = 0, \dots, n$$

(we suppose  $K^{\otimes 0} = \text{Id}_{\mathbb{C}}$ ).

Introduce the mapping  $A: \mathcal{F}_{\text{fin}}(T_+) \rightarrow \mathcal{F}(H)$  via the formula

$$\mathcal{F}_n(T_+) \ni F_n \mapsto AF_n = \frac{1}{\sqrt{n!}} \text{Pr}_{\mathcal{V}_n \ominus \mathcal{V}_{n-1}} V_n K^{\otimes n} F_n \in \mathcal{F}(H), \quad n \in \mathbb{Z}_+$$

(we suppose  $\mathcal{V}_{-1} = \{0\}$ ). It is easy to see that  $\text{Ker}(A) = \{0\}$ . We use this mapping to identify the scalar product in our desired space  $\mathcal{F}^{\text{ext}}(T_+, K)$ .

Having the required notation at hand, it is now possible to proceed with the definition. Let  $\mathcal{F}_n^{\text{ext}}(T_+, K)$  denote the completion of  $\mathcal{F}_n(T_+)$  with respect to the scalar product

$$(F_n, G_n)_{\mathcal{F}_n^{\text{ext}}(T_+, K)} = (AF_n, AG_n)_{\mathcal{F}(H)}, \quad F_n, G_n \in \mathcal{F}_n(T_+), \quad n \in \mathbb{Z}_+.$$

Put

$$\mathcal{F}^{\text{ext}}(T_+, K) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n^{\text{ext}}(T_+, K).$$

This space has an evident Fock-type structure of an infinite orthogonal sum. Once can say that it was constructed on the basis of the Fock space  $\mathcal{F}(T_+)$ . Obviously, the mapping  $A$  can be extended to an isometric operator acting from  $\mathcal{F}^{\text{ext}}(T_+, K)$  to  $\mathcal{F}(H)$ . We preserve the notation  $A$  for this operator.

**Remark 3.4.** If  $\text{Ran}(K)$  is dense in  $H_+$ , then  $\mathcal{V}_n = \bigoplus_{j=0}^n \mathcal{F}_j(H)$  and

$$AF_n = \frac{1}{\sqrt{n!}} V_{n,n} K^{\otimes n} F_n, \quad F_n \in \mathcal{F}_n(T_+), \quad n \in \mathbb{Z}_+.$$

If, additionally,  $J$  is the classical free field or the Poisson field, then

$$AF_n = K^{\otimes n} F_n, \quad F_n \in \mathcal{F}_n(T_+), \quad n \in \mathbb{Z}_+,$$

and the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  satisfies the equality

$$(F_n, G_n)_{\mathcal{F}_n^{\text{ext}}(T_+, K)} = (K^{\otimes n} F_n, K^{\otimes n} G_n)_{\mathcal{F}_n(H)}, \quad F_n, G_n \in \mathcal{F}_n(T_+), \quad n \in \mathbb{Z}_+.$$

This case has undergone a detailed study in [22].

**Remark 3.5.** One can easily verify that

$$A \left( \bigoplus_{j=0}^n \mathcal{F}_j^{\text{ext}}(T_+, K) \right) = \mathcal{V}_n, \quad n \in \mathbb{Z}_+.$$

Now we have to construct a rigging of the space  $\mathcal{F}^{\text{ext}}(T_+, K)$ . Recall that we are aiming for the spectral theory of the family  $J_K$ . Clearly, the rigging of the space plays a crucial role in this pursuit. Our next definition may seem a little clumsy at the moment, but it will become natural when we present the explicit form of  $J_K$ .

Consider a linear topological space

$$\mathcal{F}_+^{\text{ext}}(T_+, K) = A^{-1}(\mathcal{F}_{\text{fin}}(H_+) \cap \text{Ran}(A)).$$

The sequence  $(F_n)_{n=0}^\infty$  converges to  $F$  in  $\mathcal{F}_+^{\text{ext}}(T_+, K)$  if and only if the sequence  $(AF_n)_{n=0}^\infty$  converges to  $AF$  in  $\mathcal{F}_{\text{fin}}(H_+)$ .

The space  $\mathcal{F}_+^{\text{ext}}(T_+, K)$  is a dense subset of  $\mathcal{F}^{\text{ext}}(T_+, K)$ . Indeed,  $\mathcal{F}_{\text{fin}}(H_+) \cap \mathcal{V}_n$  is dense in  $\mathcal{V}_n$  for all  $n \in \mathbb{Z}_+$ . Using Remark 3.5, we can conclude that  $\mathcal{F}_{\text{fin}}(H_+) \cap \text{Ran}(A)$  is dense in  $\text{Ran}(A)$ . Hence  $\mathcal{F}_+^{\text{ext}}(T_+, K)$  is dense in  $\mathcal{F}^{\text{ext}}(T_+, K)$ .

Construct a rigging

$$(\mathcal{F}_+^{\text{ext}}(T_+, K))' \supset \mathcal{F}^{\text{ext}}(T_+, K) \supset \mathcal{F}_+^{\text{ext}}(T_+, K).$$

Denote the corresponding pairing by  $\langle \cdot, \cdot \rangle_A$ .

**3.2. The family  $J_K$  and its spectral theory.** We will now define the family  $J_K$  and prove the spectral theorem (an analogue of Theorem 2.1) for it. This would illuminate the connection between  $J_K$  and  $\rho_K$ , thereby leading us to our goal. In particular, this would yield an analogue of the Wiener – Itô decomposition for the measure  $\rho_K$ .

Generally,  $J_K$  is not a Jacobi field, which is an important thing to understand. We begin with a simple technical lemma needed for the further definitions.

**Lemma 3.1.** *The set  $\mathcal{F}_{\text{fin}}(H_+) \cap \text{Ran}(A)$  is invariant with respect to the operators  $J(Kf)$ ,  $f \in T_+$ .*

**Proof.** According to the properties of a Jacobi field, the space  $\mathcal{F}_{\text{fin}}(H_+)$  is invariant with respect to  $J(Kf)$ .

Remark 3.5 implies that  $\text{Ran}(A)$  is generated by the vectors

$$V_n K^{\otimes n} F_n, \quad F_n \in \mathcal{F}_n(T_+), \quad n \in \mathbb{Z}_+.$$

Evidently,

$$J(Kf)V_n K^{\otimes n} F_n = V_{n+1} K^{\otimes(n+1)}(f \hat{\otimes} F_n) \in \text{Ran}(A), \quad f \in T_+.$$

Since the operators  $J(Kf) \upharpoonright \mathcal{F}_n(H)$  are bounded for any  $n \in \mathbb{Z}_+$ , we can conclude that  $\mathcal{F}_{\text{fin}}(H_+) \cap \text{Ran}(A)$  is also invariant with respect to  $J(Kf)$ .

The lemma is proved.

In the space  $\mathcal{F}^{\text{ext}}(T_+, K)$ , consider the operators

$$J_K(f) = A^{-1}J(Kf)A, \quad \text{Dom}(J_K(f)) = \mathcal{F}_+^{\text{ext}}(T_+, K), \quad f \in T_+.$$

Evidently, these operators are essentially selfadjoint and their closures  $\tilde{J}_K(f)$  are strong commuting.

Define  $J_K = (\tilde{J}_K(f))_{f \in T_+}$ . Once again,  $J_K$  is not a Jacobi field (at least, in general). Noteworthy, Lemma 3.1 implies that the space  $\mathcal{F}_+^{\text{ext}}(T_+, K)$  is invariant with respect to the operators  $\tilde{J}_K(f)$ ,  $f \in T_+$ .

As before, we need to introduce the polynomials on our Hilbert space. Given  $n \in \mathbb{Z}_+$ , let  $\mathcal{Q}_n$  stand for the set of all continuous polynomials

$$T_- \ni \omega \mapsto \sum_{j=0}^n \langle \omega^{\otimes j}, c_j \rangle_T \in \mathbb{C}, \quad c_j \in \mathcal{F}_j(T_+)$$

(we suppose  $\omega^{\otimes 0} = 1$ ). The closure of  $\mathcal{Q}_n$  in  $L^2(T_-, d\rho_K)$  will be denoted by  $\tilde{\mathcal{Q}}_n$ . The spectral theorem for  $J_K$  proceeds as follows.

**Theorem 3.1.** *Assume  $\mathcal{Q} = \bigcup_{n=0}^\infty \mathcal{Q}_n$  to be a dense subset of  $L^2(T_-, d\rho_K)$ . There exists a vector-valued function  $T_- \ni \omega \mapsto Q(\omega) \in (\mathcal{F}_+^{\text{ext}}(T_+, K))'$  such that the following statements hold:*

1. *For  $\rho_K$ -almost all  $\omega \in T_-$ , the vector  $Q(\omega) \in (\mathcal{F}_+^{\text{ext}}(T_+, K))'$  is a generalized joint eigenvector of the family  $J_K$  with the eigenvalue  $\omega$ , i.e.,*

$$\langle Q(\omega), \tilde{J}_K(f)F \rangle_A = \langle \omega, f \rangle_T \langle Q(\omega), F \rangle_A, \quad F \in \mathcal{F}_+^{\text{ext}}(T_+, K).$$

2. *The Fourier transform*

$$\mathcal{F}_+^{\text{ext}}(T_+, K) \ni F \mapsto I_K F = \langle F, Q(\cdot) \rangle_A \in L^2(T_-, d\rho_K)$$

*can be extended to a unitary operator acting from  $\mathcal{F}_+^{\text{ext}}(T_+, K)$  to  $L^2(T_-, d\rho_K)$ . We preserve the notation  $I_K$  for this operator.*

3. *The Fourier transform  $I_K$  satisfies the equality*

$$I_K F_n = \frac{1}{\sqrt{n!}} \text{Pr}_{\tilde{\mathcal{Q}}_n \ominus \tilde{\mathcal{Q}}_{n-1}} \langle F_n, \cdot^{\otimes n} \rangle_T, \quad F_n \in \mathcal{F}_n(T_+), \quad n \in \mathbb{Z}_+ \tag{3.2}$$

(we suppose  $\mathcal{Q}_{-1} = \{0\}$ ).

Introduce the isometric operator

$$L^2(T_-, d\rho_K) \ni G \mapsto UG = G(K^+ \cdot) \in L^2(H_-, d\rho).$$

Before proving Theorem 3.1, we have to state an auxiliary lemma.

**Lemma 3.2.** *The equality  $U\tilde{\mathcal{Q}}_n = IV_n$  holds true.*

**Proof.** The space  $U\tilde{\mathcal{Q}}_n$  is generated by the vectors

$$U \langle F_j, \cdot^{\otimes j} \rangle_T, \quad F_j \in \mathcal{F}_j(T_+), \quad j = 0, \dots, n.$$

In virtue of Remark 2.2,

$$U \langle F_j, \cdot^{\otimes j} \rangle_T = \langle K^{\otimes j} F_j, \cdot^{\otimes j} \rangle_H = IV_j K^{\otimes j} F_j.$$

Hence  $U\tilde{\mathcal{Q}}_n = IV_n$ .

The lemma is proved.

**Proof of Theorem 3.1.** Let us construct the function  $Q(\omega)$ . Fix a dense subset  $(F^n)_{n=1}^\infty$  of the space  $\mathcal{F}_+^{\text{ext}}(T_+, K)$ . According to the definition of  $I$ ,

$$IAF^n = \langle AF^n, P(\cdot) \rangle_H.$$

Lemma 3.2 and Remark 3.5 imply the equality  $\text{Ran}(IA) = \text{Ran}(U)$ . Therefore we can find a set  $\Xi \subset H_-$  and a sequence of functions  $(q_n)_{n=1}^\infty \subset L^2(T_-, d\rho_K)$  such that  $\rho(\Xi) = 1$  and

$$\langle AF^n, P(\xi) \rangle_H = q_n(K^+\xi), \quad \xi \in \Xi, \quad n \in \mathbb{N}. \tag{3.3}$$

There exists a unique operator  $A^+ : (\mathcal{F}_{\text{fin}}(H_+))' \rightarrow (\mathcal{F}_+^{\text{ext}}(T_+, K))'$  such that

$$\langle AF^n, P(\xi) \rangle_H = \langle F^n, A^+P(\xi) \rangle_A, \quad \xi \in H_-, \quad n \in \mathbb{N}.$$

Formula (3.3) implies the equality  $A^+P(\xi_1) = A^+P(\xi_2)$  for the vectors  $\xi_1, \xi_2 \in \Xi$  such that  $K^+\xi_1 = K^+\xi_2$ . Define  $Q(K^+\xi) = A^+P(\xi)$  for  $\xi \in \Xi$ .

We have constructed the function  $Q(\omega)$  on the set  $K^+\Xi \subset T_-$ . Evidently,  $\rho_K(K^+\Xi) = 1$ . We extend  $Q(\omega)$  to the whole of the space  $T_-$  arbitrarily.

One can easily verify that

$$I_K F = \langle F, Q(\cdot) \rangle_A = U^{-1} I A F, \quad F \in \mathcal{F}_+^{\text{ext}}(T_+, K).$$

The operator  $U^{-1} I A$  is a unitary acting from  $\mathcal{F}^{\text{ext}}(T_+, K)$  to  $L^2(T_-, d\rho_K)$ . Hence  $I_K$  can be extended to a unitary operator acting from  $\mathcal{F}^{\text{ext}}(T_+, K)$  to  $L^2(T_-, d\rho_K)$ .

Let us show that  $Q(\omega)$  is a generalized joint eigenvector of  $J_K$  with the eigenvalue  $\omega$ . Given  $F \in \mathcal{F}_+^{\text{ext}}(T_+, K)$ , we obtain

$$\begin{aligned} I_K \tilde{J}_K(f)F &= U^{-1} I A \tilde{J}_K(f)F = U^{-1} I A A^{-1} \tilde{J}(Kf)AF = \\ &= U^{-1} I \tilde{J}(Kf)AF = U^{-1} \langle \tilde{J}(Kf)AF, P(\cdot) \rangle_H = \\ &= U^{-1} (\langle Kf, \cdot \rangle_H \langle AF, P(\cdot) \rangle_H) = (U^{-1} \langle Kf, \cdot \rangle_H) (U^{-1} \langle AF, P(\cdot) \rangle_H) = \\ &= \langle \cdot, f \rangle_T (U^{-1} I A F) = \langle \cdot, f \rangle_T (I_K F) \end{aligned}$$

(overbars denoting the complex conjugacy). The latter implies

$$\langle Q(\omega), \tilde{J}_K(f)F \rangle_A = \langle \omega, f \rangle_T \langle Q(\omega), F \rangle_A$$

for  $\rho_K$ -almost all  $\omega \in T_-$ .

To complete the proof, we have to show that  $I_K$  satisfies equality (3.2). In accordance with Lemma 3.2 and Remark 2.2,

$$\begin{aligned} I_K F_n &= U^{-1} I A F_n = \frac{1}{\sqrt{n!}} U^{-1} I \text{Pr}_{\mathcal{V}_n \ominus \mathcal{V}_{n-1}} V_n K^{\otimes n} F_n = \\ &= \frac{1}{\sqrt{n!}} U^{-1} \text{Pr}_{I\mathcal{V}_n \ominus I\mathcal{V}_{n-1}} I V_n K^{\otimes n} F_n = \\ &= \frac{1}{\sqrt{n!}} U^{-1} \text{Pr}_{U\tilde{\mathcal{Q}}_n \ominus U\tilde{\mathcal{Q}}_{n-1}} \langle K^{\otimes n} F_n, \cdot^{\otimes n} \rangle_H = \\ &= \frac{1}{\sqrt{n!}} U^{-1} \text{Pr}_{U\tilde{\mathcal{Q}}_n \ominus U\tilde{\mathcal{Q}}_{n-1}} U \langle F_n, \cdot^{\otimes n} \rangle_T = \\ &= \frac{1}{\sqrt{n!}} \text{Pr}_{\tilde{\mathcal{Q}}_n \ominus \tilde{\mathcal{Q}}_{n-1}} \langle F_n, \cdot^{\otimes n} \rangle_T, \quad F_n \in \mathcal{F}_n(T_+), \quad n \in \mathbb{Z}_+. \end{aligned}$$

The theorem is proved.

**Corollary 3.1.** *If  $\mathcal{Q} = \bigcup_{n=0}^{\infty} \mathcal{Q}_n$  is a dense subset of  $L^2(T_-, d\rho_K)$ , then the equality*

$$L^2(T_-, d\rho_K) = \bigoplus_{n=0}^{\infty} (\tilde{\mathcal{Q}}_n \ominus \tilde{\mathcal{Q}}_{n-1}) = \bigoplus_{n=0}^{\infty} I_K(\mathcal{F}_n^{\text{ext}}(T_+, K))$$

hold true.

Corollary 3.1 constitutes an analogue of the Wiener – Itô decomposition for the measure  $\rho_K$ .

**4. The operator of multiplication and the Poisson field.** As mentioned in Section 1, it remains quite a challenge to carry out the calculations for a particular operator  $K$  and a particular field  $J$ . In this section, we illustrate the constructions of Section 3. We obtain an explicit formula for the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  assuming  $K$  to be the operator of multiplication by a function of a new independent variable and  $J$  to be the Poisson field. This situation was considered in many works, and one of our objectives here is to give an operator theory approach to the problem. We hope this would be helpful in illuminating the complicated phenomena arising in the case under discussion. As we point out below, this case does not fall under the construction of [22]. Now  $\mathcal{F}^{\text{ext}}(T_+, K)$  will not be a classical Fock space but rather an extended Fock space of a certain kind.

Firstly, the abstract objects that appear in the previous section must be specified. Of course,  $T$  and  $H$  should now be function spaces. Consider a real separable Hilbert space  $S = L^2(\mathbb{R}^{d_1}, d\sigma)$ . Let  $T$  equal  $L^2(\mathbb{R}^{d_2}, d\tau)$ . We assume the Borel measures  $\sigma$  and  $\tau$  to be finite on compact sets. We also assume  $\tau$  to be absolutely continuous with respect to the Lebesgue measure. Let the space  $H$  equal  $S \otimes T$ . Clearly,  $H$  can be identified with  $L^2(\mathbb{R}^{d_1+d_2}, d(\sigma \otimes \tau))$ .

We choose the spaces  $T_+$  and  $H_+$  arbitrarily. Recall that the embedding  $H_+ \hookrightarrow H$  is assumed to be a Hilbert – Schmidt operator. Typically, the role of  $T_+$  and  $H_+$  is played by weighted Sobolev spaces.

Define  $K$  via the formula

$$T_+ \ni f(t) \mapsto (Kf)(s, t) = \kappa(s)f(t), \quad s \in \mathbb{R}^{d_1}, \quad t \in \mathbb{R}^{d_2}. \quad (4.1)$$

The function  $\kappa \in S$  has to be chosen so that  $K$  would be a bounded operator acting from  $T_+$  to  $H_+$ . We emphasize that  $\text{Ran}(K)$  is not dense in  $H_+$  now, which prevents the example under discussion from being described by [22].

Suppose  $J = (J(\phi))_{\phi \in H_+}$  to be the Poisson field in  $\mathcal{F}(H)$ . The operators of the Poisson field are defined via the formula

$$J(\phi) = J_+(\phi) + J_0(\phi) + J_-(\phi), \quad \phi \in H_+.$$

The operators  $J_+(\phi)$  and  $J_-(\phi)$  are the classical creation and annihilation operators in  $\mathcal{F}(H)$ , i.e.,

$$J_+(\phi)F_n = \sqrt{n+1} \phi \hat{\otimes} F_n, \quad \phi \in H_+, \quad F_n \in \mathcal{F}_n(H), \quad n \in \mathbb{Z}_+,$$

and  $J_-(\phi) = (J_+(\phi))^*$ . In order to define  $J_0(\phi)$ , consider the operator  $b(\phi)$  of multiplication by the function  $\phi \in H_+$  in the space  $H_{\mathbb{C}}$ . We assume  $b(\phi)$  to be bounded for any  $\phi \in H_+$ . For an arbitrary  $F_n \in \mathcal{F}_n(H)$ , define

$$J_0(\phi)F_0 = 0,$$

$$\begin{aligned}
 J_0(\phi)F_n &= (b(\phi) \otimes \text{Id}_H \otimes \dots \otimes \text{Id}_H)F_n + \\
 &+ (\text{Id}_H \otimes b(\phi) \otimes \text{Id}_H \otimes \dots \otimes \text{Id}_H)F_n + \dots \\
 &\dots + (\text{Id}_H \otimes \dots \otimes \text{Id}_H \otimes b(\phi))F_n, \quad \phi \in H_+, \quad n \in \mathbb{N}.
 \end{aligned}$$

In other words,  $J_0(\phi)$  equals the second (differential) quantization of  $b(\phi)$ .

We assume  $J$  to satisfy the definition of a Jacobi field. The spectral measure  $\rho$  of the field  $J$  equals the centered Poisson measure with the intensity  $\sigma \otimes \tau$ . Its characteristic functional is given by the formula

$$\hat{\rho}(\phi) = \exp \left( \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} (e^{i\phi(s,t)} - 1 - i\phi(s,t)) \, d\sigma(s)d\tau(t) \right), \quad \phi \in H_+.$$

The operator  $K^+$  takes  $\rho$  to a probability measure  $\rho_K$  on  $T_-$ . According to Remark 3.2, the characteristic functional of  $\rho_K$  is now given by the formula

$$\hat{\rho}_K(f) = \exp \left( \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} (e^{i\kappa(s)f(t)} - 1 - i\kappa(s)f(t)) \, d\sigma(s)d\tau(t) \right), \quad f \in T_+.$$

Denote by  $\sigma_\kappa$  the image of  $\sigma$  under  $\kappa$ . The above formula implies that  $\rho_K$  is the Lévy noise measure on  $T_-$  with the Lévy measure  $\sigma_\kappa$  and the intensity measure  $\tau$ .

**4.1. Preliminary constructions.** Before identifying the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  explicitly, we have to carry out some preliminary constructions. Basically, we need to introduce all the ingredients in the formula for  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$ .

Given  $n \in \mathbb{N}$ , define  $\kappa^n(s) = (\kappa(s))^n$ ,  $s \in \mathbb{R}^{d_1}$ .

**Lemma 4.1.** *The function  $\kappa^n$  belongs to the space  $S$  for any  $n \in \mathbb{N}$ .*

**Proof.** Consider a compact set  $\Delta \subset \mathbb{R}^{d_2}$  such that  $\tau(\Delta) \neq 0$ . Fix a smooth compactly supported function  $f_0 \in T_+$  such that  $f_0(t) = 1$  for  $t \in \Delta$ . In the space  $H$ , consider the bounded operator  $b(Kf_0)$  of multiplication by  $Kf_0 \in H_+$ .

The function

$$(b^n(Kf_0)Kf_0)(s, t) = \kappa^{n+1}(s)(f_0(t))^{n+1}, \quad s \in \mathbb{R}^{d_1}, \quad t \in \mathbb{R}^{d_2},$$

belongs to  $H$  for any  $n \in \mathbb{N}$ . Hence the function  $k_{t_0}(s) = \kappa^{n+1}(s)(f_0(t_0))^{n+1}$  belongs to  $S$  for  $\tau$ -almost all  $t_0 \in \Delta$ . Since  $k_{t_0}(s) = \kappa^{n+1}(s)$  for  $t_0 \in \Delta$ , the latter implies  $\kappa^{n+1} \in S$ .

The lemma is proved.

Applying the Schmidt orthogonalization procedure to the sequence  $(\kappa^n)_{n=1}^\infty$ , we obtain an orthogonal sequence  $(\kappa_n)_{n=0}^\infty$  in the space  $S$ . Each  $\kappa_n$  is a polynomial of degree  $n$  with respect to  $\kappa$ . We normalize  $\kappa_n$  so that the leading coefficient of this polynomial would equal 1.

A vector  $F \in T_{\mathbb{C}}^{\otimes n}$ ,  $n \in \mathbb{N}$ , can be treated as a complex-valued function  $F(t_1, \dots, t_n)$  depending on the variables  $t_1, \dots, t_n \in \mathbb{R}^{d_2}$ . Analogously, a vector  $\Phi \in H_{\mathbb{C}}^{\otimes n}$ ,  $n \in \mathbb{N}$ , can be treated as a complex-valued function  $\Phi(s_1, \dots, s_n, t_1, \dots, t_n)$  depending on the variables  $s_1, \dots, s_n \in \mathbb{R}^{d_1}$  and  $t_1, \dots, t_n \in \mathbb{R}^{d_2}$ . Vectors from  $\mathcal{F}_n(T)$  and  $\mathcal{F}_n(H)$  appear as symmetric functions. We assume the set of all smooth compactly supported functions on  $\mathbb{R}^{d_2 n}$  to be a dense subset of  $(T_+)_{\mathbb{C}}^{\otimes n}$ .

Consider an ordered partition  $\omega = (\omega_1, \dots, \omega_k)$  of the set  $\{1, \dots, n\}$  into  $k$  nonempty sets  $\omega_1, \dots, \omega_k$ . Let  $\Omega_n^k$  stand for the set of all such partitions and let  $|\omega_k|$  stand for the cardinality of  $\omega_k$ . Introduce the mapping

$$\mathbb{R}^{d_2 k} \ni (t_1, \dots, t_k) \mapsto \pi_\omega(t_1, \dots, t_k) = (t_{i_1}, \dots, t_{i_n}) \in \mathbb{R}^{d_2 n}$$

with  $i_j = l$  for  $j \in \omega_l$ .

**4.2. An explicit formula for the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$ .** We are now ready to identify the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  explicitly. This will show the similarity between  $\mathcal{F}^{\text{ext}}(T_+, K)$  and the extended Fock space, on which we will elaborate after proving the theorem. It is worthy to remark the relevance to the construction in [30].

Given a smooth compactly supported symmetric function  $F \in \mathcal{F}_n(T_+)$ ,  $n \in \mathbb{N}$ , denote  $D_F = (0, D_F^1, \dots, D_F^n, 0, 0, \dots) \in \mathcal{F}(H)$  with

$$\begin{aligned} D_F^k(s_1, \dots, s_k, t_1, \dots, t_k) &= \\ &= \sum_{\omega \in \Omega_n^k} \frac{1}{\sqrt{k!}} (\kappa_{|\omega_1|}(s_1) \dots \kappa_{|\omega_k|}(s_k)) F(\pi_\omega(t_1, \dots, t_k)), \quad k = 1, \dots, n. \end{aligned}$$

**Theorem 4.1.** *In the specific framework of this section, when  $K$  is the multiplication operator and  $J$  is the Poisson field, the scalar product  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$  can be computed explicitly. It satisfies the equality*

$$(F, G)_{\mathcal{F}^{\text{ext}}(T_+, K)} = \frac{1}{n!} \sum_{k=1}^n \int_{\mathbb{R}^{d_2 k}} \int_{\mathbb{R}^{d_1 k}} D_F^k(s, t) \overline{D_G^k(s, t)} d\sigma^{\otimes k}(s) d\tau^{\otimes k}(t) \quad (4.2)$$

for any smooth compactly supported symmetric functions  $F, G \in \mathcal{F}_n(T_+)$ ,  $n \in \mathbb{N}$  (the overbar denoting the complex conjugacy). This describes the space  $\mathcal{F}^{\text{ext}}(T_+, K)$  and leads to the Wiener–Itô decomposition for  $\rho_K$ .

Before proving Theorem 4.1, we have to introduce some notations and make some remarks. We also have to state three auxiliary lemmas.

According to the definition of  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$ , it suffices to show that  $AF = \frac{1}{\sqrt{n!}} D_F$ .

Calculating  $AF$  involves identifying the subspaces  $\mathcal{V}_l$ ,  $l \in \mathbb{Z}_+$ , and the vector  $V_n K^{\otimes n} F$ .

We represent  $\mathcal{V}_l$  as the subspace of  $\mathcal{F}(H)$  generated by an explicitly described set of functions. More precisely, fix an orthonormal basis  $(c_i)_{i=1}^\infty$  of the space  $T$  such that each function  $c_i$  is smooth and compactly supported. Define the symmetrization operator  $\hat{\cdot}$  on  $T_{\mathbb{C}}^{\otimes k}$  and  $H_{\mathbb{C}}^{\otimes k}$ ,  $k \in \mathbb{N}$ , via the formulas

$$\hat{F}(t_1, \dots, t_k) = \frac{1}{k!} \sum_{\iota \in S_k} F(t_{\iota(1)}, \dots, t_{\iota(k)}), \quad F \in T_{\mathbb{C}}^{\otimes k},$$

$$\hat{\Phi}(s_1, \dots, s_k, t_1, \dots, t_k) = \frac{1}{k!} \sum_{\iota \in S_k} \Phi(s_{\iota(1)}, \dots, s_{\iota(k)}, t_{\iota(1)}, \dots, t_{\iota(k)}), \quad \Phi \in H_{\mathbb{C}}^{\otimes k}$$

( $S_k$  denoting the group of all permutations of the set  $\{1, \dots, k\}$ ). Given  $l \in \mathbb{N}$ , let  $\mathcal{W}_l$  stand for the subspace of  $\mathcal{F}(H)$  generated by the vacuum  $\Omega$  and the functions

$$(\kappa_{i_1}(s_1)c_{j_1}(t_1) \dots \kappa_{i_k}(s_k)c_{j_k}(t_k))^\wedge, \quad k, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{N},$$

$$i_1 + \dots + i_k \leq l.$$

We suppose  $\mathcal{W}_0 = \mathbb{C}$ . Lemma 4.2 establishes the necessary properties of the subspaces  $\mathcal{W}_l$ . Lemma 4.4 shows that  $\mathcal{V}_l = \mathcal{W}_l$  for any  $l \in \mathbb{Z}_+$ .

It should be noted that the proof of Lemma 4.4 is based on the arguments from [24].

We represent the vector  $V_n K^{\otimes n} F$  as the sum of an explicitly described vector  $B_F$  and a vector  $N_F \in \mathcal{W}_{n-1}$ . Lemma 4.3 establishes the corresponding formula.

When put together, the assertions of Lemma 4.3 and Lemma 4.4 enable us to show that  $AF = \frac{1}{\sqrt{n!}} D_F$ . As mentioned above, this completes the proof of Theorem 4.1.

**Lemma 4.2.** *The inclusions*

$$J_+(Kc_i)\mathcal{W}_l \subset \mathcal{W}_{l+1}, \quad J_0(Kc_i)\mathcal{W}_l \subset \mathcal{W}_{l+1}, \quad J_-(Kc_i)\mathcal{W}_l \subset \mathcal{W}_{l-1}, \quad i \in \mathbb{N},$$

hold true. The first and the second inclusion hold for  $l \in \mathbb{Z}_+$ . The third one holds for  $l \in \mathbb{N}$ .

**Proof.** Let us prove the first inclusion. The second and the third one can be proved analogously.

Evidently,  $J_+(Kc_i)\Omega \in \mathcal{W}_{l+1}$  for any  $l \in \mathbb{Z}_+$ . By calculating the function

$$J_+(Kc_i)(\kappa_{i_1}(s_1)c_{j_1}(t_1) \dots \kappa_{i_k}(s_k)c_{j_k}(t_k))^\wedge, \quad k, i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{N},$$

explicitly, one can show that

$$J_+(Kc_i)(\kappa_{i_1}(s_1)c_{j_1}(t_1) \dots \kappa_{i_k}(s_k)c_{j_k}(t_k))^\wedge \in \mathcal{W}_{l+1}$$

as soon as  $i_1 + \dots + i_k \leq l$ . Furthermore, the restriction  $J_+(Kc_i) \upharpoonright \mathcal{W}_l$  is a bounded operator because  $\mathcal{W}_l \subset \bigoplus_{j=0}^l \mathcal{F}_j(H)$ . Thus  $J_+(Kc_i)\mathcal{W}_l \subset \mathcal{W}_{l+1}$  for any  $l \in \mathbb{Z}_+$ .

The lemma is proved.

Denote  $B_F = (0, B_F^1, \dots, B_F^n, 0, 0, \dots) \in \mathcal{F}(H)$  with

$$\begin{aligned} B_F^k(s_1, \dots, s_k, t_1, \dots, t_k) &= \\ &= \sum_{\omega \in \Omega_n^k} \frac{1}{\sqrt{k!}} \left( \kappa^{|\omega_1|}(s_1) \dots \kappa^{|\omega_k|}(s_k) \right) F(\pi_\omega(t_1, \dots, t_k)), \quad k = 1, \dots, n. \end{aligned}$$

Observe that although  $B_F$  resembles  $D_F$ , these vectors are not identical.

**Lemma 4.3.** *The equality*

$$V_n K^{\otimes n} F = B_F + N_F,$$

holds with  $N_F \in \mathcal{W}_{n-1}$ .

**Proof.** The statement is evident for  $n = 1$ . Assume it to hold for  $n = m$ .

Fix the basis vectors  $c_{j_1}, \dots, c_{j_{m+1}}$ . Denote

$$F_m(t_1, \dots, t_m) = (c_{j_1}(t_1) \dots c_{j_m}(t_m))^\wedge.$$

It suffices to carry out the proof for the function

$$F_{m+1}(t_1, \dots, t_{m+1}) = (c_{j_{m+1}}(t_{m+1})F_m(t_1, \dots, t_m))^\wedge.$$

According to the induction hypothesis,



$$\begin{aligned} V_{m+1}K^{\otimes m+1}F_{m+1} &= J(Kc_{j_{m+1}})V_mK^{\otimes m}F_m = \\ &= J_+(Kc_{j_{m+1}})B_{F_m} + J_0(Kc_{j_{m+1}})B_{F_m} + J_-(Kc_{j_{m+1}})B_{F_m} + J(Kc_{j_{m+1}})N_{F_m}. \end{aligned}$$

A straightforward calculation shows that

$$J_+(Kc_{j_{m+1}})B_{F_m} + J_0(Kc_{j_{m+1}})B_{F_m} = B_{F_{m+1}}.$$

Define

$$N_{F_{m+1}} = J_-(Kc_{j_{m+1}})B_{F_m} + J(Kc_{j_{m+1}})N_{F_m}.$$

One can easily verify that  $B_{F_m} \in \mathcal{W}_m$ . Furthermore,  $N_{F_m} \in \mathcal{W}_{m-1}$  by the induction hypothesis. Hence, according to Lemma 4.2, the vector  $N_{F_{m+1}} \in \mathcal{W}_m$ .

The lemma is proved.

**Lemma 4.4.** *The equality  $\mathcal{V}_l = \mathcal{W}_l$  holds for any  $l \in \mathbb{Z}_+$ .*

*Proof.* The statement is evident for  $l = 0$ . Assume it to hold for  $l = m$ .

The inclusion  $\mathcal{V}_{m+1} \subset \mathcal{W}_{m+1}$  is a direct consequence of Lemma 4.3. In order to prove the converse inclusion, consider the function

$$\begin{aligned} G(s_1, \dots, s_k, t_1, \dots, t_k) &= \kappa^{i_1}(s_1) \dots \kappa^{i_k}(s_k) g(t_1, \dots, t_k), \\ i_1 + \dots + i_k &= m + 1, \quad k, i_1, \dots, i_k \in \mathbb{N}, \end{aligned}$$

with  $g$  being smooth and compactly supported. It suffices to prove that  $\hat{G} \in \mathcal{V}_{m+1}$ . In order to do this, we will now construct a sequence  $(G_j)_{j=1}^\infty \subset \mathcal{V}_{m+1}$  which converges to  $\hat{G}$  in the space  $\mathcal{F}(H)$ .

Given a number  $j \in \mathbb{N}$  and a partition  $\omega \in \Omega_{m+1}^i$  with  $i = 1, \dots, m + 1$ , introduce a smooth function  $h_{\omega,j}$  on  $\mathbb{R}^{d_2(m+1)}$  satisfying the following requirements:

- 1) the estimate  $0 \leq h_{\omega,j}(t) \leq 1$  holds for all  $t \in \mathbb{R}^{d_2(m+1)}$ ;
- 2) the equality  $h_{\omega,j}(t) = 1$  holds for  $t \in \text{Ran}(\pi_\omega)$ ;
- 3) the equality  $h_{\omega,j}(t) = 0$  holds for all  $t$  such that the distance between  $t$  and  $\text{Ran}(\pi_\omega)$  is greater than  $\frac{1}{j}$ .

The existence of  $h_{\omega,j}$  is easy to prove.

Fix  $\omega' \in \Omega_{m+1}^k$  such that  $|\omega'_1| = i_1, \dots, |\omega'_k| = i_k$ . Introduce the functions

$$\begin{aligned} \Gamma_j(t) &= g(\pi_{\omega'}^{-1}(\text{Pr}_{\text{Ran}(\pi_{\omega'})} t)) h_{\omega',j}(t) \prod_{i=1}^{k-1} \prod_{\omega \in \Omega_{m+1}^i} (1 - h_{\omega,j}(t)), \\ t &\in \mathbb{R}^{d_2(m+1)}, \quad j \in \mathbb{N}. \end{aligned}$$

According to Lemma 4.3, the equality

$$V_{m+1}K^{\otimes(m+1)}\hat{\Gamma}_j = B_{\hat{\Gamma}_j} + N_{\hat{\Gamma}_j}, \quad j \in \mathbb{N},$$

holds with  $N_{\hat{\Gamma}_j} \in \mathcal{W}_m = \mathcal{V}_m$ . Define  $G_j = \frac{1}{\sqrt{k!}} B_{\hat{\Gamma}_j}$ .

To complete the proof, we have to show that  $(G_j)_{j=1}^\infty$  converges to  $\hat{G}$  in the space  $\mathcal{F}(H)$ .

Clearly,  $B_{\hat{\Gamma}_j}^1 = \dots = B_{\hat{\Gamma}_j}^{k-1} = 0$ . Given  $k_0 \geq k$ , consider a partition  $\omega \in \Omega_{m+1}^{k_0}$ . If  $\omega = (\omega'_{\iota(1)}, \dots, \omega'_{\iota(k)})$  for some  $\iota \in S_k$ , then

$$\lim_{j \rightarrow \infty} \left\| \kappa^{|\omega_1|}(s_1) \dots \kappa^{|\omega_k|}(s_k) \Gamma_j(\pi_\omega(t_1, \dots, t_k)) - G(s_{\iota^{-1}(1)}, \dots, s_{\iota^{-1}(k)}, t_{\iota^{-1}(1)}, \dots, t_{\iota^{-1}(k)}) \right\|_{\mathcal{F}_k(H)} = 0.$$

Otherwise,

$$\lim_{j \rightarrow \infty} \left\| \kappa^{|\omega_1|}(s_1) \dots \kappa^{|\omega_{k_0}|}(s_{k_0}) \Gamma_j(\pi_\omega(t_1, \dots, t_{k_0})) \right\|_{\mathcal{F}_{k_0}(H)} = 0.$$

Hence

$$\lim_{j \rightarrow \infty} \|G_j - \hat{G}\|_{\mathcal{F}(H)} = \lim_{j \rightarrow \infty} \left\| \frac{1}{\sqrt{k!}} B_{\hat{\Gamma}_j}^k - \hat{G} \right\|_{\mathcal{F}_k(H)} = 0$$

and  $(G_j)_{j=1}^\infty$  converges to  $\hat{G}$  in the space  $\mathcal{F}(H)$ .

The lemma is proved.

**Proof of Theorem 4.1.** According to the definition of  $(\cdot, \cdot)_{\mathcal{F}^{\text{ext}}(T_+, K)}$ , it suffices to prove that  $AF = \frac{1}{\sqrt{n!}} D_F$  or, equivalently,

$$V_n K^{\otimes n} F - D_F = \text{Pr}_{\mathcal{V}_{n-1}} V_n K^{\otimes n} F. \tag{4.3}$$

Lemma 4.3 implies the equality

$$V_n K^{\otimes n} F - D_F = B_F + N_F - D_F$$

with  $N_F \in \mathcal{W}_{n-1}$ . One can easily see that

$$\begin{aligned} & (B_F^k - D_F^k)(s_1, \dots, s_k, t_1, \dots, t_k) = \\ &= \sum_{\omega \in \Omega_n^k} \frac{1}{\sqrt{k!}} \left( \kappa^{|\omega_1|}(s_1) \dots \kappa^{|\omega_k|}(s_k) - \kappa_{|\omega_1|}(s_1) \dots \kappa_{|\omega_k|}(s_k) \right) \times \\ & \quad \times F(\pi_\omega(t_1, \dots, t_k)) \in \mathcal{W}_{n-1} \end{aligned}$$

for any  $k = 1, \dots, n$ . Therefore

$$(V_n K^{\otimes n} F - D_F) \in \mathcal{W}_{n-1} = \mathcal{V}_{n-1}.$$

To finish the proof of (4.3), we have to show that the difference

$$V_n K^{\otimes n} F - (V_n K^{\otimes n} F - D_F) = D_F$$

is orthogonal to  $\mathcal{V}_{n-1}$ . Since  $\mathcal{V}_{n-1} = \mathcal{W}_{n-1}$ , it suffices to show that  $D_F^k$  is orthogonal to the function

$$G(s_1, \dots, s_k, t_1, \dots, t_k) = (\kappa_{i_1}(s_1)c_{j_1}(t_1) \dots \kappa_{i_k}(s_k)c_{j_k}(t_k))^\wedge,$$

$$i_1 + \dots + i_k \leq n - 1, \quad i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{N},$$

for any  $k = 1, \dots, n - 1$ .

The scalar product  $(D_F^k, G)_{\mathcal{F}_k(H)}$  can be represented as a linear combination of the expressions

$$(\kappa_{|\omega_1|}, \kappa_{i_{\nu(1)}})_S \dots (\kappa_{|\omega_k|}, \kappa_{i_{\nu(k)}})_S, \quad \nu \in S_k.$$

Using the inequality

$$|\omega_1| + \dots + |\omega_k| = n > n - 1 \geq i_1 + \dots + i_k$$

and the orthogonality of  $\kappa_i$ , one can easily prove that each of these expressions equals 0. Thus  $D_F$  is orthogonal to  $\mathcal{V}_{n-1}$ .

The theorem is proved.

Theorems 3.1 and 4.1 explain the Fock-type structure of  $L^2(T_-, d\rho_K)$ . Theorem 4.1 shows that the space  $\mathcal{F}^{\text{ext}}(T_+, K)$  coincides with the extended Fock space investigated in [11] up to scalar weights at the orthogonal components.

Using the formula  $AF = \frac{1}{\sqrt{n!}} D_F$ , one can construct an embedding of  $\mathcal{F}^{\text{ext}}(T_+, K)$  into a weighted orthogonal sum of function spaces. Using the arguments from [24], see also [7, 8, 10–12], one can extend this imbedding to a unitary operator. We do not adduce the details of the corresponding construction here.

Let us carry out a more accurate comparison of the results of the present paper with the results of [11]. As before, we suppose  $K$  to be defined by (4.1) and  $J$  to be the Poisson field. The extended Fock space is defined in [11] as the weighted orthogonal sum

$$\mathfrak{F} = \bigoplus_{n=0}^{\infty} \mathfrak{F}_n n!.$$

In this formula, the space  $\mathfrak{F}_n$  coincides with the completion of  $\mathcal{F}_n(T_+)$  with respect to the scalar product given by the right-hand side of (4.2) (the space  $\mathfrak{F}_0 = \mathbb{C}$ ). Theorem 4.1 of the present paper implies the equality  $\mathfrak{F}_n = \mathcal{F}_n^{\text{ext}}(T_+, K)$ . Corollary 5.3 of [11] yields a unitary operator  $\mathfrak{U}$  acting from  $\mathfrak{F}$  to  $L^2(T_-, d\rho_K)$ . Noteworthily, the unitarity of  $\mathfrak{U}$  is proved by confronting  $\mathfrak{U}$  with an operator  $\mathcal{I}$  acting from an auxiliary Hilbert space  $\mathbf{H}$  to  $L^2(T_-, d\rho_K)$  as a sum of operators of stochastic integration. Theorem 4.1 and Theorem 5.1 of [11] establish the crucial properties of  $\mathcal{I}$ . Theorem 3.1 of the present paper implies that  $\mathfrak{U}$  coincides with  $I_K$  on  $\mathcal{F}_n^{\text{ext}}(T_+, K)$  up to the scalar factor  $\frac{1}{\sqrt{n!}}$ .

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